ON THE CONVERGENCE OF DIAGONAL ELEMENTS AND ASYMPTOTIC CONVERGENCE RATES FOR THE SHIFTED TRIDIAGONAL QL ALGORITHM*

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Abstract

The convergence of diagonal elements of an irreducible symmetric tridiagonal matrix under QL algorithm with some kinds of shift is discussed. It is proved that if $\alpha_1-\sigma\to 0$ and $\beta_j\to 0$, $j=1, 2, \cdots, m$, then $\alpha_j\to\lambda_j$, $j=1, 2, \cdots, m$, where $\lambda_j (j=1, 2, \cdots, m)$ are m eigenvalues of the matrix, and σ is the origin shift. The asymptotic convergence rates of three kinds of shift, Rayleigh quotient shift, Wilkinson's shift and RW shift, are analysed.

§ 1. Introduction

The shifted QL algorithm is a very efficient algorithm for finding all eigenvalues of a symmetric tridiagonal matrix. The global convergence of the QL algorithm with Wilkinson's shift is proved in [1], [2]. The asymptotic convergence rate of this case is at least quadratic[1], and is often cubic or better than cubic except for special bizarre matrices if they exist[2]. The RW shift is proposed in [3]. The global convergence and at least cubic aymptotic convergence rate for the case of RW shift are proved in [3].

We apply the shifted QL algorithm to a symmetric tridiagonal matrix $T=T^{(1)}$. Let the k-th iteration matrix be

$$T^{(k)}=\begin{pmatrix} \alpha_1^{(k)} & \beta_1^{(k)} & & & \\ \beta_1^{(k)} & \alpha_2^{(k)} & \beta_2^{(k)} & & \\ & \beta_2^{(k)} & & & \\ & & & & \beta_{n-1}^{(k)} \\ & & & \beta_{n-1}^{(k)} & \alpha_n^{(k)} \end{pmatrix}.$$

The global convergence means that $\beta_1^{(k)}\to 0$. Does $\alpha_1^{(k)}$ converge at the same time? Although we know there is an eigenvalue $\lambda_1^{(k)}$ of $T^{(k)}$ such that

$$|\alpha_1^{(k)}-\lambda_1^{(k)}|<|\beta_1^{(k)}|, \tag{1}$$

it seems that no one has proved that for large enough k, $\lambda_1^{(k)}$ is independent of k.

Furthermore, if $\beta_i^{(k)}\to 0$ $(i=1, 2, \cdots, j)$ can we say $\alpha_i^{(k)}$ $(i=1, 2, \cdots, j)$ are convergent?

In this paper the following theorem is proved:

Theorem. *Let $T=T^{(1)}$ be an irreducible symmetric tridiagonal matrix. The QL algorithm with shift $\{\sigma_k\}$ is applied to $T^{(1)}$. If $\alpha_1^{(k)}-\sigma_k\to 0$ and $\beta_i^{(k)}\to 0 (i=1, 2, \cdots, j)$,*

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then $\alpha_s^{(k)}\to\lambda_s$ $(s=1, 2, \cdots, j)$, *where* $\lambda_1, \lambda_2, \cdots, \lambda_j$ *are* j *different eigenvalues of* T.

Using the above theorem, we can give an improvement on Theorem 8.11 of [4] as follows:

Theorem. *Let the QL algorithm with Wilkinson's shift be applied to an unreduced tridiagonal matrix* T. *Then as* $k\to\infty$, $\beta_1\to 0$. *If, in addition,* $\beta_2\to 0$, $\beta_3\to 0$, *then as* $k\to\infty$,

$$|\hat\beta_1/\beta_1^3\beta_2^2|\to|\lambda_2-\lambda_1|^{-3}|\lambda_3-\lambda_1|^{-1}\neq 0,$$

where $\lambda_1, \lambda_2, \lambda_3$ *are the limits of* $\alpha_1, \alpha_2, \alpha_3$.

There is also a dicussion on the asymptotic convergence rate in the case of the Rayleigh quotient shift and the RW shift.

§ 2. Some Basic Theorems

Let

$$T=\begin{pmatrix} \alpha_1 & \beta_1 & & & 0 \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & & \ddots & \beta_{n-1} \\ 0 & & & \beta_{n-1} & \alpha_n \end{pmatrix}$$

be a real tridiagonal symmetric matrix. Given a scalar σ, called the shift, consider the orthogonal-lower triangular factorization

$$T-\sigma I=QL, \tag{2}$$

where I is the identity matrix, Q is an $n\times n$ orthogonal matrix

$$Q=(q_1, q_2, \cdots, q_n),$$
$$q_i=(q_{1i}, q_{2i}, \cdots, q_{ni})^T,$$

and L is a lower triangular matrix

$$L=(l_{ij}),\ l_{ij}=0 \text{ when } j>i.$$

Let

$$\hat T=LQ+\sigma I. \tag{3}$$

Obviously $\hat T$ is a symmetric tridiagonal matrix too. Denote

$$\hat T=\begin{pmatrix} \hat\alpha_1 & \hat\beta_1 & & & 0 \\ \hat\beta_1 & \hat\alpha_2 & \hat\beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & & \ddots & \hat\beta_{n-1} \\ 0 & & & \hat\beta_{n-1} & \hat\alpha_n \end{pmatrix}$$

and there is a relationship between T and $\hat T$, namely

$$\hat T=Q^TTQ. \tag{4}$$

The transformation from T to $\hat T$ is a QL transformation with shift σ.

Given a symmetric tridiagonal matrix T, let $T^{(1)}=T$. We do QL transformation with shift σ_k to $T^{(k)}$ successively and get a matrix-sequence $\{T^{(k)}\}$, such that

$$T^{(k)}-\sigma_kI=Q_kL_k,$$

$$T^{(k+1)}=L_kQ_k+\sigma_kI,$$
$$Q_k=(q_1^{(k)},\ q_2^{(k)},\ \cdots,\ q_n^{(k)}),$$
$$L_k=(l_{ij}^{(k)}).$$

From (4), we know $T^{(k)}$ is similar to T. So they have the same eigenvalues. Let

$$T_{i,n}^{(k)}=\begin{pmatrix} \alpha_i^{(k)} & \beta_i^{(k)} & & & \\ \beta_i^{(k)} & \alpha_{i+1}^{(k)} & \beta_{i+1}^{(k)} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \beta_{n-1}^{(k)} \\ & & & \beta_{n-1}^{(k)} & \alpha_n^{(k)} \end{pmatrix},\quad T^{(k)}=T_{1,n}^{(k)}.$$

For the sake of simplicity, hereafter we will often omit the index k and use $\hat{T}$ for $T^{(k+1)}$ if there is no confusion.

Lemma 1. *In the QL transformation if the included angle of q_1 and $e_1=(1,\ 0,\ 0,\ \cdots,\ 0)^T$ is θ, namely $q_{11}=\cos\theta$, then*

$$(T-\sigma I)q_1=l_{11}e_1, \tag{5}$$
$$|\hat{\beta}_1|=l_{11}|\sin\theta|. \tag{6}$$

Proof. See [4, 8-11-2 and 8-11-3].

Lemma 2. *Let*

$$d_s=\det(T_{s,n}-\sigma I),\quad s=1,\ 2,\ \cdots,\ n,$$
$$d_{n+1}=1.$$

If σ is not an eigenvalue of T, then in the QL transformation (2), (3), *the i-th component of q_1*

$$q_{i1}=(-1)^{i-1}l_{11}\beta_1\beta_2\cdots\beta_{i-1}d_{i+1}/d_1,\quad i=1,\ 2,\ \cdots,\ n; \tag{7}$$

when $i=1$,

$$\beta_1\beta_2\cdots,\ \beta_{i-1}=1.$$

Proof. By Cramer's rule, from (5) we get

$$q_{i1}=\frac{1}{d_1}\begin{vmatrix} \bar{\alpha}_1 & \beta_1 & 0 & \cdots & 0 & l_{11} & 0 & \cdots & 0 \\ \beta_1 & \bar{\alpha}_2 & \beta_2 & \cdots & 0 & 0 & & & \\ & \beta_2 & \ddots & & & & & & \\ & & \ddots & \ddots & \bar{\alpha}_{i-1} & 0 & & & \\ & & & & \beta_{i-1} & 0 & \beta_i & & \\ & & & & & 0 & \bar{\alpha}_{i+1} & \ddots & \\ & & & & & & \ddots & \ddots & \beta_{n-1} \\ & & & & & 0 & & \beta_{n-1} & \bar{\alpha}_n \end{vmatrix}$$

$$=(-1)^{i-1}\frac{l_{11}}{d_1}\beta_1\beta_2\cdots\beta_{i-1}d_{i+1},$$

where $\bar{\alpha}_i=\alpha_i-\sigma$.

Theorem 1. *Let T be an irreducible symmetric tridiagonal matrix. Then in the transformation* (2), (3), *the following equalities hold*

$$l_{11}^2=d_1^2/(d_2^2+\beta_1^2K^2), \tag{8}$$

$$\sin^2\theta = \beta_1^2 K^2/(d_2^2+\beta_1^2 K^2), \tag{9}$$

$$\hat\beta_1^2 = \beta_1^2 d_1^2 K^2/(d_2^2+\beta_1^2K^2)^2, \tag{10}$$

where

$$K^2 = d_3^2 + (\beta_2 d_4)^2 + (\beta_2\beta_3 d_5)^2 + \cdots + (\beta_2\beta_3\cdots\beta_{n-1})^2.$$

Proof. If σ is not an eigenvalue of T, then $d_1\neq 0$. So

$$q_{i1} = (-1)^{i-1} l_{11}\beta_1\beta_2\cdots\beta_{i-1}d_{i+1}/d_1.$$

Because $\sum_{i=1}^n q_{i1}^2 = 1$ and by Lemma 2 we get

$$\frac{l_{11}^2}{d_1^2}\sum_{i=1}^n (\beta_1\beta_2\cdots\beta_{i-1}d_{i+1})^2 = 1.$$

So

$$l_{11}^2 = d_1^2/(d_2^2+\beta_1^2K^2).$$

By

$$\sin^2\theta = q_{21}^2 + q_{31}^2 + \cdots + q_{n1}^2,$$

$$\sin^2\theta = \frac{l_{11}^2}{d_1^2}(\beta_1 d_3)^2 + (\beta_1\beta_2 d_4)^2 + \cdots + (\beta_1\beta_2\cdots\beta_{n-1})^2$$

$$= l_{11}^2\beta_1^2K^2/d_1^2 = \beta_1^2K^2/(d_2^2+\beta_1^2K^2).$$

At last by (6),

$$\hat\beta_1^2 = l_{11}^2\sin^2\theta = d_1^2\beta_1^2K^2/(d_2^2+\beta_1^2K^2)^2.$$

If σ is an eigenvalue of T, then (8) and (10) hold obviously. For equality (9) in this case

$$l_{11}=0,\ l_{22}\neq 0,\ l_{33}\neq 0,\ \cdots,\ l_{nn}\neq 0.$$

So q_i ($i=2, 3, \cdots, n$) are continuous functions of σ. Since q_1 is the only vector which is orthogonal with $q_2, q_3, \cdots, q_n$, unless a sign, q_1 is a continuous function of σ too. Therefore both sides of (9)

$$\sin^2\theta = \sum_{i=2}^n q_{i1}^2 \quad \text{and} \quad \beta_1^2K^2/(d_2^2+\beta_1^2K^2)$$

are continuous functions of σ. Using the limit we know (9) holds when σ is an eigenvalue of T.

Lemma 3. *Let T be unreduced. If the shift σ_k satisfies*

$$\alpha_1^{(k)} - \sigma_k \to 0 \quad \text{and} \quad \beta_1^{(k)} \to 0,$$

then

$$\sin\theta_k \to 0,$$

where

$$\sin^2\theta_k = \sum_{i=2}^n (q_{i1}^{(k)})^2.$$

Proof. By (9),

$$\sin^2\theta_k = (\beta_1^{(k)})^2(K^{(k)})^2/((d_2^{(k)})^2 + (\beta_1^{(k)})^2(K^{(k)})^2).$$

Because $\|T^{(k)}\|_2 = \|T\|_2$ is bounded, $\beta_1^{(k)}, \beta_2^{(k)}, \cdots, \beta_{n-1}^{(k)}$ are bounded uniformly, and so are $d_1^{(k)}, d_2^{(k)}, \cdots, d_n^{(k)}$ and $K^{(k)}$.

On the other hand,

$$d_2^{(k)} = \det\,(T_{2,n}^{(k)} - \sigma_k I) = \prod_{j=2}^n (\mu_j^{(k)} - \sigma_k),$$

where $\mu_2^{(k)}, \mu_3^{(k)}, \cdots, \mu_n^{(k)}$ are eigenvalues of $T_{2,n}^{(k)}$.

By the Wielandt–Hoffman theorem,

$$(\alpha_1^{(k)}-\lambda_1^{(k)})^2+\sum_{j=2}^{n}(\mu_j^{(k)}-\lambda_j^{(k)})^2=2(\beta_1^{(k)})^2,$$

where $\lambda_1^{(k)}, \lambda_2^{(k)}, \cdots, \lambda_n^{(k)}$ are eigenvalues of matrix T.

Since T is unreduced, $\lambda_1^{(k)}, \lambda_2^{(k)}, \cdots, \lambda_n^{(k)}$ are different with each other.

For a large enough natural number K_1, when $k>K_1$, we have

$$|\alpha_1^{(k)}-\sigma_k|<\min_{i\neq j}|\lambda_i^{(k)}-\lambda_j^{(k)}|/10=\delta$$

and $\beta_1^{(k)}<\delta/2$. Hence

$$|\mu_j^{(k)}-\sigma_k|=|\mu_j^{(k)}-\lambda_j^{(k)}+\lambda_j^{(k)}-\lambda_1^{(k)}+\lambda_1^{(k)}-\alpha_1^{(k)}+\alpha_1^{(k)}-\sigma_k|$$
$$\geqslant|\lambda_j^{(k)}-\lambda_1^{(k)}|-3\delta\geqslant 7\delta,$$

and there is a constant C independent of k such that

$$|d_2^{(k)}|=\prod_{j=2}^{n}|\mu_j^{(k)}-\sigma_k|\geqslant C>0.$$

By (9) we have

$$\lim_{k\to\infty}\sin^2\theta_k=0.$$

Corollary. If σ_k is the Wilkinson shift or the RW shift, then

$$\sin\theta_k\to 0.$$

Proof. It is known that in the case of the Wilkinson shift or the RW shift, we have

$$\alpha_1^{(k)}-\sigma_k\to 0 \quad \text{and} \quad \beta_1^{(k)}\to 0.$$

So $\sin\theta_k\to 0$.

Theorem 2. *Let T be an irreducible symmetric tridiagonal matrix. From $T^{(1)}=T$, successively do QL transformation with shift σ_k. If $\alpha_1^{(k)}-\sigma_k\to 0$ and there is an integer j $(1\leqslant j<n)$, such that*

$$\beta_i^{(k)}\to 0, \quad i=1, 2, \cdots, j,$$

then

$$q_{i,i+1}^{(k)}\to 0, \quad i=1, 2, \cdots, j.$$

Proof.

$$\hat{T}=LQ+\sigma I,$$
$$\hat{\beta}_1=q_{12}l_{11}.$$

By Lemma 1

$$|\hat{\beta}_1|=l_{11}|\sin\theta|;$$

so

$$|q_{12}|=|\sin\theta|.$$

In the case $j=1$, the conditions of this theorem are

$$\alpha_1^{(k)}-\sigma_k\to 0 \quad \text{and} \quad \beta_1^{(k)}\to 0.$$

By Lemma 3, we have $\sin\theta\to 0$. Hence $q_{12}\to 0$. It shows that Theorem 2 holds when $j=1$.

Now let us use the Principle of Finite Induction. Suppose the proposition is true when $j=m-1$. From

$$(T-\sigma I)Q=L^T,$$

we have

$$(T-\sigma I)q_m=l_{m1}e_1+l_{m2}e_2+\cdots+l_{mm}e_m.$$

Let

$$\tilde{q}_m=(q_{m,m}, q_{m-1,m}, \cdots, q_{n,m})^T$$

and $e_1=(1, 0, \cdots, 0)^T \in R^{n-m+1}$. So

$$(T_{m,n}-\sigma I)\tilde{q}_m=(l_{mm}-\beta_{m-1}q_{m-1,m})e_1.$$

By Lemma 2,

$$q_{i,m}=(-1)^{i-m}(l_{mm}-\beta_{m-1}q_{m-1,m})\beta_m\beta_{m+1}\cdots\beta_{i-1}d_{i+1}/d_m, \quad i=m, m+1, \cdots, n.$$

Since

$$q_m=(0, 0, \cdots, 0, q_{m-1,m}, \tilde{q}_m^T)^T$$

and

$$q_{m-1,m}^2+\sum_{i=m}^{n} q_{i,m}^2=1,$$

so

$$\sum_{i=m}^{n} q_{i,m}^2=\frac{(l_{m,m}-\beta_{m-1}q_{m-1,m})^2}{d_m^2}(d_{m+1}^2+\beta_m^2G_m^2)=1-q_{m-1,m}^2,$$

where $G_m^2=d_{m+2}^2+(\beta_{m+1}d_{m+3})^2+\cdots+(\beta_{m+1}\beta_{m+2}\cdots\beta_{n-1})^2$. Therefore

$$(l_{m,m}-\beta_{m-1}q_{m-1,m})^2=\frac{(1-q_{m-1,m}^2)d_m^2}{d_{m+1}^2+\beta_m^2G_m^2},$$

$$(l_{m,m}-\beta_{m-1}q_{m-1,m})=\pm(1-q_{m-1,m}^2)^{1/2}d_m/(d_{m+1}^2+\beta_m^2G_m^2)^{1/2},$$

and

$$l_{mm}=\beta_{m-1}q_{m-1,m}\pm(1-q_{m-1,m}^2)^{1/2}d_m/(d_{m+1}^2+\beta_m^2G_m^2)^{1/2}.$$

Since $\beta_i\to 0$ $(i=1, 2, \cdots, m)$, it is easy to know

$$|d_m|\geqslant C>0,$$

$$|d_{m+1}|\geqslant C>0$$

for large enough k. So by the hypothesis of the induction $q_{m-1,m}\to 0$

$$|l_{mm}|\geqslant C'>0$$

for large enough k, where C' is independent of k.

The m-th row and $m+1$-th column of the equality

$$\hat{T}=LQ+\sigma I$$

is

$$\hat{\beta}_m=l_{m1}q_{1,m+1}+\cdots+l_{m,m-1}q_{m-1,m+1}+l_{mm}q_{m,m+1}=l_{mm}q_{m,m+1}.$$

Thus

$$q_{m,m+1}=\hat{\beta}_m/l_{mm}\to 0.$$

Corollary. The asymptotic convergence rate of $q_{m,m+1}$ is the same as that of $\hat{\beta}_m$.

Theorem 3. *Let the QL algorithm with shift $\{\sigma_k\}$ be applied to an irreducible symmetric tridiagonal matrix $T=T^{(1)}$. If there is an index j $(1\leqslant j<n)$ such that*

$$\alpha_1^{(k)}-\sigma_k\to 0 \quad and \quad \beta_i^{(k)}\to 0, \quad i=1, 2, \cdots, j,$$

then

$$q_s\to\pm e_s \quad and \quad \alpha_s\to\lambda_s, \quad s=1, 2, \cdots, j,$$

where $\lambda_1, \lambda_2, \cdots, \lambda_j$ are j different eigenvalues of T.

Proof. For $j=1$, by $\sin\theta\to 0$, we have $q_1\to\pm e_1$. Now we prove

$$q_m\to\pm e_m, \quad m=2, 3, \cdots, j.$$

By

$$(T-\sigma I)q_m=l_{m1}e_1+l_{m2}e_2+\cdots+l_{mm}e_m,$$

$$q_{mm}^2=(l_{mm}-\beta_{m-1}q_{m-1,m})^2d_{m+1}^2/d_m^2.$$

In the proof of Theorem 2, we know

$$(l_{mm}-\beta_{m-1}q_{m-1,m})^2=(1-q_{m-1,m}^2)d_m^2/(d_{m+1}^2+\beta_m^2G_m^2).$$

So

$$q_{mm}^2=(1-q_{m-1,m}^2)d_{m+1}^2/(d_{m+1}^2+\beta_m^2G_m^2)\to 1,$$

namely

$$q_m\to\pm e_m.$$

Now we come to prove $\alpha_s\to\lambda_s$:

$$T^{(k)}=\begin{pmatrix}\alpha_1^{(k)} & & & 0\\ & \alpha_2^{(k)} & & \\ & & \ddots \ \alpha_j^{(k)} & \\ 0 & & & T_{j+1,n}^{(k)}\end{pmatrix}+\begin{pmatrix}B_{j+1} & 0\\ 0 & 0\end{pmatrix},$$

where

$$B_{j+1}=\begin{pmatrix}0 & \beta_1^{(k)} & & & \\ \beta_1^{(k)} & 0 & \beta_2^{(k)} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \beta_j^{(k)}\\ & & & \beta_j^{(k)} & 0\end{pmatrix}.$$

Obviously $\|B_{j+1}\|_F\to 0$ under the hypotheses of Theorem 3. By the Wielandt-Hoffman theorem, we know there is a natural number K_1, such that when $k>K_1$,

$$|\alpha_s^{(k)}-\lambda_s|\leqslant\delta_1,\quad s=1,\ 2,\ \cdots,\ j,\tag{11}$$

where

$$\delta_1<\min_{i\neq j}|\lambda_i-\lambda_j|/10,$$

$\lambda_j(j=1,\ 2,\ \cdots,\ n)$ are eigenvalues of T, which are different since T is an irreducible matrix.

On the z-plane, there are n circles

$$|z-\lambda_j|\leqslant\delta_1,\quad j=1,\ 2,\ \cdots,\ n\tag{12}$$

disjointed with each other.

We will prove that for large enough k, the eigenvalue λ_s in (11), $s=1,\ 2,\ \cdots,\ j$, are independent of k. For this aim, we will show that if $\alpha_s^{(k)}$ falls in the circle

$$|z-\lambda_s|\leqslant\delta_1\tag{13}$$

then $\alpha_s^{(k+1)}$ falls in the same circle (13).

By

$$\hat{T}=Q^TTQ,$$

so

$$\hat{\alpha}_s=q_s^TTq_s.$$

By $q_s\to\pm e_s$, we can write $q_s=\pm e_s+\varepsilon_s$, where $\varepsilon_s\to 0$ $(k\to\infty)$,

$$\hat{\alpha}_s=(\pm e_s^T+\varepsilon_s^T)T(\pm e_s+\varepsilon_s)=e_s^TTe_s\pm 2\varepsilon_s^TTe_s+\varepsilon_s^TT\varepsilon_s=\alpha_s+2\varepsilon_s^TTe_s+\varepsilon_s^TT\varepsilon_s.$$

We can find $K_2>K_1$. When $k\geqslant K_2$ there holds

$$|\alpha_s^{(k+1)}-\alpha_s^{(k)}|\leqslant\delta_1.$$

Therefore

$$|\alpha_s^{(k+1)}-\lambda_s|=|\alpha_s^{(k+1)}-\alpha_s^{(k)}+\alpha_s^{(k)}-\lambda_s|\leqslant|\alpha_s^{(k+1)}-\alpha_s^{(k)}|+|\alpha_s^{(k)}-\lambda_s|\leqslant 2\delta_1.$$

It says that $\alpha_s^{(k+1)}$ falls in no other circle of (12) than the circle (13). It shows that when $k\geqslant K_2$, λ_s in (11) is independent of k. Since δ_1 can be arbitrarily small, hence

$$\lim_{k\to\infty} \alpha_s^{(k)} = \lambda_s.$$

§ 3. Asymptotic Convergence Rates for Some Kinds of Shift

Lemma 4. *If* $\alpha_1^{(k)} - \sigma_k \to 0$ *and* $\beta_i \to 0$ $(i=1, 2, \cdots, j)$, *then*

$$d_{j+1} \to \prod_{m=j+1}^{n} (\lambda_m - \lambda_1).$$

Proof. By Theorem 3,

$$\alpha_s^{(k)} \to \lambda_s, \quad s=1, 2, \cdots, j.$$

Let the n eigenvalues of T be $\lambda_1, \lambda_2, \cdots, \lambda_s, \lambda_{s+1}, \cdots, \lambda_n$. Denote $n-j$ eigenvalues of $T_{j+1,n}$ as $\mu_{j+1}, \mu_{j+2}, \cdots, \mu_n$. By the Wielandt–Hoffman theorem,

$$(\mu_{j+1} - \lambda_{j+1})^2 + (\mu_{j+2} - \lambda_{j+2})^2 + \cdots + (\mu_n - \lambda_n)^2 \leqslant \|B_{j-1}\|_F^2.$$

$$d_{j+1} = \prod_{m=j+1}^{n} (\mu_m - \sigma) = \prod_{m=j+1}^{n} (\lambda_m - \lambda_1) + \varepsilon,$$

where $\varepsilon \to 0$ while $\|B_{j+1}\|_F^2 \to 0$. Since $\prod_{m=j+1}^{n} (\lambda_m - \lambda_1)$ is independent k, when k is large enough,

$$d_{j+1} \to \prod_{m=j+1}^{n} (\lambda_m - \lambda_1).$$

Corollary. When $j \geqslant 2$, $K^2 \to \prod_{m=3}^{n} (\lambda_m - \lambda_1)^2$.

Now we turn to the asymptotic convergence rates for some kinds of shift.

(1) Rayleigh quotient shift. In this case

$$\sigma_k = \alpha_1^{(k)}.$$

It satisfies the condition $\alpha_1^{(k)} - \sigma_k \to 0$ obviously. By (10),

$$\hat{\beta}_1^2 = \beta_1^2 d_1^2 K^2 / (d_2^2 + \beta_1^2 K^2)^2.$$

Since

$$d_1 = (\alpha_1 - \sigma) d_2 - \beta_1^2 d_3 = -\beta_1^2 d_3,$$

therefore

$$\hat{\beta}_1^2 = \beta_1^6 d_3^2 K^2 / (d_2^2 + \beta_1^2 K^2)^2.$$

Moreover if $\beta_1 \to 0$, then

$$\hat{\beta}_1^2 / (\beta_1^6 d_3^2 K^2) \to \prod_{m=2}^{n} (\lambda_m - \lambda_1)^{-4} \neq 0.$$

If in addition $\beta_2 \to 0$, then

$$d_3^2 \to \prod_{m=3}^{n} (\lambda_m - \lambda_1)^2,$$

$$K^2 \to \prod_{m=3}^{n} (\lambda_m - \lambda_1)^2,$$

and

$$\hat{\beta}_1^2 / \beta_1^6 \to (\lambda_2 - \lambda_1)^{-4} \neq 0.$$

Therefore we have

Theorem 4. *Suppose the symmetric tridiagonal matrix T is irreducible. We do the QL transformation with Rayleigh quotient shift successively. If $\beta_1 \to 0$, then*

$$\hat{\beta}_1^2 / (\beta_1^6 d_3^2 K^2) \to \prod_{m=2}^{n} (\lambda_m - \lambda_1)^{-4} \neq 0.$$

If in addition $\beta_2\to 0$, *then*

$$\hat\beta_1^2/\beta_1^6\to((\lambda_2-\lambda_1)^4)^{-1}\neq 0.$$

(2) Wilkinson's shift. In this case σ is the root of the equation

$$(\alpha_1-\sigma)(\alpha_2-\sigma)-\beta_1^2=0,$$

which is the one nearer to α_1, namely

$$\sigma=\alpha_1-\mathrm{sign}(\delta)\beta_1^2/(|\delta|+\sqrt{\delta^2+\beta_1^2}),$$

where $\delta=(\alpha_2-\alpha_1)/2$. We have $|\alpha_1-\sigma|\leqslant\beta_1$. By [1], [2], we know $\beta_1\to 0$ in this case. So it satisfies the conditions of Theorem 3 when $j=1$, namely

$$\alpha_1-\sigma\to 0 \quad\text{and}\quad \beta_1\to 0.$$

By Theorem 3, $\alpha_1\to\lambda_1$. On the other hand

$$\hat\beta_1^2=\beta_1^2d_1^2K^2/(d_2^2+\beta_1^2K^2)^2$$

and

$$d_1=(\alpha_1-\sigma)d_2-\beta_1^2d_3,$$
$$d_2=(\alpha_2-\sigma)d_3-\beta_2^2d_4.$$

So

$$d_1=((\alpha_1-\sigma)(\alpha_2-\sigma)-\beta_1^2)d_3-(\alpha_1-\sigma)\beta_2^2d_4=-(\alpha_1-\sigma)\beta_2^2d_4$$

and

$$\hat\beta_1^2=\beta_1^2(\alpha_1-\sigma)^2\beta_2^4d_4^2K^2/(d_2^2+\beta_1^2K^2)^2.$$

Therefore

$$\hat\beta_1^2/(\beta_1^2(\alpha_1-\sigma)^2\beta_2^4d_4^2K^2)\to\prod_{m=2}^{n}(\lambda_m-\lambda_1)^{-4}\neq 0.$$

Moreover

$$\alpha_1-\sigma=\beta_1^2/(\alpha_2-\sigma).$$

So

$$\hat\beta_1^2=\beta_1^6\beta_2^4d_4^2K^2/((\alpha_2-\sigma)^2(d_2^2+\beta_1^2K^2)^2),$$

and

$$\hat\beta_1^2(\alpha_2-\sigma)^2/(\beta_1^6\beta_2^4d_4^2K^2)\to\prod_{m=2}^{n}(\lambda_m-\lambda_1)^{-4}\neq 0.$$

If in addition $\beta_2\to 0$, then

$$\alpha_2-\sigma\to\lambda_2-\lambda_1\neq 0$$

and

$$K^2\to\prod_{m=3}^{n}(\lambda_m-\lambda_1)^2.$$

Therefore

$$\hat\beta_1^2/(\beta_1^6\beta_2^4d_4^2)\to(\lambda_2-\lambda_1)^{-6}\prod_{m=3}^{n}(\lambda_m-\lambda_1)^{-2}\neq 0.$$

If in addition $\beta_2\to 0$, $\beta_3\to 0$, then

$$d_4^2\to\prod_{m=4}^{n}(\lambda_m-\lambda_1)^2,$$

and

$$\hat\beta_1^2/(\beta_1^6\beta_2^4)\to(\lambda_2-\lambda_1)^{-6}(\lambda_3-\lambda_1)^{-2}\neq 0.$$

We write these conclusions as follows.

Theorem 5. *Let the symmetric tridiagonal matrix T be irreducible, and make QL transformation with Wilkinson's shift successively. Then*

$$\hat\beta_1^2(\alpha_2-\sigma)^2/(\beta_1^6\beta_2^4d_4^2K^2)\to\prod_{m=2}^{n}(\lambda_m-\lambda_1)^{-4}\neq 0.$$

If in addition $\beta_2\to 0$, *then*

$$\hat{\beta}_1^2/(\beta_1^6\beta_2^4d_4^2)\to(\lambda_2-\lambda_1)^{-6}\prod_{m=3}^{n}(\lambda_m-\lambda_1)^{-2}\neq 0.$$

If in addition $\beta_2\to 0$, $\beta_3\to 0$, then

$$\hat{\beta}_1^2/(\beta_1^6\beta_2^4)\to(\lambda_2-\lambda_1)^{-6}(\lambda_3-\lambda_1)^{-2}\neq 0.$$

Theorem 5 is an improvement on Theorem 8.11 of [4], where $\alpha_i\to\lambda_i(i=1, 2, 3)$ are considered as conditions.

(3) *RW* shift (see [3]). In this case the shift σ is Wilkinson's shift, namely

$$\sigma=\alpha_1-\operatorname{sign}(\delta)\beta_1^2/(|\delta|+\sqrt{\delta^2+\beta_1^2}),\quad \delta=(\alpha_2-\alpha_1)/2$$

if $\beta_2^2<2\beta_1^2$. And the shift is the Rayleigh quotient shift, namely

$$\sigma=\alpha_1$$

if $\beta_2^2\geqslant 2\beta_1^2$.

By [3] we have $\beta_1\to 0$ in this case. From the definition of the *RW* shift, it is easy to know $\alpha_1-\sigma\to 0$. So the conditions of Theorem 3 hold in this case. We have

$$\alpha_1\to\lambda_1,\quad d_2\to\prod_{m=2}^{n}(\lambda_m-\lambda_1)^2.$$

Theorem 6. *Let the symmetric tridiagonal matrix T be irreducible, and make the QL transformation with RW shift successively. Then*

$$\hat{\beta}_1^2/(G\beta_1^6)\to\prod_{m=2}^{n}(\lambda_m-\lambda_1)^{-4}\neq 0,$$

where

$$G=\begin{cases} d_3^2K^2, & \beta_2^2\geqslant 2\beta_1^2,\\ \beta_2^4d_4^2K^2/(\alpha_2-\sigma)^2, & \beta_2^2<2\beta_1^2\end{cases}$$

and in the case $\beta_2^2<2\beta_1^2$, we have $(\alpha_2-\sigma)^2\geqslant C>0$, C being a constant independent of k, for large enough k[3].

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