APPROXIMATION OF BOUNDARY CONDITIONS AT INFINITY FOR A HARMONIC EQUATION*

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Abstract

Starting from the canonical boundary reduction, this paper studies an approximate differential boundary condition and an approximate integral boundary condition on an artificial boundary for the exterior problem of a harmonic equation, and gives an error estimate for the latter. This estimate reveals the relationship between the error and the approximate grade of boundary conditions as well as the radius of the artificial boundary.

§ 1. Approximation of the Integral Boundary Condition

The treatment of an elliptic boundary value problem over an unbounded domain by the classical finite element method is often a difficulty, because a simple replacement of the infinite domain by a bounded domain can hardly produce the demanded accuracy. The canonical boundary reduction suggested by Feng Kang[1,2] and the coupling of the canonical boundary element method with the finite element method[5] have provided an approach to this problem.

Consider the boundary value problem of a harmonic equation over an exterior domain Ω with smooth boundary Γ_i

$$\begin{cases} -\Delta u=0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}=f, & \text{on } \Gamma_i, \\ u \text{ is bounded at infinity,} \end{cases} \tag{1}$$

where $f\in H^{-\frac{1}{2}}(\Gamma_i)$ satisfies the compatibility condition. We draw a circle Γ_R with radius R enclosing Γ_i. Ω is divided into Ω_i and Ω_e (Fig. 1). Then the canonical integral equation on Γ_R, obtained from the harmonic boundary value problem over the exterior domain Ω_e by canonical boundary reduction, is just the exact boundary condition on the artificial boundary Γ_R of the original boundary value problem, i.e. the problem (1) is equivalent to

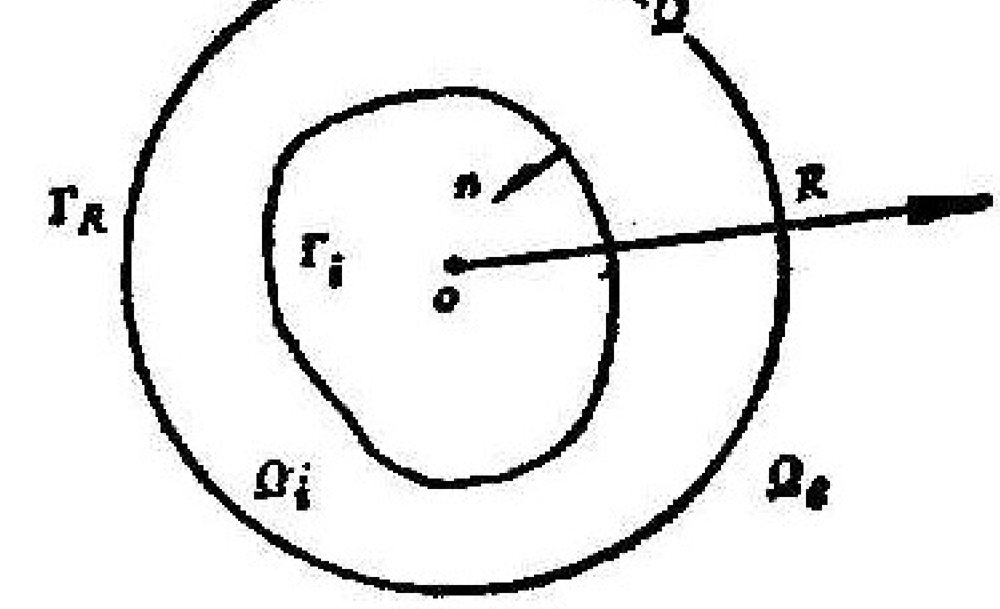

Fig. 1

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$$\begin{cases} -\Delta u=0, & \text{in } \Omega_i, \\ \dfrac{\partial u}{\partial n}=f, & \text{on } \Gamma_i, \\ \dfrac{\partial u}{\partial r}(R,\theta)=\dfrac{1}{4\pi R\sin^2\dfrac{\theta}{2}} * u(R,\theta), & \text{on } \Gamma_R, \end{cases} \tag{2}$$

where Ω_i is the bounded domain between Γ_i and Γ_R, and $*$ denotes the convolution, which can be defined by the Fourier expansion. Because[7]

$$-\frac{1}{4\pi R\sin^2\dfrac{\theta}{2}}=\frac{1}{2\pi R}\sum_{-\infty}^{\infty}|n|e^{in\theta}=\frac{1}{\pi R}\sum_{n=1}^{\infty}n\cos n\theta,$$

the integral boundary condition of (2) can be written

$$\frac{\partial u}{\partial r}(R,\theta)=-\frac{1}{\pi R}\sum_{n=1}^{\infty}n\int_0^{2\pi}u(R,\theta')\cos n(\theta-\theta')\,d\theta'. \tag{3}$$

Obviously (3) is a non-local boundary condition, and its kernel is highly singular. We attempt to simplify this boundary condition for the sake of easier application. Using the asymptotic expansion method, [3] has obtained a series of asymptotic radiation conditions for reduced wave equations. These approximate boundary conditions are differential (i.e. local) boundary conditions. But this method is not applicable to the harmonic equation, so we naturally think of replacing (3) with an approximate integral boundary condition

$$\frac{\partial u}{\partial r}(R,\theta)=-\frac{1}{\pi R}\sum_{n=1}^{N}n\int_0^{2\pi}u(R,\theta')\cos n(\theta-\theta')\,d\theta', \tag{4}$$

where N is a positive integer. The integral kernel of (4) is nonsingular of course. In particular when the Fourier series of $u(R,\theta)$ only contains the first N terms, this boundary condition can be reduced into a local boundary condition

$$\frac{\partial u}{\partial r}(R,\theta)=\frac{1}{R}\sum_{k=1}^{N}\alpha_k\frac{\partial^{2k}}{\partial\theta^{2k}}u(R,\theta), \tag{5}$$

where $\alpha_k (k=1,\cdots,N)$ are the solution of

$$\sum_{k=1}^{N}(-n^2)^k\alpha_k=-n,\quad n=1,2,\cdots,N.$$

We call (4) and (5) an approximate integral boundary condition and an approximate differential boundary condition of grade N of (3) respectively. The first four approximate differential boundary conditions are as follows:

$$N=1:\quad \frac{\partial u}{\partial r}=\frac{1}{R}\frac{\partial^2 u}{\partial\theta^2}, \tag{6$_1$}$$

$$N=2:\quad \frac{\partial u}{\partial r}=\frac{1}{R}\left(\frac{7}{6}\frac{\partial^2 u}{\partial\theta^2}+\frac{1}{6}\frac{\partial^4 u}{\partial\theta^4}\right), \tag{6$_2$}$$

$$N=3:\quad \frac{\partial u}{\partial r}=\frac{1}{R}\left(\frac{74}{60}\frac{\partial^2 u}{\partial\theta^2}+\frac{15}{60}\frac{\partial^4 u}{\partial\theta^4}+\frac{1}{60}\frac{\partial^6 u}{\partial\theta^6}\right), \tag{6$_3$}$$

$$N=4:\quad \frac{\partial u}{\partial r}=\frac{1}{R}\left(\frac{533}{420}\frac{\partial^2 u}{\partial\theta^2}+\frac{43}{144}\frac{\partial^4 u}{\partial\theta^4}+\frac{11}{360}\frac{\partial^6 u}{\partial\theta^6}+\frac{1}{1008}\frac{\partial^8 u}{\partial\theta^8}\right). \tag{6$_4$}$$

Let $\quad D_I(u,v)=\int_{\Omega_i}\nabla u\cdot\nabla v\,dx,$

$$\hat{D}(u, v)=\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi}\sum_{n=1}^{\infty} n\cos n(\theta-\theta')u(R, \theta')v(R, \theta)d\theta'\, d\theta,$$

$$\hat{D}^N(u,v)=\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi}\sum_{n=1}^{N} n\cos n(\theta-\theta')u(R, \theta')v(R, \theta)d\theta'\, d\theta,$$

$$\tilde{D}^N(u,v)=\int_0^{2\pi}\sum_{k=1}^{N}(-1)^{k-1}\alpha_k\frac{\partial^k}{\partial\theta^k}u(R,\theta)\frac{\partial^k}{\partial\theta^k}v(R, \theta)d\theta,$$

$$f(v)=\int_{\Gamma_i} fv\, ds.$$

Then the boundary value problem (2) and the problem corresponding to the approximate integral boundary condition (4) or the approximate differential boundary condition (5) are respectively equivalent to the variational problems

$$\begin{cases}\text{Find } u\in H^1(\Omega_i) \quad \text{such that}\\ D_I(u, v)+\hat{D}(u, v)=f(v), \quad \forall v\in H^1(\Omega_i)\end{cases} \tag{7}$$

and

$$\begin{cases}\text{Find } u^N\in H^1(\Omega_i) \quad \text{such that}\\ D_I(u^N, v)+\hat{D}^N(u^N, v)=f(v), \quad \forall v\in H^1(\Omega_i)\end{cases} \tag{8}$$

or

$$\begin{cases}\text{Find } \tilde{u}^N\in H^1(\Omega_i)\cap H^N(\Gamma_R) \quad \text{such that}\\ D_I(\tilde{u}^N,v)+\tilde{D}^N(\tilde{u}^N, v)=f(v), \quad \forall v\in H^1(\Omega_i)\cap H^N(\Gamma_R).\end{cases} \tag{9}$$

Proposition 1. If $\alpha_N>0$, then $\tilde{D}^N(u, v)$ is a nonnegative definite symmetric bilinear form when N is a positive odd number, and it is not nonnegative definite when N is a positive even number.

Proof. If N is a positive odd number, because the polynomial $P_{2N}(x)-x=\sum_{k=1}^{N}(-1)^{k-1}\alpha_k x^{2k}-x$ has at most $N+1$ different nonnegative roots and $x=0, 1, \cdots, N$ are its roots and $(-1)^{N-1}\alpha_N>0$, we know that $P_{2N}(x)>N>0$ for $x>N$. Then

$$\tilde{D}^N(\cos n\theta, \cos n\theta)=\tilde{D}^N(\sin n\theta, \sin n\theta)=\sum_{k=1}^{N}(-1)^{k-1}\alpha_k n^{2k}\pi\geqslant 0, \quad n=0, 1, \cdots.$$

Hence $\tilde{D}^N(v, v)\geqslant 0$, $\forall v\in H^1(\Omega_i)$.

If N is a positive even number, then $(-1)^{N-1}\alpha_N<0$. Let x_M be the largest root of $P_{2N}(x)$. We have $P_{2N}(x)<0$ for $x>x_M$. Therefore $\tilde{D}^N(u, v)$ is not nonnegative definite.

Corollary. $\tilde{D}^1(u, v)$ and $\tilde{D}^3(u, v)$, associated with the boundary conditions $(6)_1$ and $(6)_3$, are nonnegative definite symmetric bilinear forms. $\tilde{D}^2(u, v)$ and $\tilde{D}^4(u,v)$, associated with $(6)_2$ and $(6)_4$, are not nonnegative definite.

We now consider another series of approximate differential boundary conditions:

$$\frac{\partial u}{\partial r}(R, \theta)=\frac{1}{R}\sum_{k=0}^{N-1}\beta_k\frac{\partial^{2k}}{\partial\theta^{2k}}u(R, \theta), \tag{10}$$

where β_k, $k=0, 1, \cdots, N-1$, are the solution of

$$\sum_{k=0}^{N-1}(-n^2)^k\beta_k=-n, \quad n=1, 2, \cdots, N,$$

and u satisfies $\int_0^{2\pi}u(R, \theta)d\theta=0$, which is not a restriction because the solution of (1) is unique up to a constant. When the Fourier series of $u(R, \theta)$ only contains the first N terms, (10) is also equivalent to (4). The first four approximate differential boundary conditions of form (10) are as follows:

$$N=1: \quad \frac{\partial u}{\partial r}=-\frac{1}{R}u, \tag{11$_1$}$$

$$N=2: \quad \frac{\partial u}{\partial r}=\frac{1}{R}\left(-\frac{2}{3}u+\frac{1}{3}\frac{\partial^2 u}{\partial\theta^2}\right), \tag{11$_2$}$$

$$N=3: \quad \frac{\partial u}{\partial r}=\frac{1}{R}\left(-\frac{3}{5}u+\frac{5}{12}\frac{\partial^2 u}{\partial\theta^2}+\frac{1}{60}\frac{\partial^4 u}{\partial\theta^4}\right), \tag{11$_3$}$$

$$N=4: \quad \frac{\partial u}{\partial r}=\frac{1}{R}\left(-\frac{4}{7}u+\frac{41}{90}\frac{\partial^2 u}{\partial\theta^2}+\frac{1}{36}\frac{\partial^4 u}{\partial\theta^4}+\frac{1}{1260}\frac{\partial^6 u}{\partial\theta^6}\right). \tag{11$_4$}$$

Let $\tilde{\tilde{D}}^N(u, v)=\int_0^{2\pi}\sum_{k=0}^{N-1}(-1)^{k-1}\beta_k\frac{\partial^k u}{\partial\theta^k}(R, \theta)\frac{\partial^k v}{\partial\theta^k}(R, \theta)d\theta$. Using the same method, we can obtain

Proposition 2. If $\beta_{N-1}>0$, then $\tilde{\tilde{D}}^N(u, v)$ is a nonnegative definite symmetric bilinear form when N is a positive even number, and it is not nonnegative definite when N is a positive odd number.

Corollary. $\tilde{\tilde{D}}^2(u, v)$ and $\tilde{\tilde{D}}^4(u, v)$, associated with the boundary conditions $(11)_2$ and $(11)_4$, are nonnegative definite symmetric bilinear forms. $\tilde{\tilde{D}}^3(u, v)$, associated with $(11)_3$, are not nonnegative definite.

It is easy to see that $\tilde{\tilde{D}}^1(u, v)$, associated with $(11)_1$, is nonnegative definite:

$$\tilde{\tilde{D}}^1(u, u)=\int_0^{2\pi}[u(R, \theta)]^2 d\theta\geqslant 0.$$

Therefore, in order to preserve the nonnegative definite symmetry of the bilinear forms, we use the approximate differential boundary conditions as follows: $(6)_1$ or $(11)_1$ for $N=1$, $(11)_2$ for $N=2$, $(6)_3$ for $N=3$, $(11)_4$ for $N=4$, and so on.

§ 2. Error Estimate

Consider the following boundary value problem with approximate integral boundary condition (4)

$$\begin{cases} -\Delta u=0, & \text{in } \Omega_i, \\ \dfrac{\partial u}{\partial n}=f, & \text{on } \Gamma_i, \\ \dfrac{\partial u}{\partial r}=-\dfrac{1}{\pi R}\sum_{n=1}^{N} n\cos n\theta * u(R, \theta), & \text{on } \Gamma_R. \end{cases} \tag{12}$$

It is equivalent to the variational problem (8).

Proposition 3. $\hat{D}(u_0, v_0)$ and $\hat{D}^N(u_0, v_0)$ are two nonnegative definite symmetric continuous bilinear forms on $H^{\frac{1}{2}}(\Gamma_R)$.

Proof. Let
$$u_0=\sum_{-\infty}^{\infty} a_n e^{in\theta}, \quad a_{-n}=\bar{a}_n,$$
$$v_0=\sum_{-\infty}^{\infty} b_n e^{in\theta}, \quad b_{-n}=\bar{b}_n, \quad n=0, 1, \cdots.$$

Then

$$\hat{D}(u_0, v_0)=2\pi\sum_{-\infty}^{\infty}|n|a_n\bar{b}_n\leqslant\left(2\pi\sum_{-\infty}^{\infty}|n||a_n|^2\right)^{\frac{1}{2}}\left(2\pi\sum_{-\infty}^{\infty}|n||b_n|^2\right)^{\frac{1}{2}}$$

$$\leqslant\frac{1}{R}\left(2\pi R\sum_{-\infty}^{\infty}(1+n^2)^{\frac{1}{2}}|a_n|^2\right)^{\frac{1}{2}}\left(2\pi R\sum_{-\infty}^{\infty}(1+n^2)^{\frac{1}{2}}|b_n|^2\right)^{\frac{1}{2}}$$

$$=\frac{1}{R}\|u_0\|_{\frac{1}{2},\Gamma_R}\|v_0\|_{\frac{1}{2},\Gamma_R},$$

$$\hat{D}^N(u_0, v_0)=2\pi\sum_{-N}^{N}|n|a_n\bar{b}_n\leqslant\left(2\pi\sum_{-N}^{N}|n||a_n|^2\right)^{\frac{1}{2}}\left(2\pi\sum_{-N}^{N}|n||b_n|^2\right)^{\frac{1}{2}}$$

$$\leqslant\frac{1}{R}\left(2\pi R\sum_{-N}^{N}(1+n^2)^{\frac{1}{2}}|a_n|^2\right)^{\frac{1}{2}}\left(2\pi R\sum_{-N}^{N}(1+n^2)^{\frac{1}{2}}|b_n|^2\right)^{\frac{1}{2}}$$

$$\leqslant\frac{1}{R}\|u_0\|_{\frac{1}{2},\Gamma_R}\|v_0\|_{\frac{1}{2},\Gamma_R}.$$

Taking $v_0=u_0$, we have

$$\hat{D}(u_0, u_0)=2\pi\sum_{-\infty}^{\infty}|n||a_n|^2\geqslant 0,$$

$$\hat{D}^N(u_0, u_0)=2\pi\sum_{-N}^{N}|n||a_n|^2\geqslant 0.$$

The proof is complete. Moreover,

$$\hat{D}(u_0, u_0)=2\pi\sum_{-\infty}^{\infty}|n||a_n|^2\geqslant\frac{2\pi R}{\sqrt{2}\,R}\sum_{\substack{-\infty\\ n\neq 0}}^{\infty}(1+n^2)^{\frac{1}{2}}|a_n|^2=\frac{1}{\sqrt{2}\,R}\|u_0\|^2_{H^{\frac{1}{2}}(\Gamma_R)/P_0},$$

where P_0 is the set of all constants. Then $\hat{D}(u_0, v_0)$ is $H^{\frac{1}{2}}(\Gamma_R)/P_0$-elliptic.

Proposition 4. The variational problems (7) and (8) have one and only one solution in $H^1(\Omega_i)/P_0$.

Proof. Since f satisfies the compatibility condition, we can consider problems (7) and (8) in the quotient space $H^1(\Omega_i)/P_0$. From the symmetric continuous V-ellipticity of $D_I(u, v)$ in $H^1(\Omega_i)/P_0$ and Proposition 3, using the trace theorem, we obtain that $D_I(u, v)+\hat{D}(u, v)$ and $D_I(u, v)+\hat{D}^N(u, v)$ are symmetric continuous V-elliptic bilinear forms on $H^1(\Omega_i)/P_0$. Moreover, $f(v)$ is a continuous linear functional on $H^1(\Omega_i)/P_0$. Then according to the Lax-Milgram lemma, the variational problems (7) and (8) have respectively one and only one solution in $H^1(\Omega_i)/P_0$. The proof is complete.

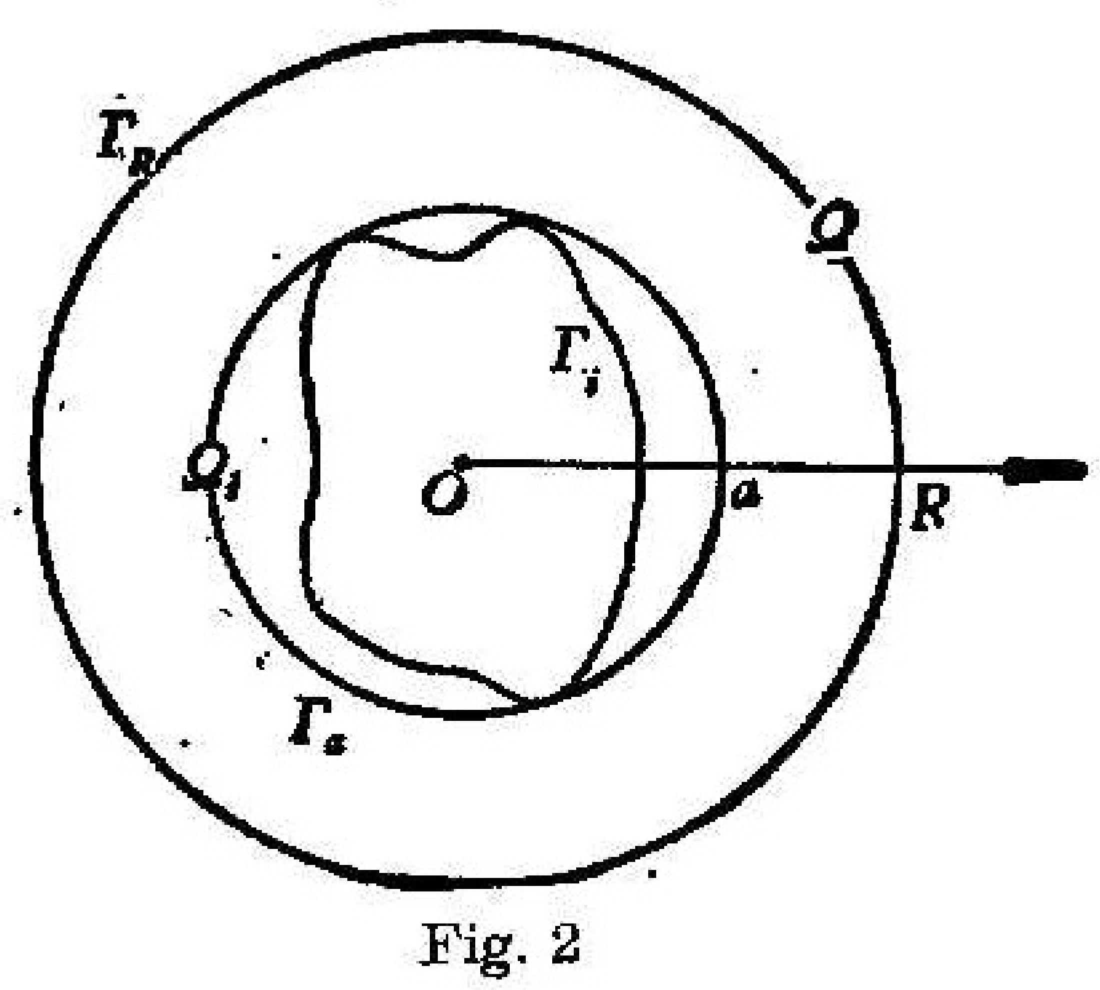

Fig. 2

In order to obtain the estimate of u^N-u in energy norm, where u^N is the solution of (8) and u the solution of (7), we first prove the following

Lemma 1. *Let a be the radius of Γ_a which is the smallest circle enclosing Γ_i. If $R\geqslant\sigma a$, $\sigma>1$, $w\in H^1(\Omega_i)$ is a harmonic function in Ω_i, we have*

$$[\hat{D}(w, w)]^{\frac{1}{2}}\leqslant\frac{\sqrt{2}\,\sigma}{\sqrt{\sigma^2-1}}|w|_{1,\Omega_i}.$$

Proof. Because w is a harmonic function in Ω_i, it is also a harmonic function in the ring domain between Γ_a and Γ_R. So we can let

$$w=\sum_{\substack{-\infty\\ n\neq 0}}^{\infty}\left(\frac{b_n}{r^{|n|}}+c_n r^{|n|}\right)e^{in\theta}+c_0+b_0\ln r,\quad b_{-n}=\bar{b}_n,\ c_{-n}=\bar{c}_n,\ n=0,1,\cdots.$$

Then from

$$\begin{aligned}|w|^2_{1,\Omega_i}&\geqslant\int_a^R\int_0^{2\pi}\left(\left|\frac{\partial w}{\partial r}\right|^2+\frac{1}{r^2}\left|\frac{\partial w}{\partial\theta}\right|^2\right)r\,d\theta\,dr\\&=\int_a^R\int_0^{2\pi}\left\{\sum_{\substack{-\infty\\ n\neq 0}}^{\infty}\left||n|c_n r^{|n|-1}-|n|\frac{b_n}{r^{|n|+1}}\right|^2\right.\\&\quad\left.+\left(\frac{b_0}{r}\right)^2+\sum_{\substack{-\infty\\ n\neq 0}}^{\infty}\frac{n^2}{r^2}\left|c_n r^{|n|}+\frac{b_n}{r^{|n|}}\right|^2\right\}r\,d\theta\,dr\\&\geqslant 4\pi\sum_{\substack{-\infty\\ n\neq 0}}^{\infty}n^2\int_a^R\left\{\frac{|b_n|^2}{r^{2|n|+1}}+r^{2|n|-1}|c_n|^2\right\}dr\\&=2\pi\sum_{\substack{-\infty\\ n\neq 0}}^{\infty}|n|\left\{\left(\frac{1}{a^{2|n|}}-\frac{1}{R^{2|n|}}\right)|b_n|^2+(R^{2|n|}-a^{2|n|})|c_n|^2\right\},\end{aligned}$$

and

$$\begin{aligned}\hat{D}(w,w)&=2\pi\sum_{\substack{-\infty\\ n\neq 0}}^{\infty}|n|\left|\frac{b_n}{R^{|n|}}+c_nR^{|n|}\right|^2\leqslant 4\pi\sum_{\substack{-\infty\\ n\neq 0}}^{\infty}|n|\left(\frac{|b_n|^2}{R^{2|n|}}+|c_n|^2R^{2|n|}\right)\\&\leqslant 4\pi\sum_{\substack{-\infty\\ n\neq 0}}^{\infty}|n|\left\{\frac{a^2}{R^2-a^2}\left(\frac{1}{a^{2|n|}}-\frac{1}{R^{2|n|}}\right)|b_n|^2\right.\\&\quad\left.+\frac{R^2}{R^2-a^2}(R^{2|n|}-a^{2|n|})|c_n|^2\right\},\end{aligned}$$

we can obtain

$$\hat{D}(w,w)\leqslant\frac{2R^2}{R^2-a^2}|w|^2_{1,\Omega_i}\leqslant\frac{2\sigma^2}{\sigma^2-1}|w|^2_{1,\Omega_i},$$

i.e.

$$[\hat{D}(w,w)]^{\frac{1}{2}}\leqslant\frac{\sqrt{2}\,\sigma}{\sqrt{\sigma^2-1}}|w|_{1,\Omega_i}.$$

The proof is complete.

As a result of the $H^{\frac{1}{2}}(\Gamma_R)/P_0$-ellipticity of $\hat{D}(u_0,v_0)$

$$\|w\|^2_{H^{\frac{1}{2}}(\Gamma_R)/P_0}\leqslant\sqrt{2}\,R\hat{D}(w,w),$$

we can further obtain

$$\|w\|_{H^{\frac{1}{2}}(\Gamma_R)/P_0}\leqslant\sqrt[4]{8}\frac{\sigma}{\sqrt{\sigma^2-1}}\sqrt{R}\,|w|_{1,\Omega_i}.$$

Lemma 2. *Let $R\geqslant\sigma a$, $\sigma>1$. Then*

$$\int_{\Omega_i}w^2\,dx\leqslant\frac{1}{(2R)^2}\left(\int_{\Omega_i}w\,dx\right)^2+M^2(2R)^2\int_{\Omega_i}|\nabla w|^2dx,\quad\forall w\in H^1(\Omega_i),$$

where $M=M(\sigma)$ is a constant.

Proof. It suffices to prove it for all continuously differentiable functions. By zero continuation we can regard w as a function over the square $[-R, R]^2$. Because $R\geqslant\sigma a$, $\sigma>1$, any two points in Ω_i can be connected by a broken line in Ω_i which consists of at most $2M$ segments parallel to one of the coordinate axes, where $M=M(\sigma)$ is a constant (Fig. 3). Then

$$w(\xi_M, \eta_M)-w(\xi_0, \eta_0)$$
$$=\int_{\eta_0}^{\eta_1}\frac{\partial w}{\partial\eta}(\xi_0, \eta)d\eta+\int_{\xi_0}^{\xi_1}\frac{\partial w}{\partial\xi}(\xi, \eta_1)d\xi+\cdots$$
$$+\int_{\xi_{M-1}}^{\xi_M}\frac{\partial w}{\partial\xi}(\xi, \eta_M)d\xi.$$

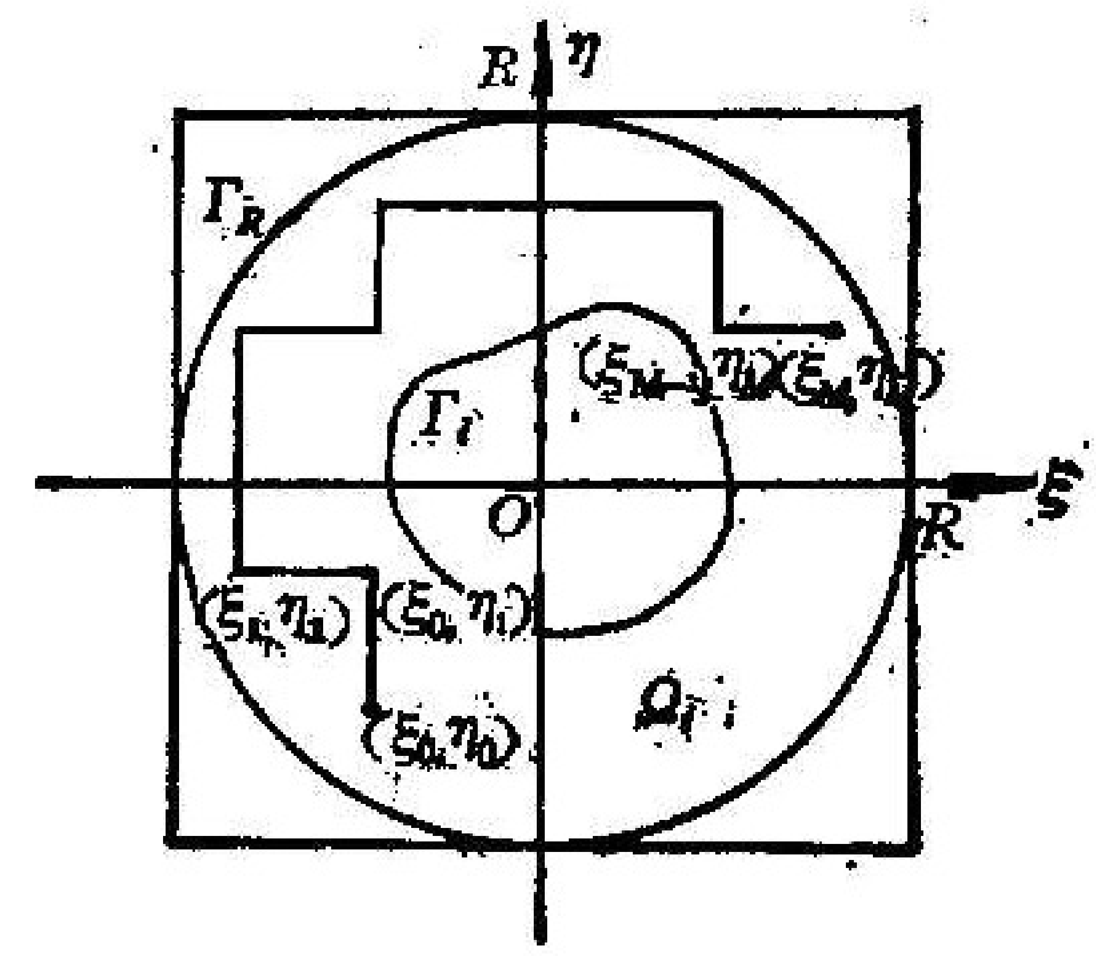

Fig. 3

Square it and we obtain

$$w(\xi_M, \eta_M)^2+w(\xi_0, \eta_0)^2-2w(\xi_M, \eta_M)w(\xi_0, \eta_0)$$
$$\leqslant 2M\left\{\left(\int_{\eta_0}^{\eta_1}\frac{\partial w}{\partial\eta}(\xi_0, \eta)d\eta\right)^2+\left(\int_{\xi_0}^{\xi_1}\frac{\partial w}{\partial\xi}(\xi, \eta_1)d\xi\right)^2+\cdots+\left(\int_{\xi_{M-1}}^{\xi_M}\frac{\partial w}{\partial\xi}(\xi, \eta_M)d\xi\right)^2\right\}$$
$$\leqslant 2M\cdot 2R\left\{\int_{-R}^{R}\left[\frac{\partial w}{\partial\eta}(\xi_0, \eta)\right]^2d\eta+\int_{-R}^{R}\left[\frac{\partial w}{\partial\xi}(\xi, \eta_1)\right]^2d\xi+\cdots+\int_{-R}^{R}\left[\frac{\partial w}{\partial\xi}(\xi, \eta_M)\right]^2d\xi\right\}.$$

By integrating from $-R$ to R successively with respect to $\xi_0, \eta_0, \xi_1, \eta_1, \cdots, \xi_M$ and η_M, we have

$$(2R)^{2M}\left\{2\int_{\Omega_i}w^2\,dx\right\}-2(2R)^{2(M-1)}\left(\int_{\Omega_i}w\,dx\right)^2\leqslant 2M^2(2R)^{2(M+1)}\int_{\Omega_i}|\nabla w|^2dx,$$

i.e.

$$\int_{\Omega_i}w^2dx\leqslant\frac{1}{(2R)^2}\left(\int_{\Omega_i}w\,dx\right)^2+M^2(2R)^2\int_{\Omega_i}|\nabla w|^2\,dx.$$

The proof is complete.

Now we can obtain the main result of this paper.

Theorem. *If* $u\in H^1(\Omega_i)\cap H^{k-\frac{1}{2}}(\Gamma_a)$, $k\geqslant 1$, $R\geqslant\sigma a$, $\sigma>1$ *is a constant, then*

$$\|u-u^N\|_{H^1(\Omega_i)/P_0}\leqslant C\frac{1}{N^{k-1}}\left(\frac{a}{R}\right)^N\|u\|_{k-\frac{1}{2},\Gamma_a},$$

where C *is a constant independent of* N *and* R.

Proof. Let $\|w\|_{D^N}=[D_I(w, w)+\hat{D}^N(w, w)]^{\frac{1}{2}}$. By Lemma 2, we have

$$\|w\|^2_{L^2(\Omega_i)/P_0}=\inf_{c\in P_0}\|w-c\|^2_{L^2(\Omega_i)}\leqslant\inf_{c\in P_0}\left\{\frac{1}{(2R)^2}\left(\int_{\Omega_i}(w-c)dx\right)^2+M^2(2R)^2\int_{\Omega_i}|\nabla w|^2dx\right\}$$
$$=M^2(2R)^2|w|^2_{1,\Omega_i},\quad\forall w\in H^1(\Omega_i).$$

Then

$$\|w\|^2_{H^1(\Omega_i)/P_0}\leqslant(1+4M^2R^2)|w|^2_{1,\Omega_i}\leqslant\left(\frac{1}{\sigma^2a^2}+4M^2\right)R^2\|w\|^2_{D^N},\quad\forall w\in H^1(\Omega_i),$$

i.e.

$$\|w\|_{H^1(\Omega_i)/P_0}\leqslant C_0R\|w\|_{D^N},\quad\forall w\in H^1(\Omega_i),$$

where

$$C_0=\left(\frac{1}{\sigma^2a^2}+4M^2\right)^{\frac{1}{2}}.$$

Because u is a harmonic function in Ω, we can let

$$u(r,\theta)=\sum_{-\infty}^{\infty}\frac{a_n}{r^{|n|}}e^{in\theta},\quad a_{-n}=\bar{a}_n,\quad n=0,1,2,\cdots.$$

Moreover, let

$$(u^N-u)_{\Gamma_R}=\sum_{-\infty}^{\infty}b_ne^{in\theta},\quad b_{-n}=\bar{b}_n,\quad n=0,1,2,\cdots.$$

From (7) and (8) we obtain

$$D_I(u-u^N,v)+\hat{D}(u,v)-\hat{D}^N(u^N,v)=0,\quad \forall v\in H^1(\Omega_i).$$

Taking $v=u-u^N$, we get

$$\|u-u^N\|^2_{D^N}=D_I(u-u^N,u-u^N)+\hat{D}^N(u-u^N,u-u^N)=\hat{D}^N(u,u-u^N)-\hat{D}(u,u-u^N)$$

$$=\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi}\sum_{n=N+1}^{\infty}n\cos n(\theta-\theta')u(R,\theta')[u^N(R,\theta)-u(R,\theta)]d\theta'\,d\theta$$

$$=2\pi\sum_{|n|\geq N+1}|n|\frac{a_n}{R^{|n|}}\bar{b}_n\leqslant\left(2\pi\sum_{|n|\geq N+1}|n|\frac{|a_n|^2}{R^{2|n|}}\right)^{\frac{1}{2}}[\hat{D}(u^N-u,u^N-u)]^{\frac{1}{2}}.$$

Because u^N-u is a harmonic function in Ω_i, using Lemma 1, we have

$$\|u^N-u\|^2_{D^N}\leqslant\frac{\sqrt{2}\,\sigma}{\sqrt{\sigma^2-1}}\left(2\pi\sum_{|n|\geq N+1}|n|\frac{|a_n|^2}{R^{2|n|}}\right)^{\frac{1}{2}}|u^N-u|_{1,\Omega_i}$$

$$\leqslant\frac{\sqrt{2}\,\sigma}{\sqrt{\sigma^2-1}}\left(2\pi\sum_{|n|\geq N+1}|n|\frac{|a_n|^2}{R^{2|n|}}\right)^{\frac{1}{2}}\|u^N-u\|_{D^N}.$$

Hence

$$\|u-u^N\|_{H^1(\Omega_i)/P_0}\leqslant C_0R\|u-u^N\|_{D^N}\leqslant C_0R\frac{\sqrt{2}\,\sigma}{\sqrt{\sigma^2-1}}\left(2\pi\sum_{|n|\geq N+1}|n|\frac{|a_n|^2}{R^{2|n|}}\right)^{\frac{1}{2}}$$

$$\leqslant C_0R\frac{\sqrt{2}\,\sigma}{\sqrt{\sigma^2-1}}\frac{1}{\sqrt{a}\,N^{k-1}}\left(\frac{a}{R}\right)^{N+1}\left(2\pi a\sum_{|n|\geq N+1}|n|^{2k-1}\frac{|a_n|^2}{a^{2|n|}}\right)^{\frac{1}{2}}$$

$$\leqslant C_0\frac{\sqrt{2a}\,\sigma}{\sqrt{\sigma^2-1}}\frac{1}{N^{k-1}}\left(\frac{a}{R}\right)^N\|u\|_{k-\frac{1}{2},\Gamma_a},$$

i.e.

$$\|u-u^N\|_{H^1(\Omega_i)/P_0}\leqslant C\frac{1}{N^{k-1}}\left(\frac{a}{R}\right)^N\|u\|_{k-\frac{1}{2},\Gamma_a},$$

where $C=\left[\frac{2(4a^2\sigma^2M^2+1)}{a(\sigma^2-1)}\right]^{\frac{1}{2}}=C(a,\sigma)$ is a constant independent of N and R. The proof is complete.

This result reveals the relationship between the error and the approximate grade N of boundary conditions as well as the radius R of the artificial boundary.

It should be pointed out that, because (4) and (5) (or (10)) are not equivalent when the Fourier expansion of $u(R,\theta)$ contains some high frequency terms corresponding to $|n|>N$, the above result is only the error estimate for the approximate integral boundary condition (4), not that for the differential boundary condition (5) or (10). Once (4) and (5) (or (10)) are equivalent, they are equivalent to the exact boundary condition (3). Then $u^N = u$.

In fact, the approximate integral boundary condition (4) has been implicitly used in the coupling of the canonical boundary element method with the finite element method[2,5,7]. When we calculate the canonical boundary element stiffness matrix by the series expansion method, we always only calculate the sum of finite terms of every series. However, in those cases we have chosen a very large number for N, for example $N = 200$—400. From the above theorem we see that, provided $R > a$ and N is a very large number, the right-hand member of the estimate is very small and can be neglected. This has been indicated by numerical computations[2,5,7].

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