ON THE ERROR ESTIMATE FOR THE ISOPARAMETRIC FINITE ELEMENT METHOD*

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Abstract

In this paper, the isoparametric element of 2-degree Lagrange type for second order elliptic P. D. E. with nonhomogeneous Dirichlet boundary value is considered. We prove

$$\|u-u_h\|_{1,\Omega}=O(h^2),$$

which is the same as $\|u-u_h\|_{1,\Omega_h}=O(h^2)$, where Ω and Ω_h in R^2 are the domain of the boundary value problem and the isoparametric triangulation domain respectively.

§ 1. Introduction

We consider the nonhomogeneous Dirichlet problem:

$$\begin{cases}\text{find } u\in V, \text{ such that}\\ a(u, v)=\langle f, v\rangle, \quad \forall v\in V,\end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^2 with a sufficiently smooth boundary $\partial\Omega$,

$$a(u, v)=\int_\Omega \sum_{i,j=1}^{2} a_{ij}(x)\partial_i u\partial_j v\,dx, \tag{1.2}$$

$$\langle f, v\rangle=\int_\Omega f\cdot v\,dx, \tag{1.3}$$

$a_{ij}\in W^{2,\infty}(\Omega)$, $f\in W^{2,q}(\Omega)(q\geq 2)$, g is restriction of a function in $H^3(\Omega)$, $\beta=\text{const.}>0$, such that

$$\sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j\geq\beta\sum_{i=1}^{2}\xi_i^2, \qquad \forall x\in\Omega,\ \xi_i\in\mathbb{R},\ i=1, 2, \tag{1.4}$$

and

$$V=\{v\in H^1(\Omega):\ v=g \text{ on } \partial\Omega\}, \tag{1.5}$$

$$V=H_0^1(\Omega). \tag{1.6}$$

The notations above are introduced from [1].

In [2], [3], Ciarlet and Raviart studied isoparametric finite element approximation of the problem (1.1), in 2-degree Lagrange type, or the so-called type (2), as follows:

Let $(\mathscr{T}_h)_{h>0}$ be a family of regular isoparametric triangulations of type (2), F_K: $\hat{K}\to K$ $\forall K\in\mathscr{T}_h$ be the isoparametric mapping of type (2) (cf. Fig. 1), $\Omega_h=\bigcup_{K\in\mathscr{T}_h} K$, in general $\Omega_h\not\subset\Omega$. Let X_h be the isoparametric finite element space of type (2), and

$$V_h=\{v_h\in X_h:\ v_h=g \text{ at the boundary nodes on } \partial\Omega\}, \tag{1.7}$$

$$V_{0h}=\{v_h\in X_h:\ v_h=0 \text{ at the boundary nodes on } \partial\Omega\}. \tag{1.8}$$

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Then the approximation problem with numerical integration is the following:

$$\begin{cases} \text{find } u_h \in V_h, \text{ such that} \\ a_h(u_h, v_h) = \langle f, v_h \rangle_h, \qquad \forall v_h \in V_{0h}, \end{cases} \tag{1.9}$$

where $a_h(u_h, v_h)$ and $\langle f, v_h\rangle_h$ are formulas of numerical integrations (cf. [2], [3]). As to the isoparametric finite element approximation without numerical integrations, it is not available in practice.

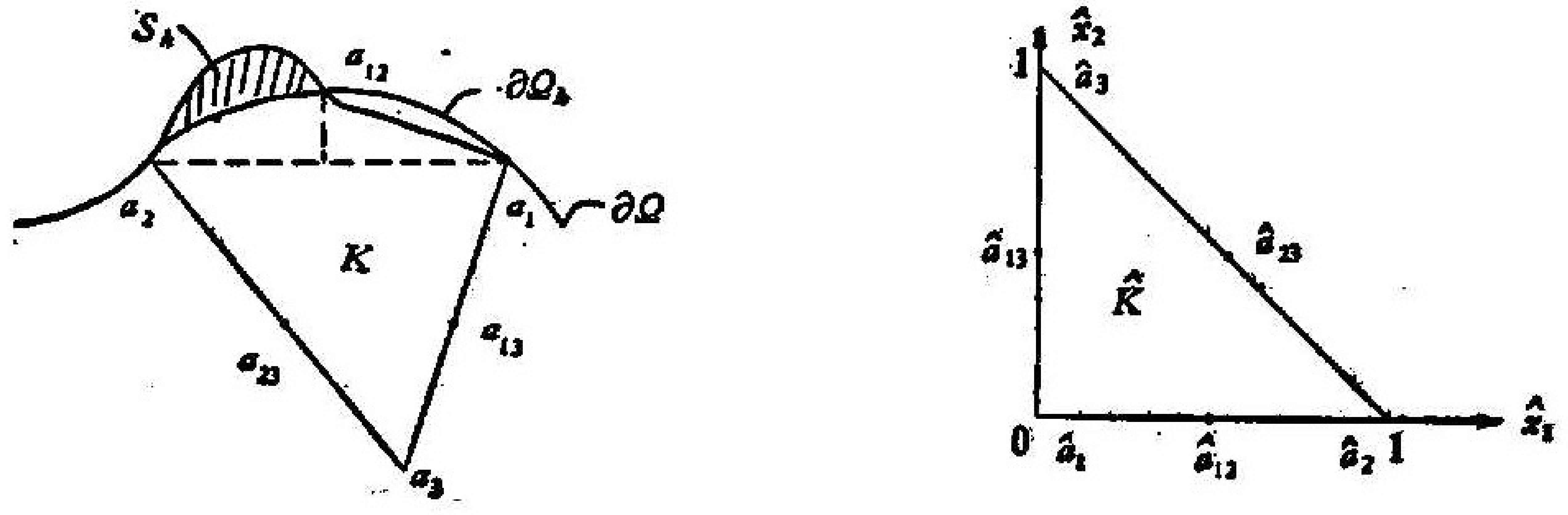

Fig. 1

Ciarlet and Raviart have shown the error estimate $\|u-u_h\|_{1,\Omega_h}=O(h^2)$ on the isoparametric triangulation domain Ω_h. In [4], in order to obtain the error estimate $\|u-u_h\|_{1,\Omega}=O(h^2)$ on the domain Ω, Li Li-kang reformed the isoparametric element approximation, which is not of the standard isoparametric type as in [2], [3], and in which it is necessary to have an expression of the boundary $\partial\Omega$ in x.

In this paper, the standard isoparametric element is considered. Section 2 contains the extension of the isoparametric finite element solution u_h. In section 3, we prove the error estimate $\|u-u_h\|_{1,\Omega}=O(h^2)$ on the domain Ω for the extension of u_h as in section 2.

§ 2. Extension of u_h

In order to estimate the error between the solutions u of (1.1) and u_h of (1.9), it is necessary to extend u_h from Ω_h onto $\Omega\cup\Omega_h$, since the solution u_h of (1.9) is defined on Ω_h only.

As well known, the estimation of $\|u-u_h\|_{1,\Omega_h}$ on the isoparametric triangulation domain Ω_h is independent of the way the solution u of (1.1) is extended from Ω onto $\Omega\cup\Omega_h$ in the case u remains in H^3 space. However the case is different for the estimation of $\|u-u_h\|_{1,\Omega}$ on the domain Ω if the solution u_h of (1.9) is extended from Ω_h onto $\Omega\cup\Omega_h$. Let us show some examples.

Example 1. Let $\mathscr{T}'_h$ be the set of boundary isoparametric triangles, and K' be a curve triangle of reflection of $K\in\mathscr{T}'_h$ with $\overline{a_1a_2}$, i.e. K consists of $\overline{a_3a_1}$, $\overline{a_3a_2}$ and $\widehat{a_1a_{12}a_2}$, and K' consists of $\overline{a'_3a_1}$, $\overline{a'_3a_2}$ and $\widehat{a_1a_{12}a_2}$ (cf. Fig. 2). Let $\tilde{\Omega}\supseteq\Omega_h\cup(\bigcup_{K\in\mathscr{T}'_h}K')\supset\Omega$, and since the boundary $\partial\Omega$ is sufficiently smooth, let $\tilde{u}\in H^3(\tilde{\Omega})$ be the extension of u. If the extension $\tilde{u}_h$ of u_h is defined as the interpolation $\tilde{u}^I$ of $\tilde{u}$ on K', then it is easy to estimate $\|u-\tilde{u}_h\|_{1,\Omega}=O(h^2)$. But it is not available, since the solution u of (1.1) is unknown.

If the extension $\tilde{u}_h$ of u_h is defined in some way on K' with the values of u_h on K, then we may not be able to obtain the error bound $O(h^2)$. Let us see the extension to be below (and cf. Remark 1).

Example 2. The homogeneous Dirichlet boundary value problem is considered here, i.e. $g=0$ in (1.1). It is somewhat natual to extend u_h on S_h with zero (cf. Fig. 2). However, the zero extension cannot enable us to obtain the error bound $O(h^2)$. There exists such function $u\in H_0^1(\Omega)$ that $|\text{grad } u|=C=$ const.>0 near $\partial\Omega$. Thus

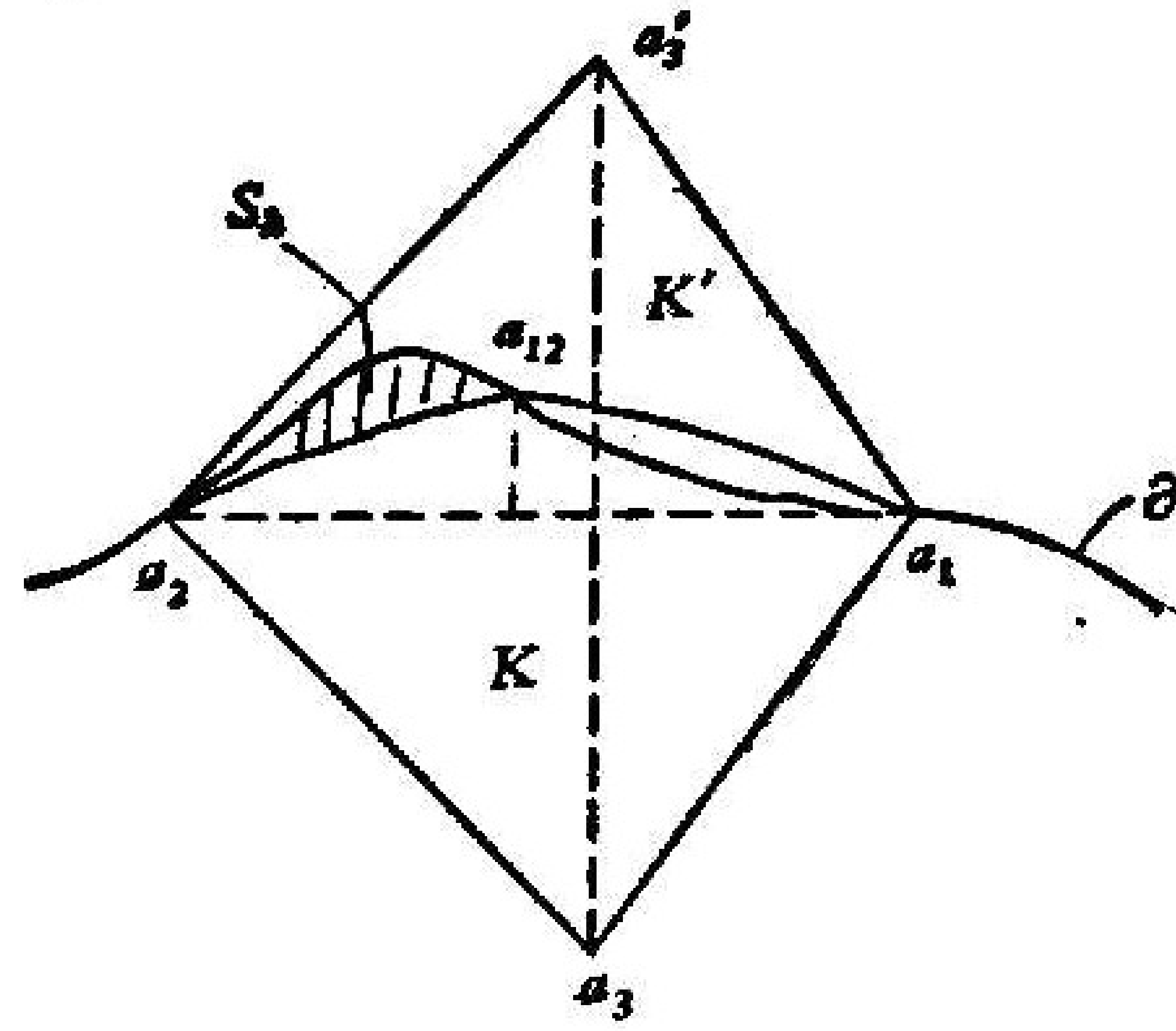

Fig. 2

$$\|u-\tilde{u}_h\|_{1,\Omega}\geqslant\|u-\tilde{u}_h\|_{1,\Omega\setminus\Omega_h}$$
$$=\|u\|_{1,\Omega\setminus\Omega_h}\geqslant|u|_{1,\Omega\setminus\Omega_h}=C\cdot[\text{meas }(\Omega\setminus\Omega_h)]^{1/2},$$

and since meas $(\Omega\setminus\Omega_h)=O(h^3)$ in general, one cannot derive the error bound $O(h^2)$.

We now show an extension of the solution u_h of (1.9) from Ω_h onto $\Omega\cup\Omega_h$.

Let $\mathscr{T}'_h$ be the set of the boundary isoparametric triangles, and K' be a curve triangle of the extension of $K\in\mathscr{T}'_h$, such that a parallelogram consists of the two isoparametric triangles K and K', where the isoparametric triangle K consists of two straight lines $\overline{a_3a_1}$, $\overline{a_3a_2}$ and one curve $\widehat{a_1a_{12}a_2}$, and the isoparametric triangle K' consists of two straight lines $\overline{a'_3a'_1}$, $\overline{a'_3a'_2}$ and one curve $\widehat{a'_1a'_{12}a'_2}$ (cf. Fig. 3).

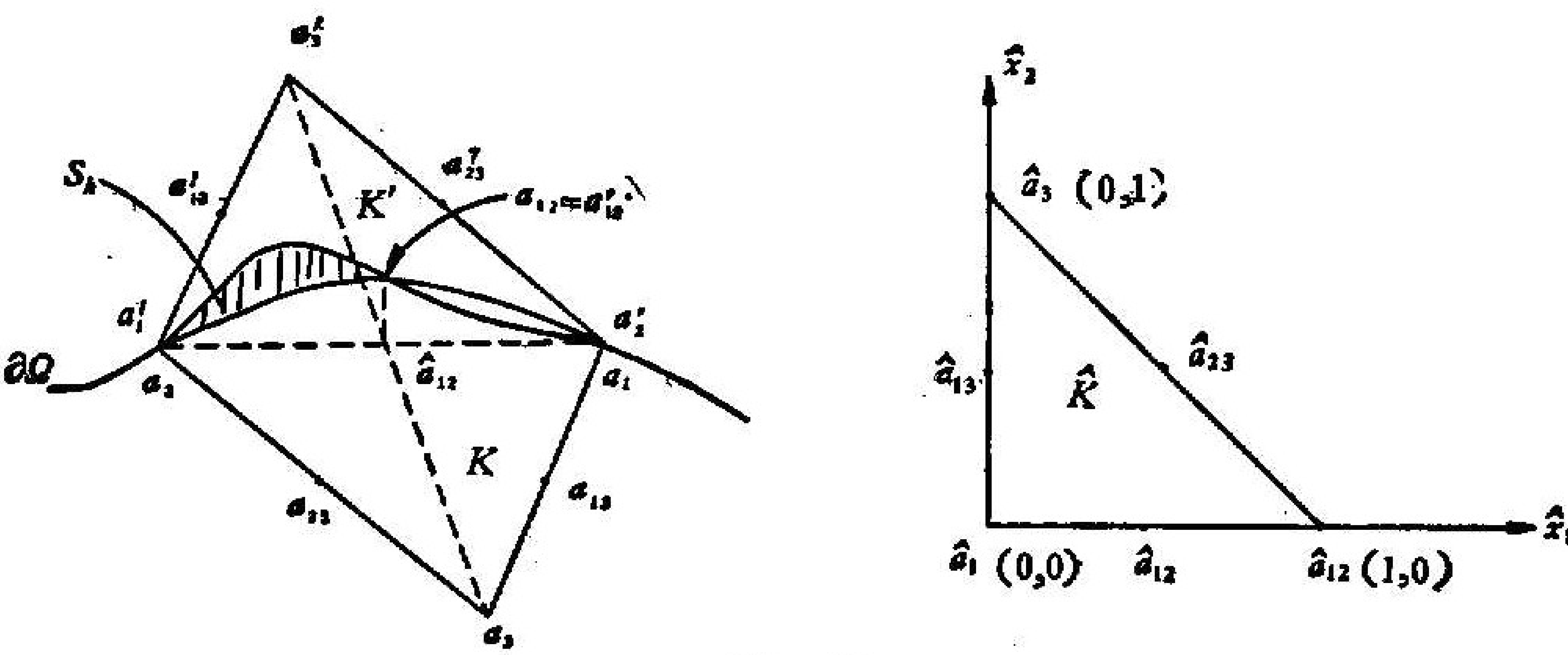

Fig. 3

Let F_K: $\hat{K}\mapsto K$ and $F_{K'}$: $\hat{K}\mapsto K'$ be the isoparametric mappings of type (2), such that

$$F_K(\hat{a}_i)=a_i,\quad 1\leqslant i\leqslant 3,\quad F_K(\hat{a}_{ij})=a_{ij},\quad 1\leqslant i<j\leqslant 3, \tag{2.1}$$

and

$$F_{K'}(\hat{a}_i)=a'_i,\quad 1\leqslant i\leqslant 3,\quad F_{K'}(\hat{a}_{ij})=a'_{ij},\quad 1\leqslant i<j\leqslant 3. \tag{2.2}$$

Then (cf. [2], [3]),

$$F_K(\hat{x})=\tilde{F}_K(\hat{x})+H_K(\hat{x}), \tag{2.3}$$
$$F_{K'}(\hat{x})=\tilde{F}_{K'}(\hat{x})+H_{K'}(\hat{x}), \tag{2.4}$$

where $\tilde{F}_K$ and $\tilde{F}_{K'}$ are the affine mappings from $\hat{K}$ onto straight triangles $\tilde{K}=\triangle a_1a_2a_3$ and $\tilde{K}'=\triangle a'_1a'_2a'_3$ respectively, such that

$$\tilde{F}_K(\hat{a}_i)=a_i,\ \tilde{F}_{K'}(\hat{a}_i)=a_i',\quad 1\leqslant i\leqslant 3, \tag{2.5}$$

and

$$H_K(\hat{x})=\hat{p}_{12}(\hat{x})\cdot(a_{12}-\tilde{a}_{12})=H_{K'}(\hat{x}), \tag{2.6}$$

with $\hat{p}_{12}\in P_2(K)$: $\hat{p}_{12}(\hat{a}_{12})=1$, $\hat{p}_{12}(\hat{a}_i)=0$, $1\leqslant i\leqslant 3$, $\hat{p}_{12}(\hat{a}_{13})=\hat{p}_{12}(\hat{a}_{23})=0$. By a simple calculation, it can be deduced that

$$\tilde{F}_K(\hat{x})=B_K\hat{x}+b_K,\quad \tilde{F}_{K'}(\hat{x})=B_{K'}\cdot\hat{x}+b_{K'}, \tag{2.7}$$

and

$$B_K=-B_{K'}=(a_2-a_1,\ a_3-a_1), \tag{2.8}$$

$$b_K=a_1,\quad b_{K'}=a_2. \tag{2.9}$$

In the derivation of (2.9), it should be noted that

$$a_3'-a_1'=a_3'-a_2=a_1-a_3, \tag{2.10}$$

since the quadrilateral $a_1a_3'a_2a_3$ is a parallelogram (cf. Fig. 3), which is the key point of our extension.

We now extend u_h. Let x denote the point on K, and x' that on K', and let

$$\begin{aligned}\tilde{u}_h(x')=&-u_h\circ F_K(\hat{x})+(u_h(a_1)+u_h(a_2))\\&+8\cdot\hat{x}_1(1-\hat{x}_1)\left(u_h(a_{12})-\frac{u_h(a_1)+u_h(a_2)}{2}\right),\end{aligned} \tag{2.11}$$

where

$$\hat{x}=F_{K'}^{-1}(x'),\quad \hat{x}=\begin{pmatrix}\hat{x}_1\\ \hat{x}_2\end{pmatrix}. \tag{2.12}$$

Then

$$\tilde{u}_h(a_1')=-u_h\circ F_K(\hat{a}_1)+(u_h(a_1)+u_h(a_2))=-u_h(a_1)+(u_h(a_1)+u_h(a_2))=u_h(a_2),$$

$$\tilde{u}_h(a_2')=-u_h\circ F_K(\hat{a}_2)+(u_h(a_1)+u_h(a_2))=u_h(a_1),$$

$$\tilde{u}_h(a_{12}')=-u_h\circ F_K(\hat{a}_{12})+(u_h(a_1)+u_h(a_2))+2\left(u_h(a_{12})-\frac{u_h(a_1)+u_h(a_2)}{2}\right)=u_h(a_{12}),$$

from which we have

$$\tilde{u}_h=u_h \text{ on the curve } \widehat{a_1a_{12}a_2}, \tag{2.13}$$

and

$$\tilde{u}_h(x)=u_h(x),\quad \forall x\in\Omega_h\cap\Omega. \tag{2.14}$$

§ 3. Error Estimate

In this section, we will assume that $u\in H^3(\tilde{\Omega})$ with $\tilde{\Omega}\supset\Omega\cup(\bigcup_{K\in\mathscr{T}_h}K')$, $\forall h>0$ by extension of the solution u of (1.1), and we will estimate error $\|u-\tilde{u}_h\|_{1,\Omega}$.

We have the main theorem.

Theorem. *Let Ω be a bounded domain in R^2 with sufficiently smooth boundary $\partial\Omega$, and $(\mathscr{T}_h)_{h>0}$ be regular isoparametric triangulations of type (2). Assume that the hypotheses concerned in problem (1.1) hold and the solution u of (1.1) is in $H^3(\Omega)$. Then*

$$\|u-\tilde{u}_h\|_{1,\Omega}=O(h^2), \tag{3.1}$$

where $\tilde{u}_h$ is defined as the extension of (2.11) and (2.14).

Proof. By use of (2.11), $\forall x'\in S_h$,

$$u(x')-\tilde{u}_h(x')=u\circ F_{K'}(\hat{x})+u_h\circ F_K(\hat{x})-(u_h(a_1)+u_h(a_2))$$
$$-8\hat{x}_1\cdot(1-\hat{x}_1)\left(u_h(a_{12})-\frac{u_h(a_1)+u_h(a_2)}{2}\right)$$
$$=\Big\{u\circ F_{K'}(\hat{x})+u\circ F_K(\hat{x})-(u(a_1)+u(a_2))$$
$$-8\hat{x}_1(1-\hat{x}_1)\left(u(a_{12})-\frac{u(a_1)+u(a_2)}{2}\right)\Big\}$$
$$+(u_h\circ F_K(\hat{x})-u\circ F_K(\hat{x})), \tag{3.2}$$

where

$$x'=F_{K'}(\hat{x}). \tag{3.3}$$

From (2.3)—(2.9), we can derive that

$$u\circ F_{K'}(\hat{x})+u\circ F_K(\hat{x})-(u(a_1)+u(a_2))$$
$$-8\hat{x}_1\cdot(1-\hat{x}_1)\left(u(a_{12})-\frac{u(a_1)+u(a_2)}{2}\right)$$
$$=[u\circ\tilde{F}_{K'}(\hat{x})+u_0\circ\tilde{F}_K(\hat{x})-(u(a_1)+u(a_2))]$$
$$-8\hat{x}_1(1-\hat{x}_1)\left(u(a_{12})-\frac{u(a_1)+u(a_2)}{2}\right)$$
$$+[u\circ F_{K'}(\hat{x})-u\circ\tilde{F}_{K'}(\hat{x})]+[u\circ F_K(\hat{x})-u\circ\tilde{F}_K(\hat{x})]. \tag{3.4}$$

And firstly, we have (cf. Fig. 4),

$$u\circ\tilde{F}_{K'}(\hat{x})+u\circ\tilde{F}_K(\hat{x})-(u(a_1)+u(a_2))$$
$$=u\circ(-\tilde{F}_K(\hat{x})+a_1+a_2)+u\circ\tilde{F}_K(\hat{x})-(u(a_1)+u(a_2))$$
$$=u(\tilde{x}')+u(\tilde{x})-(u(a_1)+u(a_2))$$
$$=2u(\tilde{a}_{12})+\nabla u(\xi')(\tilde{x}'-\tilde{a}_{12})+\nabla u(\xi)(\tilde{x}-\tilde{a}_{12})-(u(a_1)+u(a_2))$$
$$=(\nabla u(\xi')-\nabla u(\xi))\cdot(\tilde{a}_{12}-\tilde{x})+[2u(\tilde{a}_{12})-(u(a_1)+u(a_2))], \tag{3.5}$$

where $\tilde{x}=\tilde{F}_K(\hat{x})$, $\tilde{x}'=2\tilde{a}_{12}-\tilde{x}$. By using the imbedding theorem[1]

$$H^3(\tilde{\Omega})\hookrightarrow C^{1,\lambda}(\bar{\tilde{\Omega}}),\quad 0<\lambda<1,$$

we take $\lambda=1/2$. Then

$$|(\nabla u(\xi')-\nabla u(\xi))\cdot(\tilde{a}_{12}-\tilde{x})|$$
$$\leqslant C\|u\|_{3,\tilde{\Omega}}\cdot\|\xi'-\xi\|^{1/2}\cdot\|\tilde{a}_{12}-\tilde{x}\|$$
$$\leqslant C\|u\|_{3,\tilde{\Omega}}\cdot h_K^{3/2}, \tag{3.6}$$

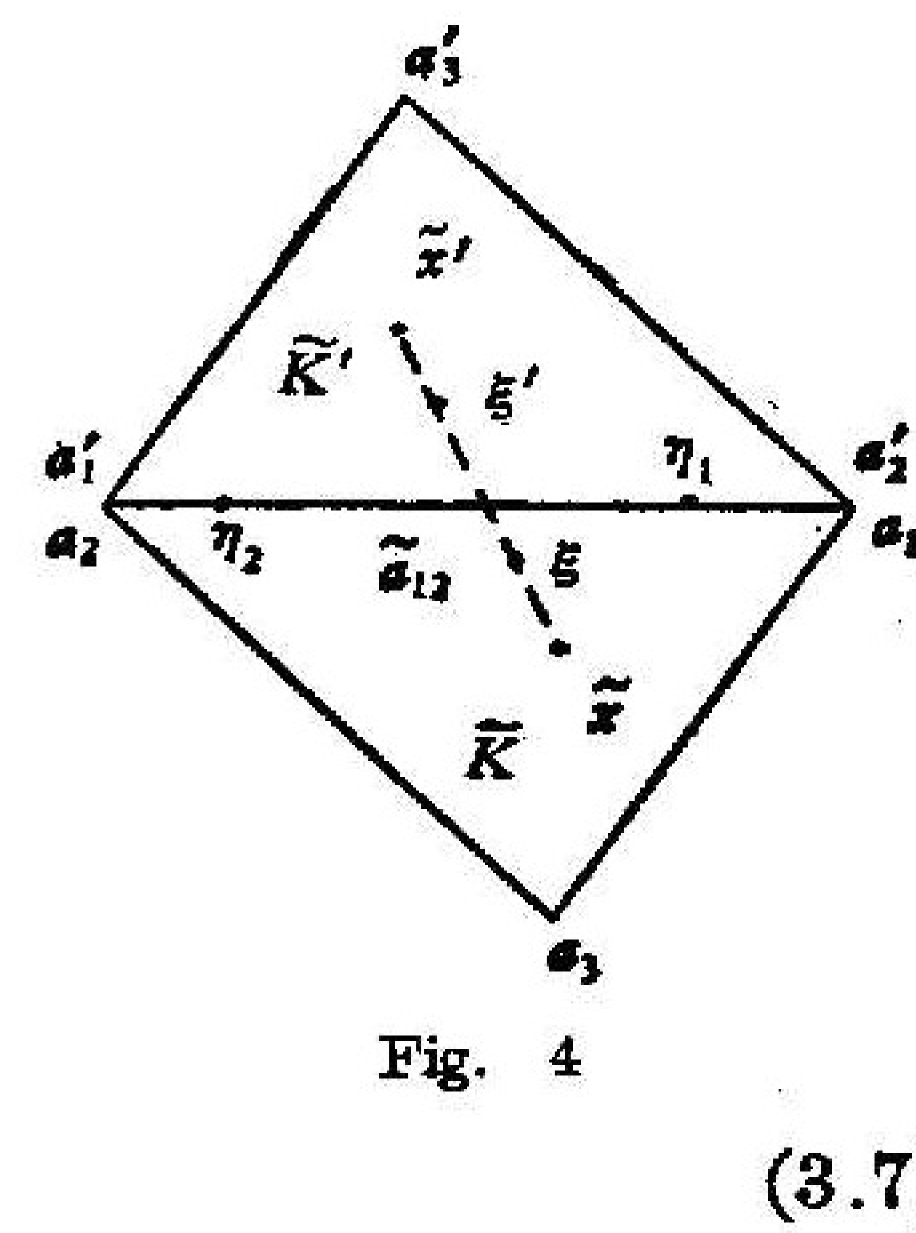

Fig. 4

and

$$|2u(\tilde{a}_{12})-(u(a_1)+u(a_2))|$$
$$=|(\nabla u(\eta_2)-\nabla u(\eta_1))\cdot(a_1-\tilde{a}_{12})|$$
$$\leqslant C\|u\|_{3,\tilde{\Omega}}\cdot\|\eta_2-\eta_1\|^{1/2}\cdot\|a_1-\tilde{a}_{12}\|$$
$$\leqslant C\|u\|_{3,\tilde{\Omega}}\cdot h_K^{3/2}. \tag{3.7}$$

From (3.5)—(3.7), we have

$$|u\circ\tilde{F}_{K'}(\hat{x})+u\circ\tilde{F}_K(\hat{x})-(u(a_1)+u(a_2))|\leqslant C\|u\|_{3,\tilde{\Omega}}\cdot h_K^{3/2}. \tag{3.8}$$

And from (3.7), taking account of $\|a_{12}-\tilde{a}_{12}\|\leqslant Ch_K^2$, we have

$$\left|u(a_{12})-\frac{u(a_1)+u(a_2)}{2}\right|\leqslant C\|u\|_{3,\tilde{\Omega}}\cdot h_K^{3/2}, \tag{3.9}$$

and from (2.3), (2.4) and (2.6), we have

$$\begin{cases} |u\circ F_{K'}(\hat{x})-u\circ \widetilde{F}_{K'}(\hat{x})|\leqslant C|u|_{1,\infty,\widetilde{\Omega}}\cdot h_{K'}^2\leqslant C\|u\|_{3,\widetilde{\Omega}}h_K^2, \\ |u\circ F_K(\hat{x})-u\circ \widetilde{F}_K(\hat{x})|\leqslant C|u|_{1,\infty,\widetilde{\Omega}}\cdot h_K^2\leqslant C\|u\|_{3,\widetilde{\Omega}}h_K^2. \end{cases} \tag{3.10}$$

Finally, by use of

$$|J_{F_{K'}}|_{0,\infty,\hat{K}}\leqslant C\ \mathrm{meas}(\widetilde{K}'),\quad |J_{F_K^{-1}}|_{0,\infty,K}\leqslant C/\mathrm{meas}(\widetilde{K}) \tag{3.11}$$

(cf. [2], [3]), we have

$$\begin{aligned} &\int_{S_h}|u_h\circ F_K(\hat{x})-u\circ F_K(\hat{x})|^2dx' \\ &\leqslant\int_{K'}|u_h\circ F_K(\hat{x})-u\circ F_K(\hat{x})|^2dx' \\ &\leqslant |J_{F_{K'}}|_{0,\infty,\hat{K}}\cdot|J_{F_K^{-1}}|_{0,\infty,K}\cdot\int_K|u_h(x)-u(x)|^2dx \\ &\leqslant C\int_K|u_h(x)-u(x)|^2dx. \end{aligned} \tag{3.12}$$

From (3.2), (3.8), (3.9), (3.10) and (3.12) and taking account of $\mathrm{meas}(S_h)\leqslant Ch_K^4$, we can deduce that

$$\begin{aligned} \|u-\tilde{u}_h\|_{0,S_h}^2&\leqslant C\|u\|_{3,\widetilde{\Omega}}^2h_K^3\cdot(\mathrm{meas}(S_h))+2\|u_h-u\|_{0,K}^2 \\ &\leqslant C\|u\|_{3,\widetilde{\Omega}}^2\cdot h_K^7+2\|u_h-u\|_{0,K}^2, \end{aligned}$$

from which, we have

$$\|u-\tilde{u}_h\|_{0,\Omega\setminus\Omega_h}^2=\sum_{S_h}\|u-\tilde{u}_h\|_{0,S_h}^2\leqslant Ch^6\|u\|_{3,\widetilde{\Omega}}^2+2\|u_h-u\|_{0,\Omega_h}^2. \tag{3.13}$$

By using the result $\|u-u_h\|_{0,\Omega_h}=O(h^3)$ (cf. [3]), we have

$$\|u-u_h\|_{0,\Omega}=O(h^3). \tag{3.14}$$

We now estimate $|u-\tilde{u}_h|_{1,\Omega}$ to complete the error estimate in H^1 norm.

By (3.2) we have $\forall x'\in S_h$,

$$\begin{aligned} \nabla_{x'}(u(x')-\tilde{u}_h(x'))=\nabla_{\hat{x}}\Big(&u\circ F_{K'}(\hat{x})+u\circ F_K(\hat{x}) \\ &-8\hat{x}_1(1-\hat{x}_1)\Big(u(a_{12})-\frac{u(a_1)+u(a_2)}{2}\Big)\Big)\cdot\nabla_{x'}F_{K'}^{-1}(x') \\ &+\nabla_{\hat{x}}(u_h\circ F_K(\hat{x})-u\circ F_K(\hat{x}))\cdot\nabla_{x'}F_{K'}^{-1}(x'). \end{aligned} \tag{3.15}$$

To begin with, we estimate the first term on the right hand of (3.15). By use of (2.3), (2.4) and (2.6)—(2.9), we can deduce that

$$\begin{aligned} &\nabla_{\hat{x}}\Big(u\circ F_{K'}(\hat{x})+u\circ F_K(\hat{x})-8\hat{x}_1(1-\hat{x}_1)\Big(u(a_{12})-\frac{u(a_1)+u(a_2)}{2}\Big)\Big)\cdot\nabla_{x'}F_{K'}^{-1}(x') \\ &=\Big(\nabla u\cdot(\nabla_{\hat{x}}F_{K'}(\hat{x})+\nabla_{\hat{x}}F_K(\hat{x}))-8(1-2\hat{x},0)\Big(u(a_{12})-\frac{u(a_1)+u(a_2)}{2}\Big)\Big)\cdot\nabla_{x'}F_{K'}^{-1}(x') \\ &=\Big(2\nabla u\cdot(\nabla\hat{p}_{12}(\hat{x})(a_{12}-\tilde{a}_{12})) \\ &\quad-8(1-2\hat{x},0)\Big(u(a_{12})-\frac{u(a_1)+u(a_2)}{2}\Big)\Big)\cdot\nabla_{x'}F_{K'}^{-1}(x'), \end{aligned}$$

from which and taking account of $\|a_{12}-\tilde{a}_{12}\|\leqslant Ch_K^2$ and (3.9) and by use of $|F_{K'}^{-1}|_{1,\infty,K'}\leqslant Ch_{K'}^{-1}$ ([2]), we have

$$\left|\nabla_{\hat{x}}\Big(u\circ F_{K'}(\hat{x})+u\circ F_K(\hat{x})-8\hat{x}_1(1-\hat{x}_1)\Big(u(a_{12})-\frac{u(a_1)+u(a_2)}{2}\Big)\Big)\cdot\nabla_{x'}F_{K'}^{-1}(x')\right|$$
$$\leqslant C(|u|_{1,\infty,\tilde{\Omega}}\cdot h_K^2+\|u\|_{3,\tilde{\Omega}}\cdot h_K^{3/2})\cdot h_{K'}^{-1}$$
$$\leqslant C\|u\|_{3,\tilde{\Omega}}\cdot h_K^{1/2}. \tag{3.16}$$

From (3.15) and (3.16), we have

$$|u-\tilde{u}_h|_{1,S_h}^2=\int_{S_h}|\nabla_{x'}(u(x')-\tilde{u}_h(x'))|^2dx'\leqslant C\|u\|_{3,\tilde{\Omega}}^2\cdot h_K\cdot\text{meas}(S_h)$$
$$+2\int_{S_h}|\nabla_{\hat{x}}(u_h\circ F_K(\hat{x})-u\circ F_K(\hat{x}))\cdot\nabla_{x'}F_{K'}^{-1}(x')|^2dx'. \tag{3.17}$$

Finally we estimate the second term on the right side of (3.17). By use of $|F_K|_{1,\infty,\hat{K}}\leqslant Ch$, $|F_{K'}^{-1}|_{1,\infty,K'}\leqslant Ch$, $|J_{F_{K'}}|_{0,\infty,\hat{K}}\leqslant C\cdot\text{meas}(K')$, $|J_{F_K^{-1}}|_{0,\infty,K}\leqslant\frac{C}{\text{meas}(K)}$, and $(\nabla_{\hat{x}}F_K(\hat{x}))^{-1}=\nabla_xF_K^{-1}(x)$ (cf. [2]),

$$\int_{S_h}|\nabla_{\hat{x}}(u_h\circ F_K(\hat{x})-u\circ F_K(\hat{x}))\cdot\nabla_{x'}F_{K'}^{-1}(x')|^2dx'$$
$$\leqslant\int_{K'}|\nabla_{\hat{x}}(u_h\circ F_K(\hat{x})-u\circ F_K(\hat{x}))\cdot\nabla_xF_K^{-1}(x)\cdot\nabla_{\hat{x}}F_K(\hat{x})\cdot\nabla_{x'}F_{K'}^{-1}(x')|^2dx'$$
$$\leqslant\int_{\hat{K}}|\nabla_{\hat{x}}(u_h\circ F_K(\hat{x})-u\circ F_K(\hat{x}))\cdot\nabla_xF_K^{-1}(x)|^2d\hat{x}$$
$$\cdot|F_K|_{1,\infty,\hat{K}}^2\cdot|F_{K'}^{-1}|_{1,\infty,K'}^2\cdot|J_{F_{K'}}|_{0,\infty,\hat{K}}$$
$$\leqslant\int_K|\nabla u_h(x)-\nabla u(x)|^2dx\cdot|F_K|_{1,\infty,\hat{K}}^2\cdot|F_{K'}^{-1}|_{1,\infty,K'}^2$$
$$\cdot|J_{F_{K'}}|_{0,\infty,\hat{K}}\cdot|J_{F_K^{-1}}|_{0,\infty,K}$$
$$\leqslant C|u-u_h|_{1,K}^2. \tag{3.18}$$

From (3.17) and (3.18), taking account of $\text{meas}(S_h)=O(h_K^4)$, we can deduce that

$$|u-\tilde{u}_h|_{1,\Omega\setminus\Omega_h}^2=\sum_{S_h}|u-\tilde{u}_h|_{1,S_h}^2\leqslant C\|u\|_{3,\tilde{\Omega}}^2\cdot h_K^4+C|u-u_h|_{1,\Omega_h}^2. \tag{3.19}$$

By using the result $|u-u_h|_{1,\Omega_h}=O(h^2)$ in [2] and [3], from (3.19), we have

$$|u-u_h|_{1,\Omega}^2\leqslant|u-u_h|_{1,\Omega_h}^2+|u-\tilde{u}|_{1,\Omega\setminus\Omega_h}^2=O(h^4). \tag{3.20}$$

Thus the theorem is proved completely.

Remark 1. As the extension of u_h at the end of example 1 in section 2 we find $B_K+B_{K'}\neq0$, in which one may not be able to obtain the error bound $\|u-\tilde{u}_h\|_{1,\Omega}=O(h^2)$.

Remark 2. For the homogeneous Dirichlet problem considered in example 2 in section 2, it is well known that the extension $\tilde{u}_h$ of the linear finite element approximation u_h on $\Omega\setminus\Omega_h$ can be taken as zero extension to retain the error bound $\|u-\tilde{u}_h\|_{1,\Omega}=O(h)$. But for the 2-degree Lagrange finite element method (with just affine but not isoparametric element) one may not be able to obtain the error bound $\|u-\tilde{u}_h\|_{1,\Omega}=O(h^{3/2})$ with use of zero extension. However, the extension considered in section 2 can be used to obtain the error bound on Ω as well as on Ω_h.

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