THE ERROR ESTIMATES FOR CRANK-NICOLSON GALERKIN METHODS FOR QUASI-LINEAR PARABOLIC EQUATIONS WITH MIXED BOUNDARY CONDITIONS*

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§ 1. Introduction

There have been a lot of papers on finite element analyses of the linear and nonlinear parabolic equations, but only a few are concerned with the problems in which the boundary conditions are of mixed type——the problems that are frequently encountered in engineering applications.

In [5], the author considered the semi-discrete Galerkin methods for quasi-linear parabolic equations with nonlinear third mixed boundary conditions. In this paper, we consider a discrete time Galerkin approximation for the same parabolic problem investigated in [5]. In § 2, a Crank-Nicolson Galerkin procedure for the problem is described and its solvability discussed. In § 3 and § 4, H^1-norm and L_2-norm error estimates with optimal approximating order with respect to the space mesh parameter h are developed respectively.

Consider the following parabolic equation and associated initial value and boundary conditions:

$$
(\mathrm{A})\begin{cases}
\dfrac{\partial u}{\partial t}=\nabla\cdot(k(x, u)\nabla u)+\boldsymbol{b}(x, u)\cdot\nabla u+f(x, t; u), \\
\qquad\qquad\qquad\qquad (x, t)\in\Omega\times(0, T], & (1.1)\\
u=0, \quad (x, t)\in\partial\Omega_1\times[0, T], \\
k(x, u)\nabla u\cdot\nu+\sigma(x, u)u=g(x, t; u), \quad (x, t)\in\partial\Omega_2\times[0, T], & (1.2)\\
u(x, 0)=u_0(x), \quad x\in\Omega, & (1.3)
\end{cases}
$$

where Ω is a bounded domain in R^n with piecewise smooth boundary and satisfies the cone condition, $\partial\Omega=\partial\Omega_1\cup\partial\Omega_2$, meas $(\partial\Omega_1)>0$, $\boldsymbol{b}(x, u)=(b_1(x, u), b_2(x, u), \cdots, b_n(x, u))$ and $\nu=(\nu_1, \nu_2, \cdots, \nu_n)$ is the unit exterior normal of $\partial\Omega_2$.

Assume that k, $\boldsymbol{b}$, σ, f and g satisfy the following

Condition (A_1).

(i) There exist constants k_*, k^* such that

$$
\begin{aligned}
&0<k_*\leqslant k(x, p)\leqslant k^*, \quad |b_i(x, p)|\leqslant k^*, \quad \forall(x, p)\in\bar{\Omega}\times R^1;\\
&0\leqslant\sigma(x, p)\leqslant k^*, \quad \forall(x, p)\in\partial\Omega_2\times R^1.
\end{aligned} \tag{1.4}
$$

(ii) k, b_i $(i=1, 2, \cdots, n)$, f, σ, g are uniformly Lipschitz continuous with

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respect to their $(n+1)$th variable with Lipschitz constant L; for each $t\in[0, T]$, $f(x, t; 0)\in L_2(\Omega)$ and $g(x, t; 0)\in L_2(\partial\Omega_2)$; and also, f, g are continuous in variable t; $u_0(x)\in H^1_*(\Omega)$, where

$$H^1_*(\Omega)=\{v: v\in H^1(\Omega),\ v|_{\partial\Omega_1}=0\}.$$

In the above notations, $H^r(\Omega)$ are usual Hilbert-Sobolev spaces on Ω with norm $\|\cdot\|_r$, the subscript will be omitted in the case $r=0$. Analogously, let $H^r(\partial\Omega)$ denote Sobolev trace spaces on $\partial\Omega$ with norm $\|\cdot\|_{r,\partial\Omega}$; specifically, in the case $r=0$, $H^0(\partial\Omega)=L_2(\partial\Omega)$ and

$$\|v\|^2_{0,\partial\Omega}=\int_{\partial\Omega}v^2\,ds.$$

Let X be a Banach space, and $\varphi(t)$ a map $[0, T]\to X$. Define

$$\|\varphi\|_{L_p(X)}=\left(\int_0^T\|\varphi\|^p_X(t)\,dt\right)^{1/p},\ 1\le p<+\infty;\quad \|\varphi\|_{L_\infty(X)}=\sup_{0<t<T}\|\varphi\|_X(t).$$

The spaces $L_p(X)$ and $L_\infty(X)$ are the set of all φ such that above norm are finite respectively.

Let J be a positive integer, and $\Delta t=T/J$ a time step. Let $t_j=j\Delta t$, and $\varphi^j=\varphi(t_j)$. Define

$$\|\varphi\|_{\tilde L_2(X)}=\left(\sum_{j=0}^{J}\|\varphi^j\|^2_X\Delta t\right)^{1/2},\quad \|\varphi\|_{\tilde L_2^\prime(X)}=\left(\sum_{j=0}^{J-1}\|\varphi^{j+1/2}\|^2_X\Delta t\right)^{1/2},$$

$$\|\varphi\|_{\tilde L_\infty(X)}=\max_{0\le j\le J}\|\varphi^j\|_X,\quad \|\varphi\|_{\tilde L_\infty^\prime(X)}=\max_{0\le j\le J-1}\|\varphi^{j+1/2}\|_X,$$

where

$$\varphi^{j+1/2}\equiv(\varphi(t_j)+\varphi(t_{j+1}))/2.$$

For convenience, we write $\|\varphi\|_{L_p(H^r(\Omega))}\equiv\|\varphi\|_{L_p(H^r)}$, $\|\varphi\|_{L_\infty(L_\infty(\Omega))}\equiv\|\varphi\|_{L_\infty(L_\infty)}$ and $u(t)\equiv u(X, t)$, $b_i(u)\equiv b_i(x, u)$, $f(u)\equiv f(x, t, u)$ etc.

The weak form of problem (A) is the following: find a differentiable map $u(t)$: $[0, T]\to H^1_*(\Omega)$ such that

$$\text{(B)}\quad\begin{cases}\left(\dfrac{\partial u}{\partial t}, v\right)+a(u; u, v)=(\boldsymbol{b}(u)\cdot\nabla u, v)+(f(u), v)+\langle g(u), v\rangle,\\ \qquad\qquad\qquad\qquad \forall v\in H^1_*(\Omega),\ 0<t\le T,\\ u(0)=u_0,\end{cases}\tag{1.5}$$

where

$$(w, v)=\int_\Omega wv\,d\Omega,\quad \langle w, v\rangle=\int_{\partial\Omega_2}wv\,ds,$$

$$a(Q; w, v)=\int_\Omega k(Q)\nabla w\cdot\nabla v\,d\Omega+\int_{\partial\Omega_2}\sigma(Q)wv\,ds.\tag{1.6}$$

From (1.4)

$$k_*|v|_1^2\le a(Q; v, v)\le k^*(|v|_1^2+\|v\|^2_{0,\partial\Omega}),\quad \forall Q,\ v\in H^1_*(\Omega),\tag{1.7}$$

where the semi-norm

$$|v|_1^2=(\nabla v, \nabla v)=\sum_{i=1}^n\|v_{x_i}\|^2.$$

Throughout this paper, we shall always suppose that the solution $u(t)$ of problem (B) exists uniquely and use letters C, C_i, C_i^*, s to denote generic constants which have different values in different inequalities.

§ 2. Crank-Nicolson Galerkin Approximation and Its Solvability

Let $S_h(\Omega)=\text{span}\{\phi_1, \phi_2, \cdots, \phi_{N_h}\}\subset H_s^1(\Omega)$ denote a finite element subspace, where the basic functions ϕ_i satisfy the hypertheses: for each $h\in(0, 1]$,

$$\phi_i\in C(\bar{\Omega})\cap H_s^1(\Omega), \quad \|\nabla\phi_i\|_{L_\infty(\Omega)}=\max_{1<j<n}\left\|\frac{\partial\phi_i}{\partial x_j}\right\|_{L_\infty(\Omega)}<+\infty, \quad i=1, 2, \cdots, N_h. \tag{2.1}$$

Set $I=\{0,1,\cdots, J\}$, the Crank-Nicolson Galerkin approximation $\{U^j\}_0^J$ for the solution $u(t)$ of problem (B) is a map: $I\to S_h(\Omega)$ such that

$$(\mathrm{C})\begin{cases}\left(\dfrac{U^{j+1}-U^j}{\Delta t}, V\right)+a(U^{\jmath}; U^{\jmath}, V)=(\boldsymbol{b}(U^{\jmath})\cdot\nabla U^{\jmath}, V)\\ \qquad +(f(U^{\jmath}), V)+\langle g(U^{\jmath}), V\rangle,\\ \qquad\qquad \forall V\in S_h(\Omega), \quad j=0, 1, \cdots, J-1,\\ U^0 \text{ is given in } S_h(\Omega) \text{ such that } U^0-u_0 \text{ is sufficiently small for}\\ \quad \text{some norm } \|\cdot\|_X{}^{1)},\end{cases} \tag{2.2}$$

where $\jmath=j+1/2$, $g^{\jmath}=(g^j+g^{j+1})/2$.

Lemma 1. *In $H_s^1(\Omega)$, the semi-norm $|v|_1$ is equivalent to norm $\|v\|_1$* ([6]),

Lemma 2. *For each fixed $Q\in H_s^1(\Omega)$, the bilinear form $a(Q; w, v)$ is symmetric positive definite and bounded on $H_s^1(\Omega)\times H_s^1(\Omega)$ under condition* (A_1) ([6]).

Theorem 1. *Suppose that condition* (A_1) *and* (2.1) *hold; then for the Crank-Nicolson Galerkin procedure there exists a unique solution $\{U^j\}_0^J$ for appropriatly small Δt.*

Proof. The existence can be shown by Brower's fixed point theorem under those conditions given above ([1], [3]).

To prove the uniqueness, let $\{U^j\}$ and $\{\tilde{U}^j\}$ be the solutions of problem (C) and $U^0=\tilde{U}^0$. Let $\beta^j=U^j-\tilde{U}^j$. From (2.2),

$$\begin{aligned}&\left(\frac{\beta^{j+1}-\beta^j}{\Delta t}, V\right)+a(U^{\jmath}; U^{\jmath}-\tilde{U}^{\jmath}, V)\\ &\quad=a(\tilde{U}^{\jmath}; \tilde{U}^{\jmath}, V)-a(U^{\jmath}; \tilde{U}^{\jmath}, V)+((\boldsymbol{b}(U^{\jmath})-\boldsymbol{b}(\tilde{U}^{\jmath}))\cdot\nabla\tilde{U}^{\jmath}, V)\\ &\qquad+(\boldsymbol{b}(U^{\jmath})\cdot\nabla\beta^{\jmath}, V)+(f(U^{\jmath})-f(\tilde{U}^{\jmath}), V)+\langle g(U^{\jmath})-g(\tilde{U}^{\jmath}), V\rangle,\\ &\qquad\qquad\forall V\in S_h(\Omega), \ j=0, 1, \cdots, J-1.\end{aligned} \tag{2.3}$$

Taking $v=\beta^{\jmath}$, using the trace inequality and the interpolation theory on Sobolev spaces and applying a treatment analogous to that used in the proof of Theorem 1 in [5], we can prove that there are positive constants k_0, C independent of h such that

$$\frac{1}{2}\,\frac{\|\beta^{j+1}\|^2-\|\beta^j\|^2}{\Delta t}+k_0\|\beta^{\jmath}\|_1^2\leqslant C\left\{\varepsilon\|\beta^{\jmath}\|_1^2+\frac{1}{4\varepsilon}\|\beta^{\jmath}\|^2\right\}, \tag{2.4}$$

where ε is an arbitrary positive constant.

Choose ε such that $C\,\varepsilon<k_0$ and restrict Δt to being suitably small. Note that $\beta^0=0$. Then from

1) For the detailed description on $\|U^0-u_0\|_X$, see (3.17).

$$\frac{\|\beta^{j+1}\|^2-\|\beta^j\|^2}{\Delta t}\leqslant C(\|\beta^{j+1}\|+\|\beta^j\|^2)$$

we see that $\|\beta^j\|=0, j=1, 2, \cdots, J$. The uniqueness is thus proved.

§ 3. H^1-Norm Estimate

In order to derive the H^1-norm estimate of error $u(t_j)-U^j$, we make some assumptions which will be referred to as condition (A_2).

Condition (A_2).

(i) $\|\nabla u\|_{L_\infty(L_\infty)}<+\infty$, $\|u\|_{L_\infty(L_\infty(\partial\Omega))}<+\infty$,

(ii) u_{tt} and u_{ttt} are continuous in variable t and $u_{tt}\in L_\infty(H^1)$, $u_{ttt}\in L_\infty(L_2)$.

Let $g_{\bar{j}}=g(t_{\bar{j}})=g\left(\frac{t_{j+1}+t_j}{2}\right)$ and $\Delta_t g^j=\frac{g^{j+1}-g^j}{\Delta t}$. From condition (A_2)(ii) we see that

$$\rho^j\equiv\left(\frac{\partial u}{\partial t}\right)_{\bar{j}}-\Delta tu^j=-\frac{1}{24}\left(\frac{\partial^3 u}{\partial t^3}\right)_{j+\theta_1}\cdot\overline{\Delta t}^2,\quad 0\leqslant\theta_1\leqslant 1 \tag{3.1}$$

and that there is a constant M such that

$$\|\rho^j\|\leqslant M\overline{\Delta t}^2,\quad \forall j\in I. \tag{3.2}$$

Let $\{Y^j\}_0^J$ be an arbitrary map: $I\to S_h(\Omega)$, and set $\xi^j=U^j-Y^j$, $\eta^j=u^j-Y^j$, $e^j=u^j-U^j$. From (2.2) and (1.5) we have

$$\begin{aligned}(\Delta_t\xi^j, V)+a(U^{\bar{j}}; \xi^{\bar{j}}, V)&=a(u_{\bar{j}}; u_{\bar{j}}, V)-a(U^{\bar{j}}; Y^{\bar{j}}, V)+(\boldsymbol{b}(U^{\bar{j}})\cdot\nabla U^{\bar{j}}, V)\\&-(\boldsymbol{b}(u_{\bar{j}})\cdot\nabla u_{\bar{j}}, V)-(f(u_{\bar{j}})-f(U^{\bar{j}}), V)-\langle g(u_{\bar{j}})-g(U^{\bar{j}}), V\rangle\\&+(\Delta_t\eta^j, V)+(\rho^j, V),\quad \forall V\in S_h(\Omega),\ 0<j\leqslant J-1.\end{aligned} \tag{3.3}$$

Set $\omega^j=u(t_{\bar{j}})-u^{\bar{j}}$. Then

$$\omega^j=-\frac{1}{8}\left(\frac{\partial^2 u}{\partial t^2}\right)_{j+\theta_2}\cdot\overline{\Delta t}^2,\quad 0\leqslant\theta_2\leqslant 1. \tag{3.4}$$

From condition (A_2)(ii),

$$\|w^j\|_1\leqslant M\overline{\Delta t}^2,\quad \forall j\in I. \tag{3.5}$$

With $V=\xi^{\bar{j}}$ in (3.3), applying condition (A_2) and the inequality $ab\leqslant\varepsilon a^2+b^2/4\varepsilon$ $(\varepsilon>0)$ we can show that (cf. § 3 in [5])

$$a(u_{\bar{j}}; u_{\bar{j}}, \xi^{\bar{j}})-a(U^{\bar{j}}; Y^{\bar{j}}, \xi^{\bar{j}})\leqslant C_1\left\{\varepsilon\|\xi^{\bar{j}}\|_1^2+\frac{1}{4\varepsilon}(\|\xi^{\bar{j}}\|^2+\{\eta^{\bar{j}}\|_1^2+\|\omega^j\|_1^2)\right\}, \tag{3.6$_1$}$$

$$(\boldsymbol{b}(U^{\bar{j}})\cdot\nabla U^{\bar{j}}, \xi^{\bar{j}})-(\boldsymbol{b}(u_{\bar{j}})\cdot\nabla u_{\bar{j}}, \xi^{\bar{j}})\leqslant C_2\left\{\varepsilon\|\xi^{\bar{j}}\|_1^2+\frac{1}{4\varepsilon}(\|\xi^{\bar{j}}\|^2+\|\eta^{\bar{j}}\|_1^2+\|\omega^j\|_1^2)\right\}, \tag{3.6$_2$}$$

$$(f(u_{\bar{j}})-f(U^{\bar{j}}), \xi^{\bar{j}})\leqslant C_3\{\|\xi^{\bar{j}}\|^2+\|\eta^{\bar{j}}\|^2+\|\omega^j\|^2\}, \tag{3.6$_3$}$$

$$\begin{aligned}\langle g(u_{\bar{j}})-g(U^{\bar{j}}), \xi^{\bar{j}}\rangle&\leqslant L\int_{\partial\Omega}|e^{\bar{j}}+\omega^j|\cdot|\xi^{\bar{j}}|ds\\&\leqslant C_4\left\{\varepsilon\|\xi^{\bar{j}}\|_1^2+\frac{1}{4\varepsilon}(\|\xi^{\bar{j}}\|^2+\|\eta^{\bar{j}}\|_1^2+\|\omega^j\|_1^2)\right\},\end{aligned} \tag{3.6$_4$}$$

$$(\Delta_t\eta^j, \xi^{\bar{j}})\leqslant\varepsilon\|\xi^{\bar{j}}\|_1^2+\frac{1}{4\varepsilon}\|\Delta_t\eta^j\|_{-1}^2, \tag{3.6$_5$}$$

$$(\rho^j, \xi^j) \leqslant \varepsilon\|\xi^j\|_1^2+\frac{1}{4\varepsilon}\|\rho^j\|_{-1}^2 \leqslant \varepsilon\|\xi^j\|_1^2+\frac{1}{4\varepsilon}\|\rho^j\|^2. \tag{3.6}$$

Combining (3.3) with (3.2), (3.5) and (3.6) we obtain

$$\frac{1}{2}\cdot\frac{\|\xi^{j+1}\|^2-\|\xi^j\|^2}{\Delta t}+k_0\|\xi^j\|_1^2 \leqslant C\left\{\varepsilon\|\xi^j\|_1^2+\frac{1}{4\varepsilon}(\|\xi^j\|^2+\|\eta^j\|_1^2+\|\Delta_t\eta^j\|_{-1}^2+M^2\overline{\Delta t}^4)\right\}.$$

Choose ε small enough. Then there is a constant $\alpha_0>0$ such that

$$\|\xi^{j+1}\|^2-\|\xi^j\|^2+\alpha_0\Delta t\|\xi^j\|_1^2 \leqslant C_1\{\|\xi^{j+1}\|^2+\|\xi^j\|^2\}\Delta t+C_2(\|\eta^j\|_1^2+\|\Delta_t\eta^j\|_{-1}^2+M^2\overline{\Delta t}^4)\Delta t.$$

Therefore

$$\|\xi^{j+1}\|^2-\|\xi^0\|^2+\alpha_0\sum_{k=0}^{j}\|\xi^k\|_1\cdot\Delta t \leqslant 2C_1\sum_{k=0}^{j+1}\|\xi^k\|^2\cdot\Delta t+C_2(\|\eta\|_{\bar{L}_\Delta(H^1)}^2$$
$$+\|\Delta_t\eta\|_{\bar{L}_2(H^{-1})}^2+M^2T\cdot\overline{\Delta t}^4\},\quad \forall j:\ (j+1)\Delta t\leqslant T. \tag{3.7}$$

Applying Gronwall's inequality in discrete form and taking Δt to be small enough we see that there exists a constant $\beta_0>0$ such that

$$\|\xi^j\|^2+\beta_0\sum_{k=0}^{j-1}\|\xi^k\|_1^2\cdot\Delta t \leqslant C\{\|\eta\|_{\bar{L}_2^j(H^1)}^2+\|\Delta_t\eta\|_{\bar{L}_2(H^{-1})}^2+\overline{\Delta t}^4+\|\xi^0\|^2\}.$$

Hence

$$\|\xi\|_{\bar{L}_\infty(L_2)}+\|\xi\|_{\bar{L}_2^j(H^1)} \leqslant C\{\|\eta\|_{\bar{L}_2^j(H^1)}+\|\Delta_t\eta\|_{\bar{L}_2(H^{-1})}+\overline{\Delta t}^2+\|\xi^0\|\}. \tag{3.8}$$

By the triangle inequality we have

Theorem 2. *Assume that conditions* (A_1), (2.1) *and* (A_2) *hold, then for any map* $\{Y^j\}_0^J: I\to S_h(\Omega)$, *the error* $e^j\equiv u^j-U^j$ *be bounded by*

$$\|u-U\|_{\bar{L}_\infty(L_2)}+\|u-U\|_{\bar{L}_2^j(H^1)} \leqslant C\{\|u-Y\|_{\bar{L}_\Delta(L_2)}+\|u-Y\|_{\bar{L}_2^j(H^1)}$$
$$+\|\Delta_t(u-Y)\|_{\bar{L}_2(H^{-1})}+\overline{\Delta t}^2+\|\xi^0\|\}, \tag{3.9}$$

where C is a constant independent of h and Y.

Now we want to estimate the approximating order of error e^j. To this end, assume that the following conditions are satisfied.

Condition (A_3).

(i) $u\in L_\infty(H^r)$, $\frac{\partial u}{\partial t}\in L(H^{r-1})$, $u_0\in H^{r-1}(\Omega)$, $(r\geqslant 2)$ and condition (A_2) holds.

(ii) Condition (2.1) holds and $S_h(\Omega)$ is taken from a family of spaces of class $\tilde{S}_{1,r}(\Omega)$, $r\geqslant 2$, that is, $S_h(\Omega)\subset H_s^1(\Omega)$ and there exists a constant $C>0$ such that for each $v\in H_s^1(\Omega)\cap H^1(\Omega)$,

$$\inf_{x\in S_h(\Omega)}\|v-x\|_p \leqslant Ch^{l-p}\|v\|_l,\quad p<l\leqslant r,\ p=0,1. \tag{3.10}$$

(iii) Boundary $\partial\Omega$ is regular enough such that for every $\psi\in H^1(\Omega)$, the unique weak solution φ for the following boundary-value problem

$$\begin{cases} -\Delta\varphi+\varphi=\psi & \text{in } \Omega,\\ \varphi|_{\partial\Omega_1}=0;\ \left.\frac{\partial\varphi}{\partial\nu}\right|_{\partial\Omega_2}=0 \end{cases} \tag{3.11}$$

obeys the priori-estimate

$$\|\varphi\|_3 \leqslant C\|\psi\|_1\quad (C \text{ is independent of } \psi \text{ and } \varphi). \tag{3.12}$$

Let $Y(t)$ be H^1-projection into $S_h(\Omega)$ of $u(t)$, that is, $Y(t)$ is a map $[0, T]\to$

$S_h(\Omega)$ defined by

$$(u-Y,\ v)+(\nabla(u-Y),\ \nabla v)=0,\quad \forall v\in S_h(\Omega),\ 0\leqslant t\leqslant T. \tag{3.13}$$

It is proved in [5] and [8] that there exist C_1, C_2 such that

$$\|\eta\|_1(t)\leqslant C_1h^{r-1}\|u\|_r(t),\ \left\|\frac{\partial\eta}{\partial t}\right\|_{-1}(t)\leqslant C_2h^{r-1}\left\|\frac{\partial u}{\partial t}\right\|_{r-1}(t),\quad \forall t\in[0,\ T].$$

Thus

$$\|\eta\|_{\tilde{L}_2^j(H^1)}\leqslant Ch^{r-1}\|u\|_{L_\infty(H^r)}. \tag{3.14}$$

Noting that

$$\|\Delta_t\eta^j\|_{-1}^2\leqslant\frac{1}{\Delta t}\int_{t_j}^{t_{j+1}}\left\|\frac{\partial\eta}{\partial t}\right\|_{-1}^2dt,$$

we have

$$\|\Delta_t\eta\|_{\tilde{L}_2(H^{-1})}\leqslant Ch^{r-1}\left\|\frac{\partial u}{\partial t}\right\|_{L_\infty(H^{r-1})}. \tag{3.15}$$

Since

$$\|\eta^j\|^2\leqslant\|\eta^0\|^2+\sum_{k=0}^{j-1}\left\|\frac{(\eta^{k+1})^2-(\eta^k)^2}{\Delta t}\right\|\cdot\Delta t=\|\eta^0\|^2+2\sum_{k=0}^{j-1}(\Delta_t\eta^k,\ \eta^{\bar{k}})\Delta t$$
$$\leqslant\|\eta^0\|^2+\sum_{k=0}^{j-1}(\|\Delta_t\eta^k\|_{-1}^2+\|\eta^{\bar{k}}\|_1^2)\Delta t,$$

from (3.14) and (3.15), and noting that $\|\eta^0\|=\|\eta(0)\|\leqslant\|\eta(0)\|_1$ we get

$$\|\eta\|_{\tilde{L}_\infty^j(L_2)}\leqslant Ch^{r-1}\left(\|u\|_{L_\infty(H^r)}+\left\|\frac{\partial u}{\partial t}\right\|_{L_\infty(H^{r-1})}\right). \tag{3.16}$$

Choose U^0 such that

$$\|u_0-U^0\|\leqslant Ch^{r-1}, \tag{3.17}$$

then

$$\|\xi^0\|=\|U^0-Y^0\|\leqslant\|u_0-U^0\|+\|\eta^0\|\leqslant Ch^{r-1}. \tag{3.18}$$

In order to get (3.17), it is sufficient to choose U^0 to be the L_2-projection into $S_h(\Omega)$ of u_0.

Substituting (3.14), (3.15), (3.16) and (3.18) into (3.9) we obtain

Theorem 3. *Assume that conditions* (A_1), (A_3) *and* (3.18) *are satisfied; then*

$$\|u-U\|_{\tilde{L}_\infty(L_2)}+\|u-U\|_{\tilde{L}_2^j(H^1)}\leqslant C(\overline{\Delta t}^2+h^{r-1}), \tag{3.19}$$

where C is a constant independent of U, h and Δt.

§ 4. L_2-Norm Estimate

We now turn to an L_2-estimate for error $e^j=u(t_j)-U^j$.

Taking $Y(t)$ to be a Galerkin projection into $S_h(\Omega)$ of $u(t)$, that is, $Y(t)$ is a map $[0,\ T]\to S_h(\Omega)$ satisfying

$$a(u(t);\ Y(t),\ V)=a(u(t),\ u(t),\ V),\quad \forall V\in S_h(\Omega),\ 0\leqslant t\leqslant T. \tag{4.1}$$

By Lemma 2 and the Lax-Milgram theorem, the solution $Y(t)$ of (4.1) is existent uniquely and differentiable in variable t.

Let $\eta=u-Y$, $\xi=U-Y$ again. Since $a(u(t);\ \cdot,\ \cdot)$ is positive definite and bounded, and $S_h(\Omega)\subset\bar{S}_{1,r}(\Omega)$,

$$\|\eta\|_1(t)\leqslant C_1\inf_{x\in S_h(\Omega)}\|u-x\|_1(t)\leqslant Ch^{r-1}\|u\|_r(t).$$

Using Nitsche's method we get

$$\|\eta\|(t)\leqslant Ch^r\|u\|_r(t).$$

Hence

$$\|\eta\|_{\tilde{L}_\beta(L_2)}\leqslant C_1h^r\|u\|_{L_\infty(H^r)},\ \|\eta\|_{\tilde{L}_\infty(L_2)}\leqslant C_2h^r\|u\|_{L_\infty(H^r)}. \tag{4.2}$$

Now, assume that the following condition is satisfied.

Condition (A_4).

(a) Assumptions (i) and (ii) hold in condition (A_3);

(b) $(k(\cdot, u(x, t)))_t$, $(k(\cdot, u(\cdot, t)))_{tt}$ and $(k(x, u(x, t)))_{x_it}(i=1, 2, \cdots, n)$ are continuous in variable t and belong to $L_\infty(\Omega\times[0, T])$; $(\sigma(\cdot, u(\cdot, t)))_t$; $(\sigma(\cdot, u(\cdot,t)))_{tt}$ are in $L_\infty(\partial\Omega\times[0, T])$ and continuous with respect to t; $(b_i(x, u(x, t)))_{x_i}\in L_\infty(\Omega\times[0, T])$, $i=1, 2, \cdots, n$;

(c) For every $\psi\in H^1(\Omega)$ and very $\nu\in H^{l+1/2}(\partial\Omega)$, $l=0, 1$, the solution of the linear problem

$$a(u(t); \phi, v)=(\psi, v)+\langle\nu, v\rangle, \quad \forall v\in H_s^1(\Omega),\ 0\leqslant t\leqslant T \tag{4.3}$$

obeys the regularity estimate

$$\|\phi\|_{l+2}\leqslant C\{\|\psi\|_l+\|\nu\|_{l+1/2,\partial\Omega}\}, \tag{4.4}$$

here constant C is independent of ψ and ϕ.

According to Lemma 3 and Lemma 5 in [5] we have

Lemma 3. *Let $S_h(\Omega)\subset\tilde{S}_{1,r}(\Omega)$, $u\in L_p(H^r)$, $p=2, +\infty$, and $Y(t)$ be the solution of problem* (4.1). *If conditions* (A_1) (i) *and* (A_4) (c) *hold, then there is a constant C such that*

$$\|\eta\|_{\tilde{L}_\beta(H^{-\frac{1}{2}}(\partial\Omega))}=\|u-Y\|_{\tilde{L}_\beta(H^{-\frac{1}{2}}(\partial\Omega))}\leqslant Ch^r\|u\|_{L_p(H^r)},\quad p=2, +\infty. \tag{4.5}$$

Lemma 4. *Let $u(t)$ and $Y(t)$ be the solutions for problems* (B) *and* (4.1) *respectively. If conditions* (A_1) *and* (A_4) *hold, then there is a constant C such that*

$$\|\varDelta_t\eta\|_{\tilde{L}_2(H^{-1})}\leqslant Ch^r\Big(\|u\|_{L_\infty(H^r)}+\Big\|\frac{\partial u}{\partial t}\Big\|_{L_\infty(H^{r-1})}\Big). \tag{4.6}$$

Using the trace inequality we can prove the following lemma in a way similar to the proof of Lemma 4 in [6] (cf. § 3 in [6]).

Lemma 5. *Let $Y(t)$ be the solution of* (4.1), *then under conditions* (A_1) *and* (a), (b) *in* (A_4), $\Big\|\frac{\partial^2 Y}{\partial t^2}\Big\|_{L_\infty(H^1)}$ *can be bounded by* $\|u\|_{L_\infty(H^1)}$, $\Big\|\frac{\partial u}{\partial t}\Big\|_{L_\infty(H^1)}$ *and* $\Big\|\frac{\partial^2 u}{\partial t^2}\Big\|_{L_\infty(H^1)}$.

To finish our derivation, as usual, we have to make the following assumption ([2], [4], [8]).

Condition (A_5). There exists a constant K which is independent of h such that for each $h\in(0, 1]$, the solution $Y(t)$ of problem (4.1) satisfies

$$\|\nabla Y\|_{L_\infty(L_\infty)}\leqslant K. \tag{4.7}$$

For the discussion on condition (4.7), see [2], [8].

Now, estimate each term on the right-hand side in (3.3) with $V=\xi^j$. From (4.1),

$$\begin{aligned}J_1&\equiv a(u_j; u_j, \xi^j)-a(U^j; Y^j, \xi^j)=a(u_j; Y_j, \xi^j)-a(U^j; Y^j, \xi^j)\\&=a(u_j; Y^j, \xi^j)-a(U^j; Y^j, \xi^j)+a(u_j; P_j, \xi^j)\equiv I_1+I_2,\end{aligned} \tag{4.8}$$

where

$$P_j = Y_j - Y^j = Y(t_{j+1/2}) - \frac{Y^{j+1}+Y^j}{2} = -\frac{1}{8}\left(\frac{\partial^2 Y}{\partial t^2}\right)_{j+\theta_3} \cdot \overline{\Delta t}^2, \quad 0 \leqslant \theta_3 \leqslant 1. \tag{4.9}$$

By Lemma 5, there is a constant M such that

$$\|P_j\|_1 \leqslant M\overline{\Delta t}^2, \quad \forall j \in I. \tag{4.10}$$

Note that $\|\nabla u\|_{L_\infty(L_\infty)} < +\infty$, $\|\nabla Y\|_{L_\infty(L_\infty)} < +\infty$ and $\|u\|_{L_\infty(L_\infty(\partial\Omega))} < +\infty$. We have

$$I_1 = a(u_j; Y^j, \xi^j) - a(U^j; Y^j, \xi^j) \leqslant C_1\left\{\varepsilon\|\xi^j\|_1^2 + \frac{1}{4\varepsilon}(\|\xi^j\|^2 + \|\eta^j\|^2 + \|\omega^j\|^2)\right\}$$

$$+\int_{\partial\Omega_1}[\sigma(u_j)-\sigma(U^j)](Y^j-u^j)\xi^j\,ds + \int_{\partial\Omega_1}[\sigma(u_j)-\sigma(U^j)]u^j\xi^j\,ds$$

$$\leqslant C_1\{\varepsilon\|\xi^j\|_1^2+\cdots\} + C_2\|\xi^j\|_{\frac{1}{2},\partial\Omega}\cdot\|\eta^j\|_{-\frac{1}{2},\partial\Omega} + C_3(\|\xi^j\|_{\frac{1}{2}}^2 + \|\xi^j\|_{\frac{1}{2},\partial\Omega}\cdot\|\eta^j\|_{-\frac{1}{2},\partial\Omega} + \|\omega^j\|_1^2)$$

$$\leqslant C\left\{\varepsilon\|\xi^j\|_1^2 + \frac{1}{4\varepsilon}(\|\xi^j\|^2 + \|\eta^j\|^2 + \|\eta^j\|_{-\frac{1}{2},\partial\Omega}^2 + \|\omega^j\|_1^2\right\}.$$

Also,

$$I_2 = a(u_j; P_j, \xi^j) \leqslant C\left\{\varepsilon\|\xi^j\|_1^2 + \frac{1}{4\varepsilon}\|P_j\|_1^2\right\}.$$

Thus,

$$J_1 \leqslant C_1\left\{\varepsilon\|\xi^j\|_1^2 + \frac{1}{4\varepsilon}\left(\|\xi^j\|^2 + \|\eta^j\|^2 + \|\eta^j\|_{-\frac{1}{2},\partial\Omega}^2 + \|\omega^j\|_1^2 + \|P_j\|_1^2\right)\right\}. \tag{4.11}$$

Set

$$J_2 \equiv (\boldsymbol{b}(U^j)\cdot\nabla U^j, \xi^j) - (\boldsymbol{b}(u_j)\cdot\nabla u_j, \xi^j) = (\boldsymbol{b}(U^j)\cdot(\nabla U^j - \nabla Y^j), \xi^j)$$

$$+(\boldsymbol{b}(U^j)\cdot(\nabla Y^j - \nabla u_j), \xi^j) + ((\boldsymbol{b}(U^j) - \boldsymbol{b}(u_j))\cdot\nabla u_j, \xi^j) \equiv S_1 + S_2 + S_3. \tag{4.12}$$

Obviously,

$$S_1 \leqslant C_1^*\left(\varepsilon\|\xi^j\|_1^2 + \frac{1}{4\varepsilon}\|\xi^j\|^2\right), \tag{4.13}$$

$$S_3 \leqslant C_3^*(\|\xi^j\|^2 + \|\eta^j\|^2 + \|\omega^j\|^2), \tag{4.14}$$

$$S_2 = -(\boldsymbol{b}(U^j)\cdot(\nabla\eta^j + \nabla\omega^j), \xi^j) \leqslant -(\boldsymbol{b}(U^j)\cdot\nabla\eta^j, \xi^j) + \widetilde{C}_2(\|\xi^j\|^2 + \|\omega^j\|_1^2).$$

Since $\|\nabla\eta\|_{L_\infty(L_\infty)} < +\infty$,

$$(\boldsymbol{b}(U^j)\cdot\nabla\eta^j, \xi^j) = ((\boldsymbol{b}(U^j) - \boldsymbol{b}(u^j))\cdot\nabla\eta^j, \xi^j) + (\boldsymbol{b}(u^j)\cdot\nabla\eta^j, \xi^j)$$

$$\leqslant \widetilde{C}_3(\|\xi^j\|^2 + \|\eta^j\|^2) + (\boldsymbol{b}(u^j)\cdot\nabla\eta^j, \xi^j).$$

Integrating by parts for term $(\boldsymbol{b}(u^j)\cdot\nabla\eta^j, \xi^j)$ and applying the duality of $H^{-1/2}(\partial\Omega)$ with $H^{1/2}(\partial\Omega)$ we obtain

$$S_2 \leqslant C_2^*\left\{\varepsilon\|\xi^j\|_1^2 + \frac{1}{4\varepsilon}(\|\xi^j\|^2 + \|\eta^j\|^2 + \|\eta^j\|_{-\frac{1}{2},\partial\Omega}^2)\right\}. \tag{4.15}$$

Therefore

$$J_2 \leqslant C_2\left\{\varepsilon\|\xi^j\|^2 + \frac{1}{4\varepsilon}(\|\xi^j\|^2 + \|\eta^j\|^2 + \|\eta^j\|_{-\frac{1}{2},\partial\Omega}^2 + \|\omega^j\|_1^2)\right\}. \tag{4.16}$$

In addition,

$$\langle g(u_j) - g(U^j), \xi^j\rangle \leqslant L(\|\xi^j\|_{\frac{1}{2}}^2 + \|\xi^j\|_{\frac{1}{2},\partial\Omega}\cdot\|\eta^j\|_{-\frac{1}{2},\partial\Omega} + \|\omega^j\|_1^2)$$

$$\leqslant C_3\left\{\varepsilon\|\xi^j\|_1^2 + \frac{1}{4\varepsilon}(\|\xi^j\|^2 + \|\eta^j\|_{-\frac{1}{2},\partial\Omega}^2 + \|\omega^j\|_1^2)\right\}. \tag{4.17}$$

With $V=\xi^j$ in (3.3) and using (4.11), (4.16), (4.17), $(3.6)_3$, $(3.6)_5$, and $(3.6)_6$ we obtain

$$\frac{\|\xi^{j+1}\|^2-\|\xi^j\|^2}{\Delta t}+k_0\|\xi^j\|_1^2\leqslant C\Big\{\varepsilon\|\xi^j\|_1^2+\frac{1}{4\varepsilon}(\|\xi^j\|^2+\|\eta^j\|^2+\|\eta^j\|^2_{-\frac{1}{2},\partial\Omega}$$
$$+\|\Delta_t\eta^j\|_{-1}^2+\|\omega^j\|_1^2+\|P_j\|^2+\|\rho^j\|^2)\Big\}. \quad (4.18)$$

Repeating the argument in Theorem 2 and recalling (3.2), (3.5) and (4.10), we see that when the time step Δt is small enough,

$$\|\xi^{j+1}\|^2+\sum_{k=0}^{j}\|\xi^k\|_1^2\Delta t\leqslant C\{\|\eta\|^2_{\tilde{L}_2(L_2)}+\|\eta\|^2_{\tilde{L}_2(H^{-\frac{1}{2}}(\partial\Omega))}+\|\Delta_t\eta\|_{2\tilde{L}_2(H^{-1})}+\overline{\Delta t}^4+\|\xi^0\|^2\}.$$

Hence

$$\|\xi\|_{\tilde{L}_\infty(L_2)}+\|\xi\|_{\tilde{L}_2(H^1)}\leqslant C\{\|\eta\|_{\tilde{L}_2(L_2)}+\|\eta\|_{\tilde{L}_2(H^{-\frac{1}{2}}(\partial\Omega))}+\|\Delta_t\eta\|_{\tilde{L}_2(H^{-1})}+\overline{\Delta t}^2+\|\xi^0\|\}, \quad (4.19)$$

and

$$\|u-U\|_{\tilde{L}_\infty(L_2)}\leqslant C\{\|\eta\|_{\tilde{L}_\infty(L_2)}+\|\eta\|_{\tilde{L}_2(L_2)}+\|\eta\|_{\tilde{L}_2(H^{-\frac{1}{2}}(\partial\Omega))}+\|\Delta_t\eta\|_{\tilde{L}_2(H^{-1})}+\overline{\Delta t}^2+\|U^0-Y^0\|\}. \quad (4.20)$$

Applying (4.2), Lemma 3 and Lemma 4, we get

$$\|\eta\|_{\tilde{L}_\infty(L_2)}+\|\eta\|_{\tilde{L}_2(L_2)}+\|\eta\|_{\tilde{L}_2(H^{-\frac{1}{2}}(\partial\Omega))}+\|\Delta_t\eta\|_{\tilde{L}_2(H^{-1})}\leqslant Ch^r. \quad (4.21)$$

Choose U^0 in problem (C) such that

$$\|U^0-Y^0\|\leqslant Ch^r. \quad (4.22)$$

Specifically, we can take U^0 to be Y^0, where Y^0 is the solution of problem (4.1) at $t=0$.

Substituting (4.21) and (4.22) into (4.20) we obtain

$$\|u-U\|_{\tilde{L}_\infty(L_2)}\leqslant C(h^r+\overline{\Delta t}^2). \quad (4.23)$$

To sum up, we have proved the following result:

Theorem 4. *Let u and $\{U^j\}$ be the solutions for problems* (B) *and* (C) *respectively and choose U^0 such that inequality* (4.22) *holds, then under conditions* (A_1), (A_4) *and* (A_5), *the L_2-norm of error $u-U$ is estimated by inequality* (4.23), *here C is a constant independent of h, $\{U^j\}$, and Δt.*

For spaces $\tilde{S}_{1,r}(\Omega)$, the approximation order of h in the right-hand side of (4.23) is optimal.

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