EXTRAPOLATION CORRECTION AND COMBINED ALGORITHMS FOR SOLVING PARABOLIC EQUATIONS BY THE DIFFERENCE METHOD*

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Introduction

The primary motivation of the work described in this paper comes from [1], in which Lin Qun and Lü Tao have successfully presented the so-called splitting extrapolation process and obtained a series of extrapolation, correction and combined algorithms for the solution of multidimensional integral equations and elliptic equations. These algorithms can give higher accuracy results and are especially suitable for parallel computation.

In this paper, the idea of [1] is generalized to evolution equations. It is proved that, under certain conditions, the finite-difference approximate solution of the differential problem can be expanded to power series of the mesh width, so that the splitting extrapolation process presented in [1] can also be used. Some correction and combined algorithms for the solution of a heat equation are given, so as to obtain higher accuracy results. In addition, the accuracy of the method by B. K. Saul'ev[2] taking the arithmetic mean of the non-symmetric schemes is discussed.

1.

In this section we shall confine ourselves to linear evolution equations. An expression connecting the finite-difference solution with the analytic one will be given as a starting point of later discussions. As the condition of our main theorem requires some smoothness for the solution of the differential problem, this paper is mainly concerned with the Cauchy problem and initial-boundary value problem with the boundary condition of the first kind for the parabolic equation. It is well known that these problems are properly posed[3]. As for the hyperbolic equation, our conclusion will also be true provided that the solution is smooth enough.

In a suitably chosen Banach space, the problem in question can be expressed, following the notation of [4], by

$$\begin{cases} \frac{d}{dt} U(t) = AU(t) + g(t), \quad 0<t<T, \\ U(0) = U_0, \end{cases} \tag{1.1}$$

where A is a linear operator and, as in [4], does not depend on t. The boundary

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conditions, if any, will be supposed to be homogeneous linear, and be contained in the definition of the domain $D(A)$.

In solving problem (1.1) by the difference method, we choose mesh widths $\Delta t = h_0$, $\Delta x_i = h_i = g_i(\Delta t)$, and suppose $g_i(\Delta t) \to 0$ as $\Delta t \to 0$, $i=1, 2, \cdots, p$. The corresponding difference scheme is denoted by

$$\begin{cases} B_1(\Delta t)u^{n+1} = B_0(\Delta t)u^n + g^n, \\ u^0 = U_0, \end{cases} \tag{1.2}$$

where u^n is the numerical approximation to $U(n\Delta t)$, $g^n = g(n\Delta t)$, and $B_0(\Delta t)$, $B_1(\Delta t)$ are linear difference operators.

The difference problem (1.2) is said to provide a consistent approximation for the initial-value problem (1.1) if, for any function $u(t, X)$ having continuous partial derivatives up to the order $m+1$, and $u(t) \in D(A)$,

$$\left\| B_1(\Delta t)\, u(t+\Delta t) - B_0(\Delta t)\, u(t) - \left\{ \frac{d}{dt} - A \right\} u(t) \right\|$$
$$= \Big\| \sum_{\alpha < |\beta| < m} (R_\beta u)(t) h^\beta \Big\| + O(\bar{h}^{m+1}), \tag{1.3}$$

where the integer $\alpha \geq 1$, $h = (h_0, h_1, \cdots, h_p)$, $\bar{h} = \max\limits_i h_i$, $\beta = (\beta_0, \beta_1, \cdots, \beta_p)$, β_i are positive integers, $|\beta| = \beta_0 + \beta_1 + \cdots + \beta_p$, $h^\beta = h_0^{\beta_0} h_1^{\beta_1} \cdots h_p^{\beta_p}$ and $(R_\beta u)(t)$ are elements in the Banach space (in fact they are the derivatives of the function $u(t, X)$).

Set $C(\Delta t) = B_1^{-1}(\Delta t) B_0(\Delta t)$. The finite-difference approximation (1.2) is said to be stable, if, for some constants $\tau > 0$ and $M > 0$,

$$\|C(\Delta t)^n\| \leq M, \quad 0 < \Delta t < \tau, \; 0 \leq n\Delta t \leq T. \tag{1.4}$$

In addition, it is assumed that there exists a constant $N > 0$ such that

$$\|B_1^{-1}(\Delta t)\| \leq N \Delta t. \tag{1.5}$$

This condition can be satisfied by usual schemes.

We can now prove

Theorem. *If*

1) *equation* (1.1) *and the finite-difference approximation are consistent in the sense of* (1.3);

2) *scheme* (1.2) *is stable; and*

3) *condition* (1.5) *is satisfied,*

then, for any solution of equation (1.1) *having continuous partial derivatives of order* $2m+1$, *the following equality holds:*

$$u^n = U(n\Delta t) + \sum_{\alpha < |\beta| < m} V_\beta(n\Delta t) h^\beta + Q(n\Delta t), \tag{1.6}$$

where $V_\beta(t)$ *are independent of* h *and* $\|Q(n\Delta t)\| = O(\bar{h}^{m+1})$.

Proof. For any smooth solution of (1.1), from (1.3), we have

$$B_1(\Delta t)U(t+\Delta t) - B_0(\Delta t)U(t) = g(t) + \sum_{\alpha < |\beta| < m} (R_\beta U)(t) h^\beta + R_0(t), \tag{1.7}$$

where $\|R_0(t)\| = O(\bar{h}^{m+1})$. At the mesh points $t = n\Delta t$, subtracting (1.7) from (1.2), we get

$$B_1(\Delta t)\{u^{n+1} - U(t+\Delta t)\} - B_0(\Delta t)\{u^n - U(t)\} = -\sum_{\alpha < |\beta| < m} (R_\beta U)(t) h^\beta - R_0(t). \tag{1.8}$$

Corresponding to every term of $|\beta|=\alpha$ on the right of (1.8), we consider the following linear equation

$$\begin{cases} \frac{d}{dt}V_\beta(t)=AV_\beta(t)-(R_\beta U)(t), \\ V_\beta(0)=0. \end{cases} \tag{1.9}$$

From (1.3), the solution of (1.9) satisfies

$$B_1(\Delta t)V_\beta(t+\Delta t)-B_0(\Delta t)V_\beta(t)=-(R_\beta U)(t)+\sum_{\alpha<|\mu|<m}(R_\mu V_\beta)h^\mu+R_\beta(t), \tag{1.10}$$

where $\mu=(\mu_0, \mu_1, \cdots, \mu_p)$ with integers $\mu_i\geqslant 0$, and

$$\|R_\beta(t)\|=O(\bar{h}^{m+1}).$$

Multiplying (1.10) by h^β, and subtracting the result from (1.8), we obtain

$$B_1(\Delta t)\{u^{n+1}-U(t+\Delta t)-\sum_{|\beta|=\alpha}V_\beta(t+\Delta t)h^\beta\}$$
$$-B_0(\Delta t)\{u^n-U(t)-\sum_{|\beta|=\alpha}V_\beta(t)h_\beta\}$$
$$=-\sum_{\alpha+1<|\beta|<m}(R_\beta^{(1)}U)(t)h^\beta+R_1(t)$$

and

$$\|R_1(t)\|=O(\bar{h}^{m+1}).$$

Repeating the above argument, we know there exists a series of $V_\beta(t)$, $\alpha\leqslant|\beta|\leqslant m$, such that

$$B_1(\Delta t)\{u^{n+1}-U(t+\Delta t)-\sum_{\alpha<|\beta|<m}V_\beta(t+\Delta t)h^\beta\}$$
$$-B_0(\Delta t)\{u^n-U(t)-\sum_{\alpha<|\beta|<m}V_\beta(t)h^\beta\}=R(t) \tag{1.11}$$

and

$$\|R(t)\|=O(\bar{h}^{m+1}).$$

Write

$$Q^n=u^n-U(n\Delta t)-\sum_{\alpha<|\beta|<m}V_\beta(n\Delta t)h^\beta$$

and

$$R^n=R(n\Delta t).$$

Then we have

$$\begin{cases} B_1(\Delta t)Q^{n+1}-B_0(\Delta t)Q^n=R^n, \\ Q^0=0. \end{cases}$$

From condition (1.5) for $B_1^{-1}(\Delta t)$, we have

$$\begin{cases} Q^{n+1}=C(\Delta t)Q^n+\bar{R}^n, \\ Q^0=0, \end{cases} \tag{1.12}$$

with

$$\|\bar{R}^n\|=O(h_0\bar{h}^{m+1}).$$

By induction, it is easy to get

$$Q^n=\sum_{k=1}^{n}C(\Delta t)^{n-k}\bar{R}^{k-1}.$$

Then, using the definition of stability and the estimation of $\|\bar{R}^n\|$, we have

$$\|Q^n\|=O(\bar{h}^{m+1}). \tag{1.13}$$

This completes our proof.

The splitting extrapolation method for the multi-dimensional problem presented in [1] is essentially based on a relation between the analytic solution and the numerical one, which is similar to the expression (1.6). Since (1.6) has been proved, the same method can be applied to our problem. But, since the time variable is given the same treatment as the space variables, this method will not be very efficient in practice. Therefore, we should seek other correction and combined algorithms.

2.

The foregoing discussion suggested that, if the solution of equation (1.9) for $|\beta|=\alpha$ has been found, it will be possible to improve the accuracy of the approximate solution. But the difficulty lies in that the $(R_\beta U)(t)$ on the right of (1.9) is a high order differential of the unknown U, which is not easy to be found. However, in the case $\alpha=1$, the lower order differentials of the unknown are contained among $(R_\beta U)(t)$, so that we can solve the difference equation (1.2), and by means of the numerical solution with lower accuracy, the approximation to $(R_\beta U)(t)$ for $|\beta|=1$ can be given by the difference quotients. Thus equation (1.9) can be solved numerically, and the obtained solution will be used to correct the numerical solution of (1.2), and the accuracy of the approximation will be improved.

As an illustration we consider the heat equation. We first use the explicit 4-point scheme and then construct a new scheme.

Consider the Cauchy problem of the heat equation

$$\begin{cases} U_t=U_{xx}, & 0<t<T, \\ U(x,0)=f(x), & -\infty<x<+\infty. \end{cases} \tag{2.1}$$

Denote $r=\frac{\Delta t}{\Delta x^2}$. The explicit 4-point scheme is

$$\begin{cases} u_j^{n+1}=u_j^n+r(u_{j+1}^n-2u_j^n+u_{j-1}^n), \\ u_j^0=f(j\Delta x)=f_j, \quad j=0, \pm1, \pm2, \cdots, n=0, 1, \cdots, \left[\frac{T}{\Delta t}\right], \end{cases} \tag{2.2}$$

with truncation error

$$\begin{aligned} R_j^n &= \left(\frac{\Delta t}{2}U_{t^2}-\frac{\Delta x^2}{12}U_{x^4}\right)_j^n+O(\Delta t^2+\Delta x^4) \\ &= \left(\frac{r}{2}-\frac{1}{12}\right)\Delta x^2(U_{tx^2})_j^n+O(\Delta t^2+\Delta x^4). \end{aligned}$$

The solution obtained from (2.2) satisfies

$$u_j^n=U(j\Delta x, n\Delta t)+O(\Delta t+\Delta x^2). \tag{2.3}$$

To improve the accuracy of the solution (2.3) we apply the same scheme to solve the equation

$$\begin{cases} V_t=V_{xx}+\left(\frac{r}{2}-\frac{1}{12}\right)U_{tx^2}, \\ V(x,0)=0 \end{cases} \tag{2.4}$$

with U_{tx^2} replaced by the difference quotient expressed by u_j^n. If U is smooth enough, we have

$$\frac{1}{\Delta t}\left\{\frac{\delta^2 u_j^{n+1}}{\Delta x^2}-\frac{\delta^2 u_j^n}{\Delta x^2}\right\}=(U_{tx^2})_j^n+O(\Delta t+\Delta x^2),$$

where $\delta^2 u_j^n = u_{j-1}^n - 2u_j^n + u_{j+1}^n$. Such a treatment does not change the accuracy of the numerical solution of (2.4).

Let $\hat{u}_j^n = u_j^n + V_j^n \Delta x^2$. From the above, the error of $\tilde{u}_j^n$ is $O(\Delta t^2+\Delta x^4)$.

This correction algorithm requires the solution of two different difference equations in turn. To avoid the trouble, in solving equation (2.2), the initial data of each time step are considered to be exact, and after every step the solution will be immediately corrected. Combination of these two steps will give a new scheme, namely

$$u_j^{n+1}=u_j^n+r\delta^2 u_j^n+\left(\frac{r}{2}-\frac{1}{12}\right)r\{\delta^2(u_{j+1}^n-2u_j^n+u_{j-1}^n)\}. \tag{2.5}$$

It is easy to prove that the truncation error of the scheme is $O(\Delta t^2+\Delta x^4)$. By the Fourier method the amplification factor is

$$G=1-4r\sin^2\frac{k\Delta x}{2}+\left(8r-\frac{4}{3}\right)r\sin^4\frac{k\Delta x}{2},$$

and the stability condition is

$$r\leqslant\frac{2}{3}.$$

Thus, scheme (2.5) has a rather less strict stability condition in comparison with the explicit 4-point scheme.

(2.5) is a 6-point scheme which uses five neighbouring points at the time level t_n. It can also be used to solve the initial-boundary value problem with the boundary condition of the first kind. This only requires the replacement of the second order difference along x-direction by $u^{n+1}-u^n$ in computing the approximate value at points close to the boundary.

Table 1

t	Explicit 4-point scheme	Correction algorithm	Scheme (2.5)	Analytic solution
0.005	0.475528 0.769421	0.482675 0.780984	0.482675 0.780984	0.482495 0.780693
0.015	0.311237 0.503592	0.325269 0.526296	0.325481 0.526640	0.325117 0.526050
0.025	0.203707 0.329606	0.219014 0.354373	0.219481 0.355128	0.219072 0.354466
0.035	0.133328 0.215730	0.147354 0.238424	0.148003 0.239473	0.147616 0.238848
0.045	0.872645×10^{-1} 0.141197	0.990674×10^{-1} 0.160294	0.998024×10^{-1} 0.161484	0.994677×10^{-1} 0.160942
0.100	0.847952×10^{-2} 0.137202×10^{-1}	0.110282×10^{-1} 0.178440×10^{-1}	0.114271×10^{-1} 0.184894×10^{-1}	0.113421×10^{-1} 0.183519×10^{-1}

It must be pointed out that a similar scheme for the heat equation using the same mesh points has been given in [2], but it was obtained by a different approach, and has different coefficients and accuracy.

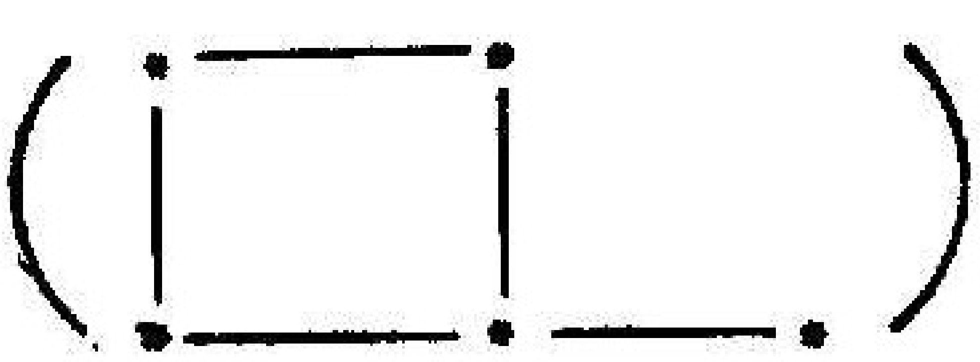
As a numerical test of the above methods we consider the heat equation (2.1) for which $f(x)=\sin 2\pi x$, $0<x<1$, and the boundary value is taken to be zero. The results of the explicit 4–point scheme, correction algorithm and scheme (2.5) are recorded in Table 1. Here $\Delta x=0.1$, $r=\frac{\Delta t}{\Delta x^2}=0.5$. Because of the periodicity of the solution, only two meaningful values are given. It is easy to see that the correction algorithm and scheme (2.5) give better results than the explicit 4–point scheme. Besides, the computation by scheme (2.5) with $r=\frac{2}{3}$ also gives satisfactory results.

3.

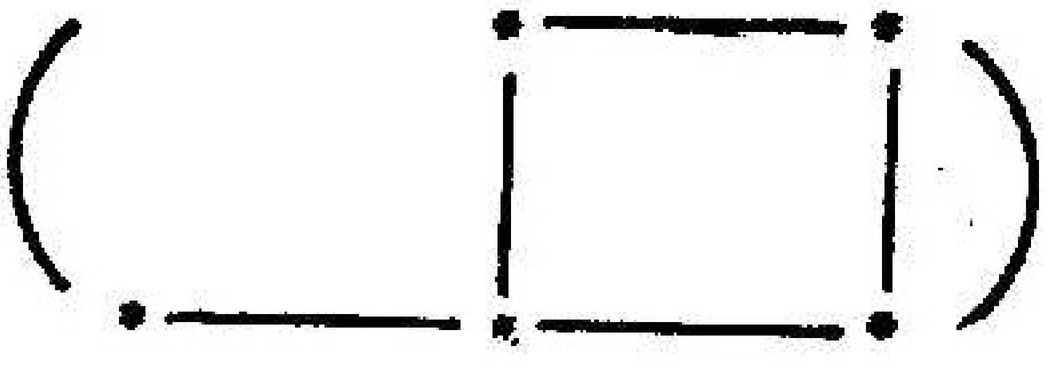
The proof of the theorem in section 1 indicates that $V_\beta(t)$ is determined by the truncation error of the difference equation. If two different schemes used to solve the same equation contain such truncation error that their lowest order terms, with opposite signs, are related to each other, then, a proper combination of the two corresponding solutions will give a result with higher accuracy. To illustrate the method we still consider the heat equation.

For a mixed problem with boundary conditions of the first kind in a rectangular domain, the non–symmetrical Saul'ev scheme for one space variable presented in [2] has two different forms as follows, with the diagrams indicating the mesh points used by each form.

a) $$\frac{u_j^{n+1}-u_j^u}{\tau}=\frac{\alpha}{h^2}(u_{j-1}^{n+1}-u_j^{n+1}-u_j^n+u_{j+1}^n)+\frac{1-\alpha}{h^2}(u_{j-1}^n-2u_j^n+u_{j+1}^n),$$



b) $$\frac{u_j^{n+1}-u_j^n}{\tau}=\frac{\alpha}{h^2}(u_{j-1}^n-u_j^n-u_j^{n+1}+u_{j+1}^{n+1})+\frac{1-\alpha}{h^2}(u_{j-1}^n-2u_j^n+u_{j+1}^n),$$



where $\tau=\Delta t$, $h=\Delta x$, α is a parameter and $0\leqslant\alpha\leqslant 1$; when $\alpha=1$, (a) and (b) have the simpliest forms.

In [2], it was shown that the arithmetic mean of the solutions of schemes (a) and (b) can be used as an approximation, and the algorithm is called the arithmetic mean method. To discuss the error of the method, the boundary value was supposed to be zero in [2]; then, the arithmetic mean expression was given in matrix form, and by Taylor's expansion, it was declared that the truncation error was $O(h^2)$ at the middle point of the interval and only $O(h)$ at other mesh points. To overcome this defect, the so-called multipoint symmetric method was suggested in [2].

However, by the foregoing discussion, it is easy to see that the error of the arithmetic mean solution is $O(h^2)$ at every mesh point, so that the multipoint symmetric method is unnecessary. To show this, we only need to calculate the truncation errors of (a) and (b),

$$\text{a)}\ R_j^n=\alpha\left(\frac{\partial^2 U}{\partial x\partial t}\right)_j^n\frac{\tau}{h}+\frac{1-\alpha}{2}\left(\frac{\partial^2 U}{\partial t^2}\right)_j^n\tau-\frac{h^2}{12}\left(\frac{\partial^4 U}{\partial x^4}\right)_j^n+(\text{higher order terms}),$$

$$\text{b)}\ R_j^n=-\alpha\left(\frac{\partial^2 U}{\partial x\partial t}\right)_j^n\frac{\tau}{h}+\frac{1-\alpha}{2}\left(\frac{\partial^2 U}{\partial t^2}\right)_j^n\tau-\frac{h^2}{12}\left(\frac{\partial^4 U}{\partial x^4}\right)_j^n+(\text{higher order terms}).$$

When $\frac{\tau}{h^2}=\text{const.}$, $\frac{\tau}{h}=O(h)$, the terms of order h in the truncation errors of both schemes (a) and (b) have the same absolute value, with opposite signs. Therefore, in the approximate solution expansions of schemes (a) and (b), which correspond to (1.6), the coefficients of the terms of order h have the same absolute value and opposite signs. In arithmetic mean, these terms are cancelled, thus, the arithmetic mean solution has the same accuracy at any point in the interval.

In [2], the algorithm using (a) and (b) alternately is called the mixed method. Especially, when $\alpha=1$, the scheme becomes

$$u_j^{2n+1}=\frac{1}{1+r}\{r(u_{j+1}^{2n}+u_{j-1}^{2n+1})+(1-r)u_j^{2n}\},$$

$$u_j^{2n+2}=\frac{1}{1+r}\{r(u_{j+1}^{2n+2}+u_{j-1}^{2n+1})+(1-r)u_j^{2n+1}\},$$

which is called the Saul'ev scheme in [4]. If the computation from time level $2n$ to $2n+2$ is regarded as one step, it is easy to know that the truncation error is

$$R_j^{2n+1}=-\frac{2}{3}\left(\frac{\partial^3 U}{\partial t^3}\right)_j^{2n+1}\tau^2-\frac{1}{6}\left(\frac{\partial^4 U}{\partial x^4}\right)_j^{2n+1}h^2$$
$$-2\left(\frac{\partial^2 U}{\partial t^2}\right)_j^{2n+1}\left(\frac{\tau}{h}\right)^2+(\text{higher order terms}).$$

Because the scheme is unconditionally stable, the main term of the error is $O\left(\frac{\tau^2}{h^2}\right)$.

Meanwhile, consider the Du Fort-Frankel scheme

$$\frac{u_j^{2n+2}-u_j^{2n}}{2\tau}=\frac{u_{j+1}^{2n+1}-u_j^{2n+2}-u_j^{2n}+u_{j-1}^{2n+1}}{h^2},$$

which has the truncation error

$$R_j^{2n+1}=\frac{1}{6}\left(\frac{\partial^3 U}{\partial t^3}\right)_j^{2n+1}\tau^2-\frac{1}{12}\left(\frac{\partial^4 U}{\partial x^4}\right)_j^{2n+1}h^2$$
$$+\left(\frac{\partial^2 U}{\partial t^2}\right)_j^{2n+1}\left(\frac{\tau}{h}\right)^2+(\text{higher order terms}).$$

This is a three-level scheme. If the initial data at two time steps are exact, it is easy to prove that its solution has an expression similar to (1.6). In view of the relation between the coefficients of the term containing $\left(\frac{\tau}{h}\right)^2$ in the truncation errors of both the Saul'ev scheme and the Du Fort-Frankel scheme, we take the linear combination

$$u_j^n = [2(u_j^n)_{D-F} + (u_j^n)_{Saul'ev}]/3$$

as the numerical solution, which has the accuracy $O(\Delta t^2 + \Delta x^2)$, although the truncation errors of both the original schemes do contain the term $O\left(\left(\frac{\Delta t}{\Delta x}\right)^2\right)$.

In Table 2, the numerical results of the Saul'ev scheme, the Du Fort-Frankel scheme, and the combined algorithm are shown. The computed problem and the parameters used are all identical with those in Table 1. The initial data at two time steps for the Du Fort-Frankel scheme are given by the analytic solution.

Table 2

t	Saul'ev scheme	D-F scheme	Combined algorithm	Analytic solution
0.015	0.350953 0.559539	0.315797 0.510970	0.327515 0.527159	0.325117 0.526050
0.025	0.242712 0.388559	0.206692 0.334434	0.218698 0.352476	0.219072 0.354466
0.035	0.168359 0.269871	0.135281 0.218890	0.146307 0.235883	0.147616 0.238848
0.045	0.116863 0.187346	0.885429×10^{-1} 0.143265	0.979828×10^{-1} 0.157959	0.994677×10^{-1} 0.160942
0.055	0.810776×10^{-1} 0.129902	0.579521×10^{-1} 0.937684×10^{-1}	0.656606×10^{-1} 0.105813	0.670239×10^{-1} 0.108447
0.100	0.139855×10^{-1} 0.235920×10^{-1}	0.860375×10^{-2} 0.139212×10^{-1}	0.103977×10^{-1} 0.171448×10^{-1}	0.113421×10^{-1} 0.183519×10^{-1}

The above combined algorithm can be generalized to the two-dimensional case. We consider the heat equation in a rectangular domain of x-y plane

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}$$

with boundary conditions of the first kind. Take $\alpha=1$, $\Delta x=\Delta y=h$. The corresponding non-symmetrical scheme of B. K. Saul'ev has four different forms

$$\text{A:}\quad \frac{u_{i,j}^{n+1}-u_{i,j}^n}{\tau} = \frac{1}{h^2}(u_{i-1,j}^{n+1}-u_{i,j}^{n+1}-u_{i,j}^n+u_{i+1,j}^n) + \frac{1}{h^2}(u_{i,j-1}^{n+1}-u_{i,j}^{n+1}-u_{i,j}^n+u_{i,j+1}^n),$$

$$\text{B:}\quad \frac{u_{i,j}^{n+1}-u_{i,j}^n}{\tau} = \frac{1}{h^2}(u_{i-1,j}^{n}-u_{i,j}^{n}-u_{i,j}^{n+1}+u_{i+1,j}^{n+1}) + \frac{1}{h^2}(u_{i,j-1}^{n}-u_{i,j}^{n}-u_{i,j}^{n+1}+u_{i,j+1}^{n+1}),$$

$$\text{C:}\quad \frac{u_{i,j}^{n+1}-u_{i,j}^n}{\tau} = \frac{1}{h^2}(u_{i-1,j}^{n+1}-u_{i,j}^{n+1}-u_{i,j}^n+u_{i+1,j}^n) + \frac{1}{h^2}(u_{i,j-1}^{n}-u_{i,j}^{n}-u_{i,j}^{n+1}+u_{i,j+1}^{n+1}),$$

$$\text{D:}\ \frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\tau}=\frac{1}{h^2}(u_{i-1,j}^{n}-u_{i,j}^{n}-u_{i,j}^{n+1}+u_{i+1,j}^{n+1})+\frac{1}{h^2}(u_{i,j-1}^{n+1}-u_{i,j}^{n+1}-u_{i,j}^{n}+u_{i,j+1}^{n}).$$

When A and B, or C and D, are used alternately, two different mixed methods can be constructed. The arithmetic mean of their solutions has the truncation error

$$R_{i,j}^{2n+1}=-\frac{2}{3}\left(\frac{\partial^3 U}{\partial t^3}\right)_{i,j}^{2n+1}\tau^2-\frac{1}{6}\left(\frac{\partial^4 U}{\partial x^4}+\frac{\partial^4 U}{\partial y^4}\right)_{i,j}^{2n+1}h^2$$
$$-2\left(\frac{\tau}{h}\right)^2\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 U\right]_{i,j}^{2n+1}+(\text{higher order terms}).$$

In the case of two space variables the Du Fort-Frankel scheme is

$$\frac{u_{i,j}^{2n+2}-u_{i,j}^{2n}}{2\tau}=\frac{u_{i+1,j}^{2n+1}-u_{i,j}^{2n+2}-u_{i,j}^{2n}+u_{i-1,j}^{2n+1}}{h^2}+\frac{u_{i,j+1}^{2n+1}-u_{i,j}^{2n+2}-u_{i,j}^{2n}+u_{i,j-1}^{2n+1}}{h^2},$$

and the corresponding truncation error is

$$R_{i,j}^{2n+1}=\frac{1}{3!}\left(\frac{\partial^3 U}{\partial t^3}\right)_{i,j}^{2n+1}\tau^2-\frac{1}{12}\left(\frac{\partial^4 U}{\partial x^4}+\frac{\partial^4 U}{\partial y^4}\right)_{i,j}^{2n+1}h^2$$
$$+\left(\frac{\tau}{h}\right)^2\left(\frac{\partial^2 U}{\partial t^2}\right)_{i,j}^{2n+1}+(\text{higher order terms}).$$

Noticing the coefficient of term $\left(\frac{\tau}{h}\right)^2$, combining the solution of the Du Fort-Frankel scheme with the arithmetic mean of the above-mentioned two mixed methods, we can get an approximate solution with the accuracy $O(\tau^2+h^2)$. The method is suitable to the parallel computation and is explicit.

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