APPROXIMATION OF INFINITE BOUNDARY CONDITION AND ITS APPLICATION TO FINITE ELEMENT METHODS*

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Abstract

The exterior boundary value problems of Laplace equation and linear elastic equations are considered. A series of approximate infinite boundary conditions are given. Then the original problem is reduced to a boundary value problem on a bounded domain. The finite element approximation of this problem and its error estimate are obtained. Finally, a numerical example shows that this method is very effective.

§ 1. Introduction

Many boundary value problems of partial differential equations involving the unbounded domain arise in practical applications, such as coupling of structures with foundation and environment and fluid flow around obstacles. In finding the numerical solutions of this kind of problems, it is often a difficulty using the classical finite element method or finite difference method. In engineering, the usual method is to cut off the unbounded part of the domain and to set up an artificial boundary condition at the new boundary of the remaining bounded domain. For example, the Dirichlet condition and Neumann condition are often used for elliptic partial differential equations. In general, the artificial boundary condition at the new boundary is only a rough approximation of the exact boundary condition. Hence the remaining bounded domain must be quite large when high accuracy is required. It is still difficult to compute the numerical solution on a quite large domain.

Combining the finite element method and the classical analytical method, Han and Ying[1] proposed the local finite element method for solving the elliptic boundary value problem on an unbounded domain. An exterior boundary value problem of model equation $\Delta u=0$ has been considered. By cutting off the exterior domain of a circle and getting the exact boundary condition at the new boundary of the remaining bounded domain by the classical analytical method, the original problem is reduced to an equivalent boundary value problem on a bounded domain with integral boundary condition. This method is closely related to the method of coupling of F. E. M. and canonical boundary reduction proposed by Feng Kang[2,3]. Their difference is in the form of the canonical integral equations. But in both methods, the integrals have singular kernels, and thus they are not

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readily available for compution. In this paper, exterior boundary value problems of Laplace equation and the linear elastic equations are considered. A series of approximate infinite boundary conditions are given and applied to the finite element method. The error estimate of the finite element approximate solution is obtained and a numerical example shows the effectiveness of this method.

§ 2. An Exterior Boundary Value Problem of Laplace Equation

2.1. The continuous problem

Let Γ_i be a bounded, simple closed curve in $\mathbb{R}^2$, and Ω be the unbounded domain with boundary Γ_i. Consider the following problem:

$$\begin{cases} -\Delta u=0, \ \Omega, \\ u|_{\Gamma_i}=f_i, \\ u \text{ is bounded, when } r\to+\infty. \end{cases} \tag{2.1}$$

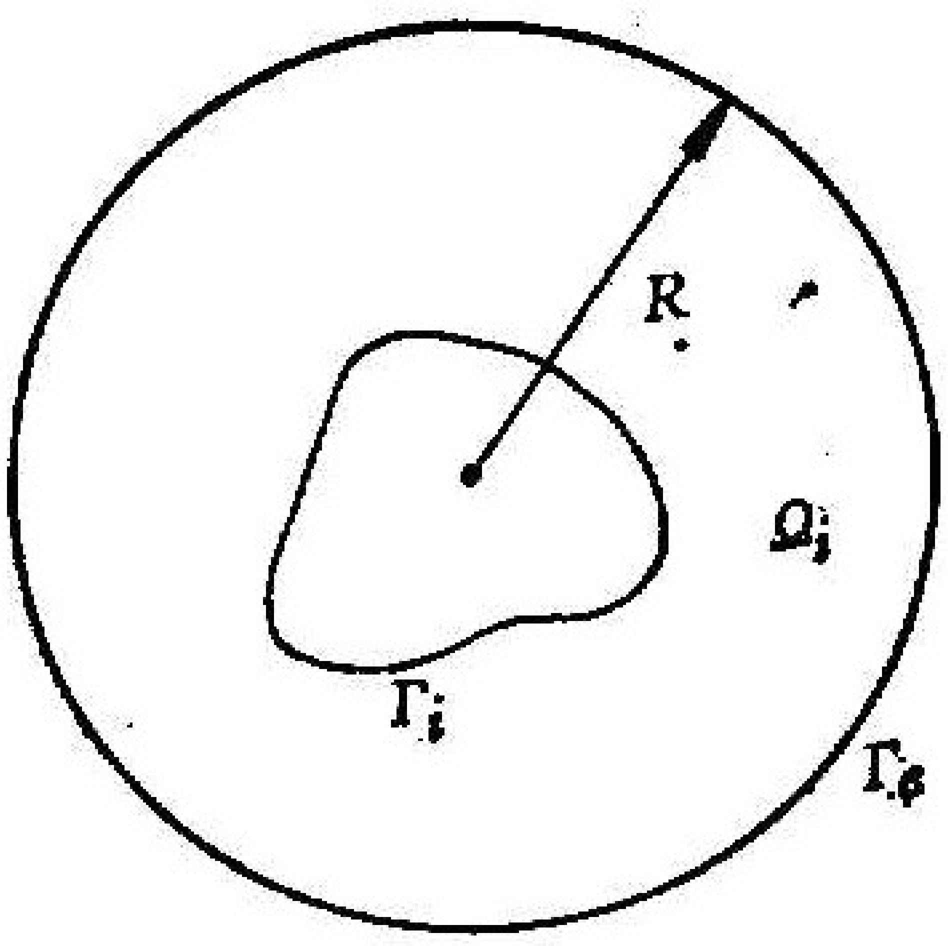

Fig. 1

This problem is defined on an unbounded domain Ω. First the problem is reduced to a boundary value problem on a bounded domain. In Ω, we draw a circumference Γ_e with radius R; then Ω is divided into two parts. The bounded part is denoted by Ω_i and $\Omega_e=\Omega\backslash\Omega_i$ is the unbounded part (see Fig. 1). Let $u(r, \theta)$ denote the solution of problem (2.1), where $x_1=r\cos\theta$, $x_2=r\sin\theta$. If a certain boundary condition of $u(r, \theta)$ on Γ_e is given, then we can consider problem (2.1) only on the bounded domain Ω_i. On domain Ω_e, $u(r, \theta)$ can be written as

$$u(r, \theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(\frac{R}{r}\right)^n(a_n\cos n\theta+b_n\sin n\theta), \tag{2.2}$$

therefore

$$\frac{\partial u(R, \theta)}{\partial r}=\sum_{n=1}^{\infty}-\frac{n}{R}(a_n\cos n\theta+b_n\sin n\theta). \tag{2.3}$$

On Γ_e, we have

$$u(R, \theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta) \tag{2.2$'$}$$

and

$$\frac{\partial^2 u(R, \theta)}{\partial\theta^2}=\sum_{n=1}^{\infty}(-n^2)(a_n\cos n\theta+b_n\sin n\theta). \tag{2.4}$$

From (2.4), we obtain the Fourier coefficients a_n, b_n $(n=1, 2, \cdots)$:

$$\begin{cases} a_n=-\dfrac{1}{\pi n^2}\displaystyle\int_0^{2\pi}\frac{\partial^2 u(R, \varphi)}{\partial\varphi^2}\cos n\varphi\, d\varphi, \\ b_n=-\dfrac{1}{\pi n^2}\displaystyle\int_0^{2\pi}\frac{\partial^2 u(R, \varphi)}{\partial\varphi^2}\sin n\varphi\, d\varphi. \end{cases} \tag{2.5}$$

And we have

$$\frac{\partial u(R,\theta)}{\partial r}=\sum_{n=1}^{\infty}\frac{1}{\pi nR}\int_0^{2\pi}\frac{\partial^2 u(R,\varphi)}{\partial\varphi^2}\cos n(\theta-\varphi)d\varphi\equiv g_2. \tag{2.6}$$

Moreover, we have

$$\frac{\partial u(R,\theta)}{\partial n}=\frac{\partial u(R,\theta)}{\partial r},$$

where $\frac{\partial}{\partial n}$ denotes the normal outer derivative on Γ_e. Therefore, we obtain the boundary condition on Γ_e

$$\frac{\partial u(R,\theta)}{\partial n}=\sum_{n=1}^{\infty}\frac{1}{\pi nR}\int_0^{2\pi}\frac{\partial^2 u(R,\varphi)}{\partial\varphi^2}\cos n(\theta-\varphi)d\varphi=g_2. \tag{2.7}$$

On Ω_i, $u(r,\theta)$ is the solution of the following boundary value problem

$$\begin{cases}-\Delta u=0,\ \Omega_i,\\ u|_{\Gamma_i}=f_i,\\ \left.\frac{\partial u}{\partial n}\right|_{\Gamma}=g_2.\end{cases} \tag{2.8}$$

Similarly, if we obtain the Fourier coefficients a_n, b_n by (2.2)′

$$\begin{cases}a_n=\frac{1}{\pi}\int_0^{2\pi}u(R,\varphi)\cos n\varphi\,d\varphi,\\ b_n=\frac{1}{\pi}\int_0^{2\pi}u(R,\varphi)\sin n\varphi\,d\varphi,\end{cases} \tag{2.9}$$

then from (2.3), we have

$$\frac{\partial u(R,\theta)}{\partial r}=\sum_{n=1}^{\infty}-\frac{n}{\pi R}\int_0^{2\pi}u(R,\varphi)\cos n(\theta-\varphi)d\varphi\equiv g_0, \tag{2.10}$$

or

$$\begin{cases}a_n=-\frac{1}{\pi n}\int_0^{2\pi}\frac{\partial u(R,\varphi)}{\partial\varphi}\sin n\varphi\,d\varphi,\\ b_n=\frac{1}{\pi n}\int_0^{2\pi}\frac{\partial u(R,\varphi)}{\partial\varphi}\cos n\varphi\,d\varphi,\end{cases} \tag{2.11}$$

and

$$\frac{\partial u(R,\theta)}{\partial r}=\sum_{n=1}^{\infty}\frac{1}{\pi R}\int_0^{2\pi}\frac{\partial u(R,\varphi)}{\partial\varphi}\sin n(\varphi-\theta)d\varphi\equiv g_1. \tag{2.12}$$

Consequently, we have obtained three kinds of boundary conditions on Γ_e. Boundary condition (2.7) corresponds to the integral boundary condition with weak singular kernel and (2.10) to the integral boundary condition with strong singular kernel (see [1, 2]).

Now, we consider the boundary value problem (2.8) when boundary condition (2.7) is used. It is convenient to consider the following problem

$$\begin{cases}-\Delta u=f,\ \Omega_i,\\ u|_{\Gamma_i}=0,\\ \left.\frac{\partial u}{\partial n}\right|_{\Gamma_e}=g_2.\end{cases} \tag{2.13}$$

Let $V=\{v\in H^1(\Omega_i);\ v|_{\Gamma_i}=0\}$. Then the boundary value problem (2.13) is equivalent to the following variational problem

Find $u\in V$, such that

$$a(u,v)+b(u,v)=f(v),\quad \forall v\in V, \tag{2.14}$$

where

$$a(u,v)=\int_{\Omega_i}\nabla u\cdot\nabla v\,dx,\qquad f(v)=\int_{\Omega_i}fv\,dx,$$

$$b(u,v)=\sum_{n=1}^{\infty}\frac{1}{n\pi}\int_0^{2\pi}\int_0^{2\pi}\cos n(\theta-\varphi)\frac{\partial u(R,\theta)}{\partial\theta}\,\frac{\partial v(R,\varphi)}{\partial\varphi}\,d\theta\,d\varphi, \tag{2.15}$$

and $\frac{\partial u(R,\theta)}{\partial\theta}$, $\frac{\partial v(R,\varphi)}{\partial\varphi}$ are understood as the distributions on Γ_e.

Theorem 2.1. *$a(u,v)+b(u,v)$ is a symmetric and continuous V-elliptic bilinear form on $V\times V$.*

Proof. We recall an equivalent definition of Sobolev space $H^s(\Gamma_e)$[4]

$$u\in H^s(\Gamma_e)\Leftrightarrow u=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta),$$

$$\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(1+n^2)^s(a_n^2+b_n^2)<\infty,\quad s\in\mathbb{R}^1.$$

Assume that

$$u(R,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta),$$

$$v(R,\varphi)=\frac{c_0}{2}+\sum_{n=1}^{\infty}(c_n\cos n\varphi+d_n\sin n\varphi).$$

Take the first derivative with respect to θ and φ

$$\frac{\partial u(R,\theta)}{\partial\theta}=\sum_{n=1}^{\infty}n(-a_n\sin n\theta+b_n\cos n\theta),$$

$$\frac{\partial v(R,\varphi)}{\partial\varphi}=\sum_{n=1}^{\infty}n(-c_n\sin n\varphi+d_n\cos n\varphi).$$

Then we have

$$\begin{aligned}b(u,v)&=\sum_{n=1}^{\infty}\frac{1}{\pi n}\int_0^{2\pi}\int_0^{2\pi}(\cos n\theta\cos n\varphi+\sin n\theta\sin n\varphi)\,\frac{\partial u(R,\theta)}{\partial\theta}\,\frac{\partial v(R,\varphi)}{\partial\varphi}\,d\theta\,d\varphi\\&=\pi\sum_{n=1}^{\infty}n(a_nc_n+b_nd_n).\end{aligned}$$

Using the Cauchy inequality, we get

$$|b(u,v)|\leqslant\pi\Big[\sum_{n=1}^{\infty}n(a_n^2+b_n^2)\Big]^{1/2}\Big[\sum_{n=1}^{\infty}n(c_n^2+d_n^2)\Big]^{1/2}\leqslant\pi\|u\|_{1/2,\Gamma_e}\|v\|_{1/2,\Gamma_e}.$$

By the trace theorem, we obtain

$$|b(u,v)|\leqslant C\|u\|_{1,\Omega_i}\|v\|_{1,\Omega_i}. \tag{2.16}$$

On the other hand, since $b(u,u)\geqslant 0$, $a(u,v)+b(u,v)$ is V-elliptic. The proof is completed.

Now, we consider the approximation of problem (2.13). Let

$$g_2^N=\frac{1}{\pi R}\int_0^{2\pi}\Big(\sum_{n=1}^{N}\frac{\cos n(\theta-\varphi)}{n}\Big)\frac{\partial^2u(R,\varphi)}{\partial\varphi^2}\,d\varphi. \tag{2.17}$$

Consider the following problem

$$\begin{cases} -\Delta u^N = f, \\ u^N|_{\Gamma_i} = 0, \\ \left.\dfrac{\partial u^N}{\partial n}\right|_{\Gamma_e} = g_2^N. \end{cases} \tag{2.18}$$

It is equivalent to the variational problem

$$\begin{aligned} &\text{Find } u^N \in V, \text{ such that} \\ &a(u^N, v) + b_N(u^N, v) = f(v), \quad \forall v \in V, \end{aligned} \tag{2.19}$$

where

$$b_N(u^N, v) = \frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi}\left(\sum_{n=1}^{N}\frac{\cos n(\theta-\varphi)}{n}\right)\frac{\partial u^N(R,\theta)}{\partial\theta}\,\frac{\partial v(R,\varphi)}{\partial\varphi}\,d\theta\,d\varphi.$$

Theorem 2.2. *Problem* (2.19) *has a unique solution* u^N.

Proof. For arbitrary $u, v \in V$, assume that

$$u(R,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n\sin n\theta),$$

$$v(R,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty}(c_n\cos n\theta + d_n\sin n\theta).$$

Then we have

$$|b_N(u, v)| = \left|\pi\sum_{n=1}^{N} n(a_nc_n + b_nd_n)\right| \leqslant \pi\|u\|_{1/2,\Gamma_e}\|v\|_{1/2,\Gamma_e} \leqslant C\|u\|_{1,\Omega_i}\|v\|_{1,\Omega_i},$$

where C is a constant independent of N. Moreover, we know that $a(u,v)+b_N(u,v)$ is a symmetric and continuous V-elliptic bilinear form on $V\times V$. The conclusion then follows from the Lax-Milgram theorem.

The boundary conditions considered above are called global boundary conditions. We can also obtain another kind of boundary conditions, the local boundary conditions. Assume that $\frac{\partial u}{\partial r}$ has the following expansion on the boundary Γ_e

$$\frac{\partial u(R,\theta)}{\partial r} = \frac{1}{R}\sum_{K=1}^{\infty} S_K\,\frac{\partial^{2K}u(R,\theta)}{\partial\theta^{2K}}, \tag{2.20}$$

where S_K $(K=1, 2, \cdots)$ are constants to be determined, and $u(r,\theta)$ is the solution of problem (2.1). From

$$\frac{\partial u(R,\theta)}{\partial r} = \sum_{n=1}^{\infty} -\frac{n}{R}(a_n\cos n\theta + b_n\sin n\theta),$$

$$\frac{\partial u^{2K}(R,\theta)}{\partial\theta^{2K}} = \sum_{n=1}^{\infty}(-n^2)^K(a_n\cos n\theta + b_n\sin n\theta),$$

we get

$$\sum_{n=1}^{\infty}\left[-n - \sum_{K=1}^{\infty}(-n^2)^K S_K\right](a_n\cos n\theta + b_n\sin n\theta) = 0.$$

Then $S_K(K=1, 2, 3, \cdots)$ satisfy the following linear algebraic equations

$$\sum_{K=1}^{\infty}(-n^2)^K S_K = -n, \quad n=1, 2, 3, \cdots. \tag{2.21}$$

Consider the approximation of the boundary condition (2.20). Assume that

$$\frac{\partial u(R,\theta)}{\partial r} = \frac{1}{R}\sum_{k=1}^{N} S_k^*\,\frac{\partial^{2k}u(R,\theta)}{\partial\theta^{2k}},$$

where S_k^* $(k=1, 2, \cdots, N)$ satisfy

$$\sum_{k=1}^{N}(-n^2)^k S_k^* = -n, \quad n=1, 2, \cdots, N.$$

For $N=1, 2, 3$, we have the following approximate boundary conditions

$$\frac{\partial u(R, \theta)}{\partial r} = \frac{1}{R}\frac{\partial^2 u(R, \theta)}{\partial \theta^2},$$

$$\frac{\partial u(R, \theta)}{\partial r} = \frac{1}{R}\left(\frac{7}{6}\frac{\partial^2 u(R, \theta)}{\partial \theta^2} + \frac{1}{6}\frac{\partial^4 u(R, \theta)}{\partial \theta^4}\right),$$

$$\frac{\partial u(R, \theta)}{\partial r} = \frac{1}{R}\left(\frac{74}{60}\frac{\partial^2 u(R, \theta)}{\partial \theta^2} + \frac{15}{60}\frac{\partial^4 u(R, \theta)}{\partial \theta^4} + \frac{1}{60}\frac{\partial^6 u(R, \theta)}{\partial \theta^6}\right).$$

Obviously, it is inconvenient to use the local boundary condition for finite element approximation when $N>1$, but it is convenient to use the global boundary condition.

2.2. Finite element approximation

For the sake of simplicity, let Γ_i be a polygonal line, and $\mathscr{T}_h$ be a triangulation on Ω_i satisfying

(i) $$\Omega_i = \left(\bigcup_{K \in \mathscr{T}_h} K\right) \cup \left(\bigcup_{\tilde{K} \in \mathscr{T}_h} \tilde{K}\right),$$

where K is a triangle, $\tilde{K}$ is a curved triangle with a curved side on Γ_e, and

(ii) $$h_K/\rho_K \leqslant \sigma, \quad \forall K, \tilde{K} \in \mathscr{T}_h,$$

where

$$h_K = \text{diameter of } K \text{ or } \tilde{K},$$

$$\rho_K = \text{diameter of the inscribed circle of } K \text{ or } \tilde{K},$$

$$h = \max_{K, \tilde{K} \in \mathscr{T}_h} h_K.$$

Let

$$V_h(\Omega_i) = \{v \in H^1(\Omega_i),\ v|_K (v|_{\tilde{K}}) \text{ is a linear polynomial}, \forall K(\tilde{K}) \in \mathscr{T}_h,\ v|_{\Gamma_i} = 0\}.$$

We consider the approximate problem of (2.19):

$$\text{Find } u_h^N \in V_h, \text{ such that}$$
$$a(u_h^N, v) + b_N(u_h^N, v) = f(v), \quad \forall v \in V_h. \tag{2.24}$$

Similarly, we have

Theorem 2.3. *The variational problem (2.24) has a unique solution u_h^N.*

The remainder of this section is devoted to estimating the error between exact solution u and approximate solution u_h^N. We have

Theorem 2.4. *There exists a constant C independent of h and N, such that*

$$\|u-u_h^N\|_{1,\Omega_i} \leqslant C\left\{\inf_{v \in V_h}\|u-v\|_{1,\Omega_i} + \sup_{w \in V_h}\frac{|b_N(u, w) - b(u, w)|}{\|w\|_{1,\Omega_i}}\right\}. \tag{2.25}$$

Proof. Equality (2.14) can be rewritten as

$$a(u, v) + b_N(u, v) = f(v) + b_N(u, v) - b(u, v), \quad \forall v \in V.$$

Combining (2.24), we obtain

$$a(u-u_h^N, v) + b_N(u-u_h^N, v) = b_N(u, v) - b(u, v), \quad \forall v \in V_h. \tag{2.26}$$

Then

$$\|u_h^N-v\|^2_{1,\Omega_i}\leqslant C[a(u_h^N-v,\ u_h^N-v)+b_N(u_h^N-v,\ u_h^N-v)]$$
$$=C[a(u-v,\ u_h^N-v)+b_N(u-v,\ u_h^N-v)+b(u,\ u_h^N-v)-b_N(u,\ u_h^N-v)]$$
$$\leqslant C[\|u-v\|_{1,\Omega_i}\|u_h^N-v\|_{1,\Omega_i}+|b_N(u,\ u_h^N-v)-b(u,\ u_h^N-v)|],\quad \forall v\in V_h.$$

Therefore, we have

$$\|u_h^N-v\|_{1,\Omega_i}\leqslant C\Big[\|u-v\|_{1,\Omega_i}+\sup_{w\in V_h}\frac{|b_N(u,\ w)-b(u,\ w)|}{\|w\|_{1,\Omega_i}}\Big],\quad \forall v\in V_h.$$

By the triangle inequality

$$\|u-u_h^N\|_{1,\Omega_i}\leqslant\|u-v\|_{1,\Omega_i}+\|u_h^N-v\|_{1,\Omega_i},$$

the proof is completed.

Remark 2.1. Theorem 2.4 is similar to the first Strang lemma[5], the difference is in the second term.

Theorem 2.5. *Assume that* $u\in H^2(\Omega_i)\cap H^{K-\frac{1}{2}}(\Gamma_e)$, $K\geqslant 2$; *then*

$$\|u-u_h^N\|_{1,\Omega_i}\leqslant C\Big(h\|u\|_{2,\Omega_i}+\frac{1}{N^{K-1}}\|u\|_{K-\frac{1}{2},\Gamma_e}\Big),\tag{2.27}$$

where C is a constant independent of h and N.

Proof. By Theorem 2.4, we obtain the estimation

$$\|u-u_h^N\|_{1,\Omega_i}\leqslant C\Big\{\inf_{v\in V_h}\|u-v\|_{1,\Omega_i}+\sup_{w\in V_h}\frac{|b_N(u,\ w)-b(u,\ w)|}{\|w\|_{1,\Omega_i}}\Big\}.$$

For the first term, we have[5]

$$\inf_{v\in V_h}\|u-v\|_{1,\Omega_i}\leqslant Ch\|u\|_{2,\Omega_i}.$$

Therefore, we only need to estimate the second term. We have

$$|b_N(u,\ w)-b(u,\ w)|\leqslant\frac{1}{\pi}\Big|\int_0^{2\pi}\int_0^{2\pi}\frac{\partial u}{\partial\theta}\frac{\partial w}{\partial\varphi}\sum_{n=N+1}^{\infty}\frac{\cos n(\theta-\varphi)}{n}\,d\theta\,d\varphi\Big|$$
$$=\pi\Big|\sum_{n=N+1}^{\infty}n(a_ne_n+b_nf_n)\Big|\leqslant\frac{\pi}{N^{K-1}}\sum_{n=N+1}^{\infty}n^K|a_ne_n+b_nf_n|$$
$$\leqslant\frac{C}{N^{K-1}}\Big[\sum_{n=N+1}^{\infty}(n^2)^{K-\frac{1}{2}}(a_n^2+b_n^2)\Big]^{1/2}\Big[\sum_{n=N+1}^{\infty}n(e_n^2+f_n^2)\Big]^{1/2},$$

where

$$u(R,\ \theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta),$$
$$w(R,\ \theta)=\frac{e_0}{2}+\sum_{n=1}^{\infty}(e_n\cos n\theta+f_n\sin n\theta).$$

Finally, we obtain

$$|b_N(u,\ w)-b(u,\ w)|\leqslant\frac{C}{N^{K-1}}\|u\|_{K-\frac{1}{2},\Gamma_e}\|w\|_{\frac{1}{2},\Gamma_e}\leqslant\frac{C}{N^{K-1}}\|u\|_{K-\frac{1}{2},\Gamma_e}\|w\|_{1,\Omega_i}.$$

Inequality (2.27) follows immediately.

§ 3. Linear Elastic Equations

Consider the boundary value problem of linear elastic equations on the unbounded domain Ω with boundary Γ_i:

$$\begin{cases} \Delta u+\frac{\lambda+\mu}{\mu}\frac{\partial\Theta}{\partial x}=0,\ \Omega, \\ \Delta v+\frac{\lambda+\mu}{\mu}\frac{\partial\Theta}{\partial y}=0,\ \Omega, \\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\Theta,\ \Omega, \\ u|_{\Gamma_i}=f_1, \\ v|_{\Gamma_i}=f_2, \\ u,\ v \text{ are bounded, when } r=(x^2+y^2)^{1/2}\to+\infty. \end{cases} \tag{3.1}$$

In this section, a series of approximate infinite boundary conditions for the linear elastic equations are given. Similarly, the finite element approximation of problem (3.1) can be obtained on a bounded domain. In Ω, we draw a circumference Γ_e with radius R. Then Ω is divided into two parts as shown in Fig. 1. The solution of problem (3.1) on the unbounded domain Ω_e satisfies

$$\begin{cases} \Delta u+\frac{\lambda+\mu}{\mu}\frac{\partial\Theta}{\partial x}=0,\ \Omega_e, \\ \Delta v+\frac{\lambda+\mu}{\mu}\frac{\partial\Theta}{\partial y}=0,\ \Omega_e, \\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\Theta,\ \Omega_e, \\ u|_{\Gamma_e}=u(R,\theta), \\ v|_{\Gamma_e}=v(R,\theta), \\ u,\ v \text{ are bounded, when } r\to+\infty. \end{cases}$$

Let $p=-\frac{\lambda+\mu}{\mu}\Theta$. The above problem can be rewritten as

$$\begin{cases} \Delta u-\frac{\partial p}{\partial x}=0,\ \Omega_e, \\ \Delta v-\frac{\partial p}{\partial y}=0,\ \Omega_e, \\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=-\varkappa p,\ \Omega_e, \\ u|_{\Gamma_e}=u(R,\theta), \\ v|_{\Gamma_e}=v(R,\theta), \\ u,\ v \text{ are bounded, when } r\to+\infty, \end{cases} \tag{3.2}$$

where $\varkappa=\frac{\mu}{\lambda+\mu}>0$. By these equations, we have

$$\varkappa(\Delta p)+\Delta p=0,\ \Omega_e,$$

namely,

$$\Delta p=0,\ \Omega_e. \tag{3.3}$$

Moreover, we obtain

$$\Delta^2 u=0,\ \Omega_e, \tag{3.4}$$

$$\Delta^2 v=0,\ \Omega_e. \tag{3.5}$$

By the Fourier series, functions u, v, p can be obtained with boundary values

$u(R,\theta)$, $v(R,\theta)$, by a method similar to the discussion for Stokes equations[6]. u, v can be written as

$$u=(r^2-R^2)W_1+G_1,$$
$$v=(r^2-R^2)W_2+G_2,$$

where W_1, G_1, W_2, G_2 are harmonic functions to be determined and

$$G_1|_{\Gamma_e}=u(R,\theta)\equiv g_1(\theta),\quad G_2|_{\Gamma_e}=v(R,\theta)\equiv g_2(\theta).$$

Hence G_1, G_2 satisfy

$$\begin{cases}\Delta G_i=0, & \Omega_e,\\ G_i|_{\Gamma_e}=g_i(\theta), \\ G_i \text{ is bounded, when } r\to\infty, & i=1,2.\end{cases}$$

$G_i(i=1, 2)$ have the following expansion

$$G_1(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)r^{-n},$$

$$G_2(r,\theta)=\frac{c_0}{2}+\sum_{n=1}^{\infty}(c_n\cos n\theta+d_n\sin n\theta)r^{-n},$$

where

$$\begin{cases}a_n=\dfrac{R^n}{\pi}\displaystyle\int_0^{2\pi}g_1(\theta)\cos n\theta\,d\theta,\\ b_n=\dfrac{R^n}{\pi}\displaystyle\int_0^{2\pi}g_1(\theta)\sin n\theta\,d\theta,\\ c_n=\dfrac{R^n}{\pi}\displaystyle\int_0^{2\pi}g_2(\theta)\cos n\theta\,d\theta,\\ d_n=\dfrac{R^n}{\pi}\displaystyle\int_0^{2\pi}g_2(\theta)\sin n\theta\,d\theta, & n=0,1,2,\cdots.\end{cases}$$

Function p also has the expansion

$$p(r,\theta)=\sum_{n=2}^{\infty}(p_{1n}\cos n\theta+p_{2n}\sin n\theta)r^{-n},$$

where $\{p_{1n}, p_{2n}: n=2, 3, \cdots\}$ are constants to be determined. By equalities

$$\Delta u=\Delta[(r^2-R^2)W_1]=4\frac{\partial}{\partial r}(rW_1),$$

$$\Delta v=\Delta[(r^2-R^2)W_2]=4\frac{\partial}{\partial r}(rW_2),$$

we have

$$\frac{\partial}{\partial r}(rW_1)=\frac{1}{4}\frac{\partial p}{\partial x},$$
$$\frac{\partial}{\partial r}(rW_2)=\frac{1}{4}\frac{\partial p}{\partial y},$$

and

$$W_1=\frac{1}{4}\sum_{n=2}^{\infty}\{p_{1n}\cos(n+1)\theta+p_{2n}\sin(n+1)\theta\}r^{-n-1},$$

$$W_2=\frac{1}{4}\sum_{n=2}^{\infty}\{p_{1n}\sin(n+1)\theta-p_{2n}\cos(n+1)\theta\}r^{-n-1}.$$

On the other hand, we have

$$-\varkappa p=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=2xW_1+2yW_2+(r^2-R^2)\left(\frac{\partial W_1}{\partial x}+\frac{\partial W_2}{\partial y}\right)+\frac{\partial G_1}{\partial x}+\frac{\partial G_2}{\partial y}.$$

On the boundary Γ_e, we obtain

$$(2xW_1+2yW_2)|_{\Gamma_e}+\varkappa p|_{\Gamma_e}=-\left(\frac{\partial G_1}{\partial x}+\frac{\partial G_2}{\partial y}\right)\Big|_{\Gamma_e}.$$

A computation shows

$$2xW_1+2yW_2=\frac{1}{2}p.$$

Therefore

$$\left(\frac{1}{2}+\varkappa\right)p\Big|_{\Gamma_e}=-\left(\frac{\partial G_1}{\partial x}+\frac{\partial G_2}{\partial y}\right)\Big|_{\Gamma_e},$$

and

$$\begin{cases}\left(\frac{1}{2}+\varkappa\right)p_{1n}=(n-1)\left[a_{n-1}-d_{n-1}\right],\\ \left(\frac{1}{2}+\varkappa\right)p_{2n}=(n-1)\left[b_{n-1}+c_{n-1}\right], \quad n=2, 3, \cdots.\end{cases}$$

Finally, we obtain the boundary conditions on Γ_e

$$\begin{aligned}\left(-\frac{\partial u}{\partial r}+p\cos\theta\right)\Big|_{\Gamma_e}&=-\frac{1}{2\varkappa+1}\left[(2\varkappa+2)\frac{\partial G_1}{\partial r}\Big|_{\Gamma_e}+\frac{1}{R}\frac{\partial G_2(R,\theta)}{\partial\theta}\right]\\ &=\frac{1}{2\varkappa+1}\left[\frac{(2\varkappa+2)}{\pi R}\sum_{n=1}^{\infty}\int_0^{2\pi}\frac{\partial^2u(R,\varphi)}{\partial\varphi^2}\frac{\cos n(\theta-\varphi)}{n}d\varphi+\frac{1}{R}\frac{\partial v(R,\theta)}{\partial\theta}\right],\\ \left(-\frac{\partial v}{\partial r}+p\sin\theta\right)\Big|_{\Gamma_e}&=-\frac{1}{2\varkappa+1}\left[(2\varkappa+2)\frac{\partial G_2}{\partial r}\Big|_{\Gamma_e}-\frac{1}{R}\frac{\partial G_1(R,\theta)}{\partial\theta}\right]\\ &=-\frac{1}{2\varkappa+1}\left[\frac{(2\varkappa+2)}{\pi R}\sum_{n=1}^{\infty}\int_0^{2\pi}\frac{\partial^2v(R,\varphi)}{\partial\varphi^2}\frac{\cos n(\theta-\varphi)}{n}d\varphi\right.\\ &\qquad\left.-\frac{1}{R}\frac{\partial u(R,\theta)}{\partial\theta}\right].\end{aligned}$$

On the other hand, $\frac{\partial u(R,\theta)}{\partial\theta}$ and $\frac{\partial v(R,\theta)}{\partial\theta}$ have the following expansions

$$\frac{\partial u(R,\theta)}{\partial\theta}=\frac{1}{\pi}\sum_{n=1}^{\infty}n\int_0^{2\pi}u(R,\varphi)\sin n(\varphi-\theta)d\varphi,$$

$$\frac{\partial v(R,\theta)}{\partial\theta}=\frac{1}{\pi}\sum_{n=1}^{\infty}n\int_0^{2\pi}v(R,\varphi)\sin n(\varphi-\theta)d\varphi.$$

Hence we obtain a series of approximate boundary conditions on Γ_e

$$\begin{aligned}\left(-\frac{\partial u}{\partial r}+p\cos\theta\right)\Big|_{\Gamma_e}&=-\frac{1}{2\varkappa+1}\left[\frac{2\varkappa+2}{\pi R}\sum_{n=1}^{N}\int_0^{2\pi}\frac{\partial^2u(R,\varphi)}{\partial\varphi^2}\frac{\cos n(\theta-\varphi)}{n}d\varphi\right.\\ &\qquad\left.+\frac{1}{\pi R}\sum_{n=1}^{N}n\int_0^{2\pi}v(R,\varphi)\sin n(\varphi-\theta)\,d\varphi\right]\equiv h_1^N,\\ \left(-\frac{\partial v}{\partial r}+p\sin\theta\right)\Big|_{\Gamma_e}&=-\frac{1}{2\varkappa+1}\left[\frac{2\varkappa+2}{\pi R}\sum_{n=1}^{N}\int_0^{2\pi}\frac{\partial^2v(R,\varphi)}{\partial\varphi^2}\frac{\cos n(\theta-\varphi)}{n}d\varphi\right.\\ &\qquad\left.-\frac{1}{\pi R}\sum_{n=1}^{N}n\int_0^{2\pi}u(R,\varphi)\sin n(\varphi-\theta)d\varphi\right]\equiv h_2^N.\end{aligned}$$

On the domain $\Omega_i=\Omega_\infty\backslash\Omega_e$, consider the approximate problem

$$\begin{cases} \Delta u-\dfrac{\partial p}{\partial x}=0, \ \Omega_i, \\ \Delta v-\dfrac{\partial p}{\partial y}=0, \ \Omega_i, \\ \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}=-\varkappa p, \ \Omega_i, \\ u|_{\Gamma_i}=f_1, \\ v|_{\Gamma_i}=f_2, \\ \left(-\dfrac{\partial u}{\partial r}+p\cos\theta\right)\Big|_{\Gamma_e}=h_1^N, \\ \left(-\dfrac{\partial v}{\partial r}+p\sin\theta\right)\Big|_{\Gamma_e}=h_2^N. \end{cases} \tag{3.6}$$

For the finite element approximation of (3.6), let $\mathscr{T}_h$ be a triangulation of domain Ω_i as in section 2. Approximate u and v by piecewise linear functions (function p disappears in the variational formulation). Then the same error estimation can be obtained as in Theorem 2.5.

§ 4. Numerical Example

Consider the following example:

$$\begin{cases} \Delta u=0, \ \Omega, \\ u|_{x=1}=u|_{x=-1}=\ln\dfrac{1}{\sqrt{1+(y-0.5)^2}}-\ln\dfrac{1}{\sqrt{1+(y+0.5)^2}}, \\ u|_{y=1}=\ln\dfrac{1}{\sqrt{x^2+0.5^2}}-\ln\dfrac{1}{\sqrt{x^2+1.5^2}}, \\ u|_{y=-1}=\ln\dfrac{1}{\sqrt{x^2+1.5^2}}-\ln\dfrac{1}{\sqrt{x^2+0.5^2}}, \\ u \text{ is bounded, when } r\to\infty, \end{cases} \tag{4.1}$$

where $\Omega=\{(x, y)\in\Omega, 1<|x| \text{ or } 1<|y|\}$ is the exterior domain of square $[-1, 1]^2$. The exact solution of problem (4.1) is

$$u(x, y)=\ln\frac{1}{\sqrt{x^2+(y-0.5)^2}}-\ln\frac{1}{\sqrt{x^2+(y+0.5)^2}}. \tag{4.2}$$

We take Γ_e as a circumference with radius 2. Then we can consider the finite element approximation of u on the bounded domain $\Omega_i=\{(x, y)\in\Omega_i|(x, y)\in\Omega$ and $r<2\}$. The triangulation of Ω_i is shown in Fig. 2; it is denoted by triangulation I. In this case $h=0.57$. Then the mesh is refined by dividing every triangle into four smaller triangles; it is denoted by triangulation II and $h=0.285$. Refine it again and the final mesh is denoted by triangulation III. In this case $h=0.1425$.

The relative errors $\left(\dfrac{|u-u_h^N|}{|u|}\right)$ are given in Figs. 3—6 for $N=1, 2, 3, 4, 5$. As shown, it is sufficient to take $N=5$. The maximum relative error is 0.0162 for I, 0.0076 for II, and 0.0028 for III. Figs. 7—10 show the effect of N on the approximation. As shown, when $N=5$ the error concerning N in Theorem 2.5 can be

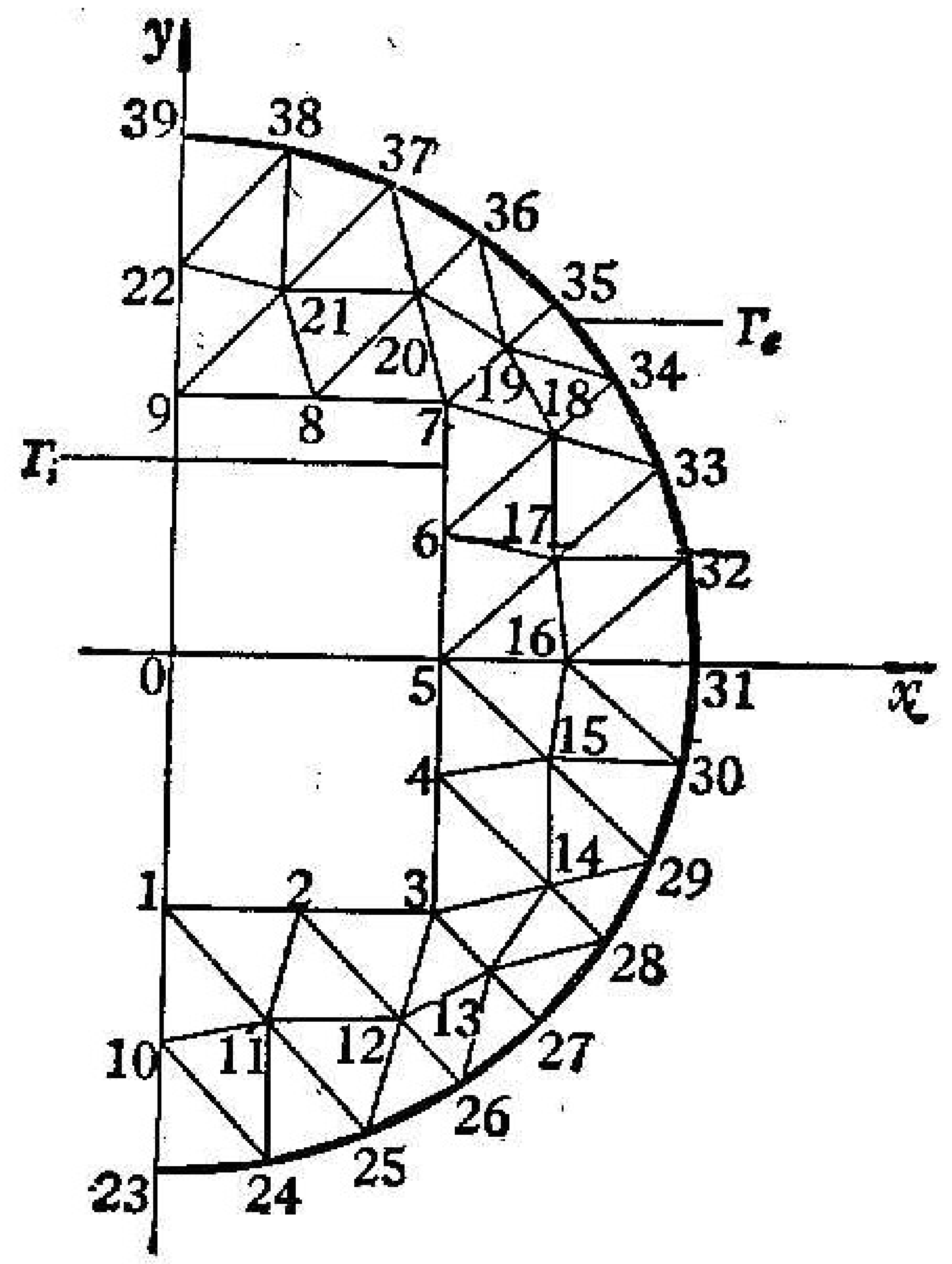

Fig. 2

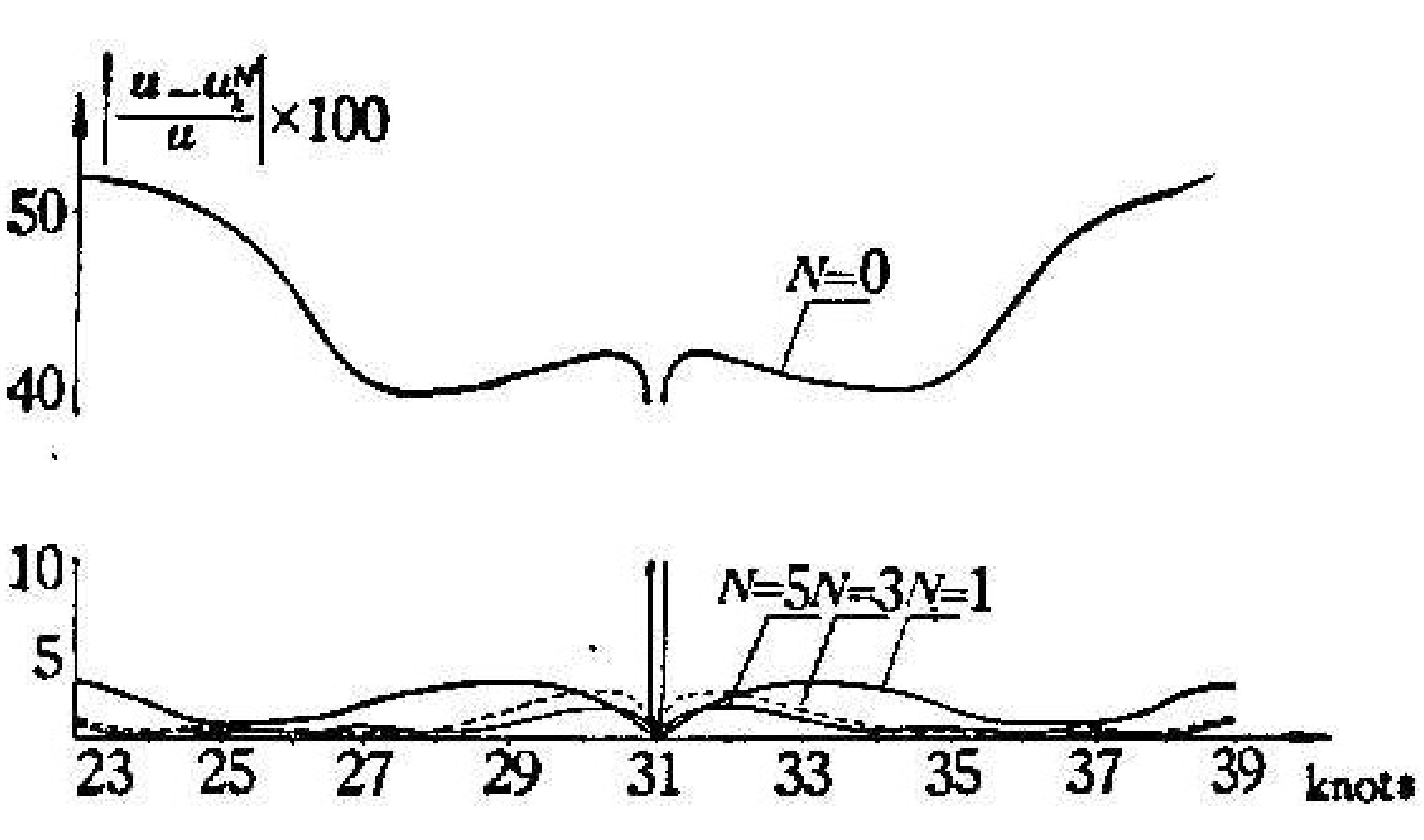

Fig. 3 $h=0.57$ boundary points

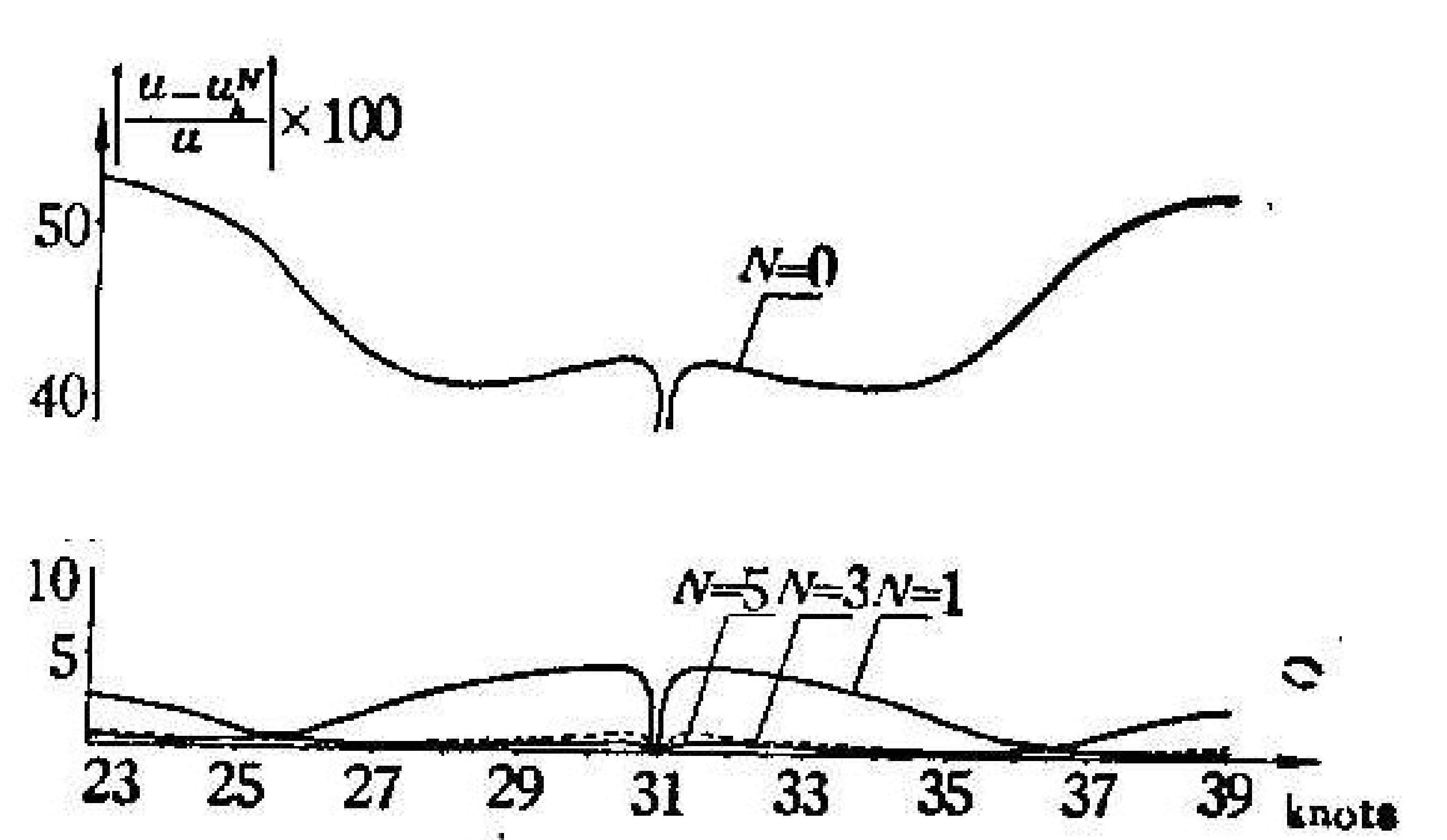

Fig. 4 $h=0.285$ boundary points

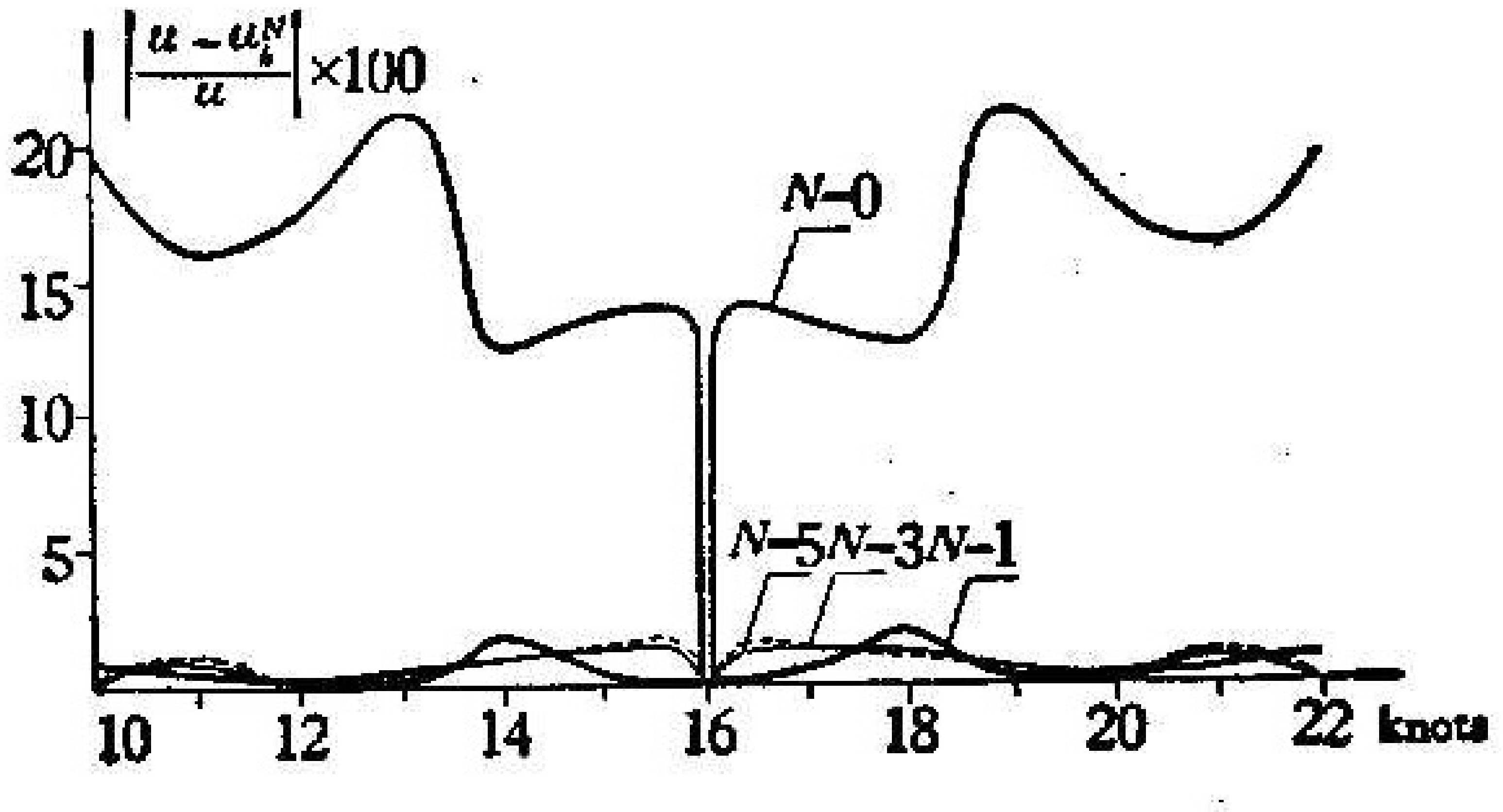

Fig. 5 $h=0.57$ interior points

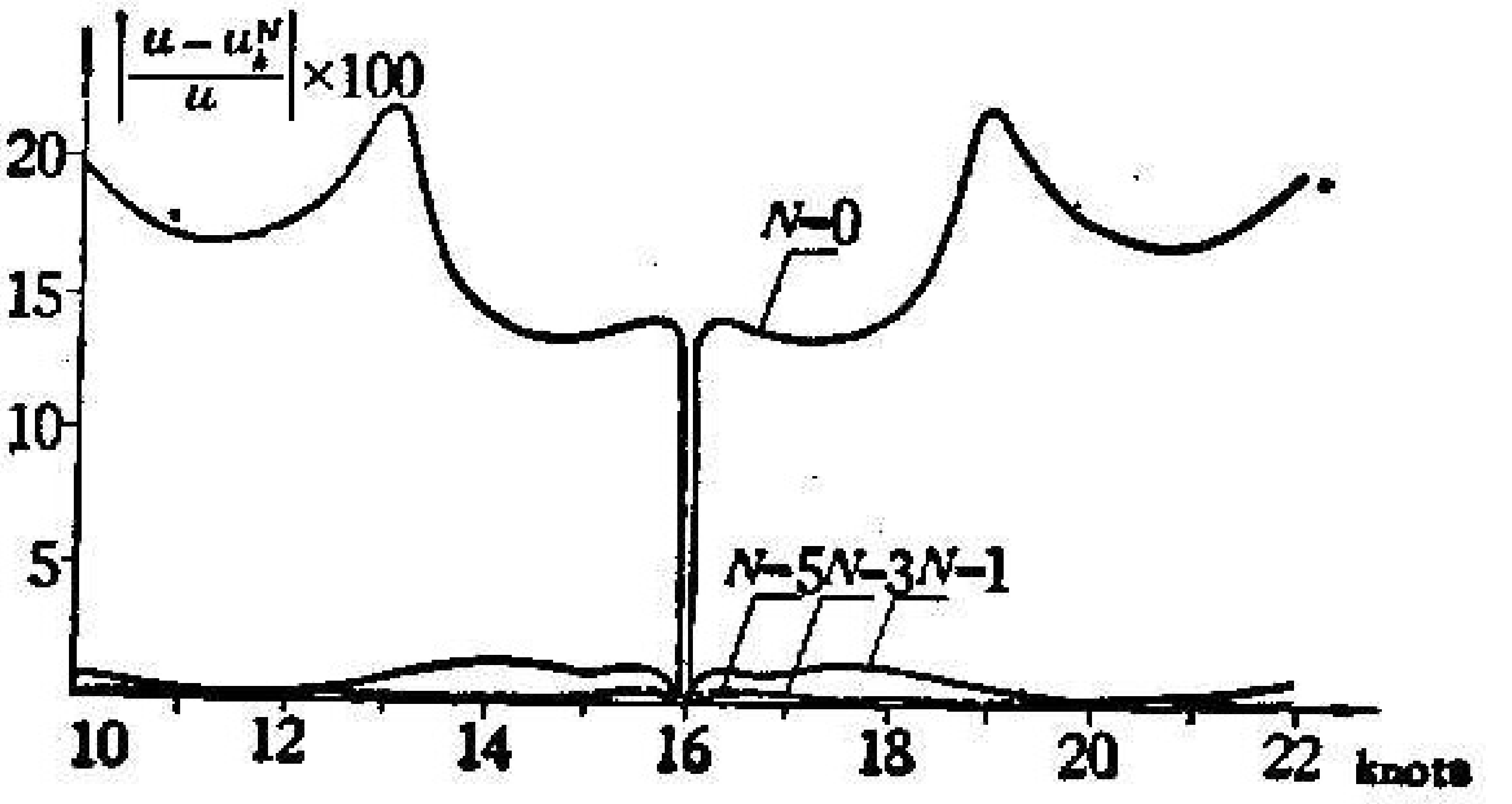

Fig. 6 $h=0.285$ interior points

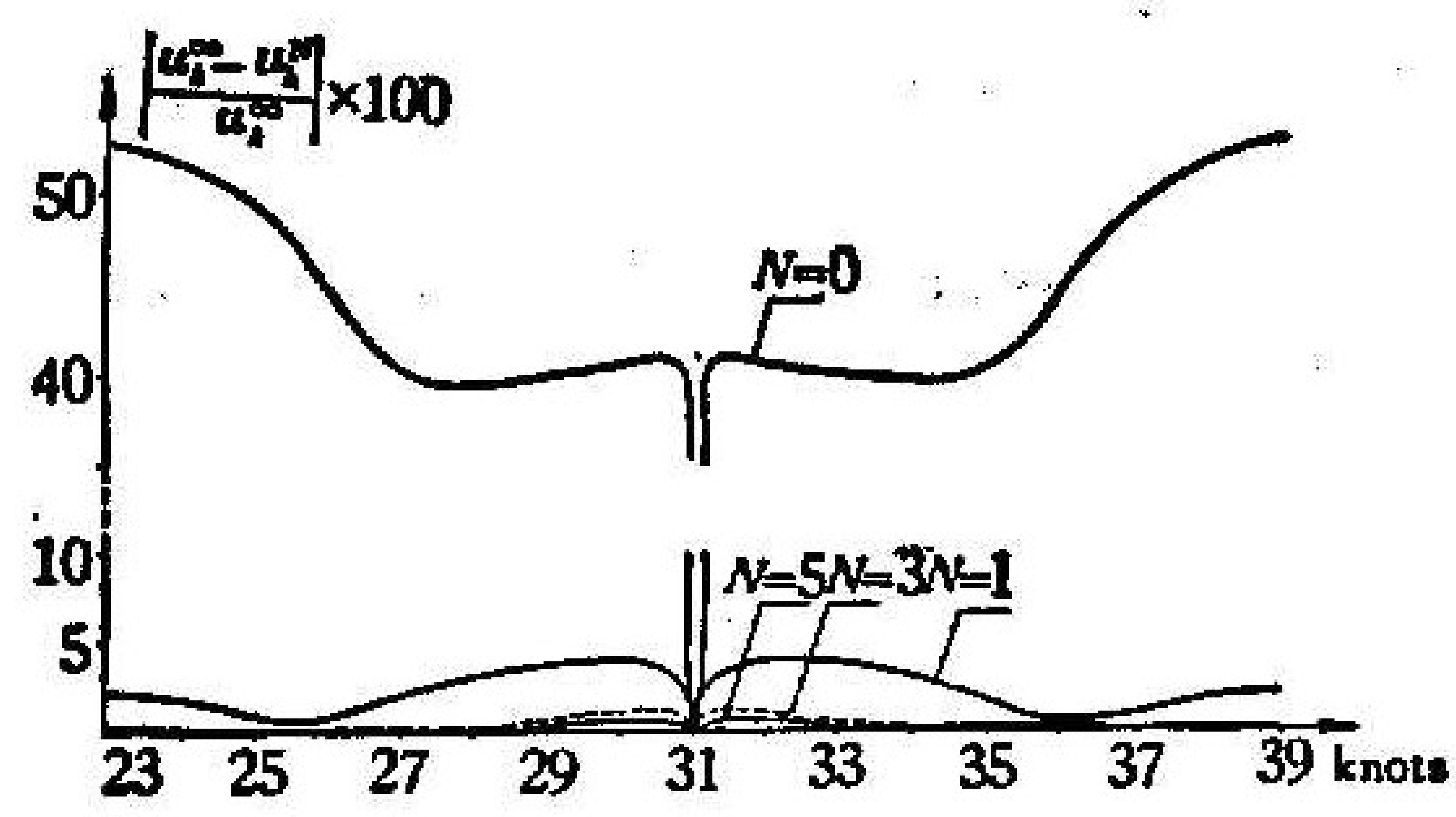

Fig. 7 $h=0.57$ boundary points

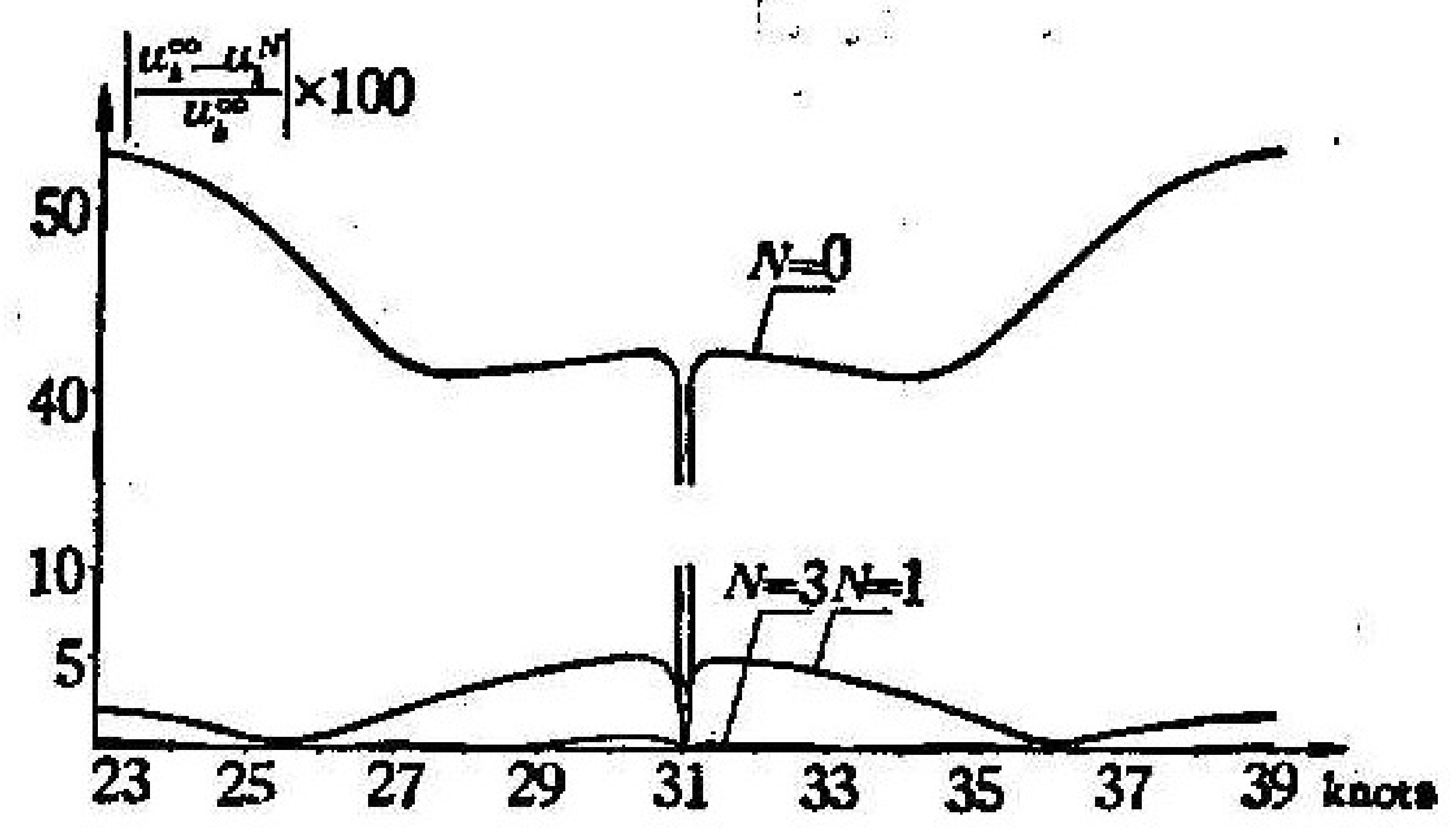

Fig. 8 $h=0.285$ boundary points

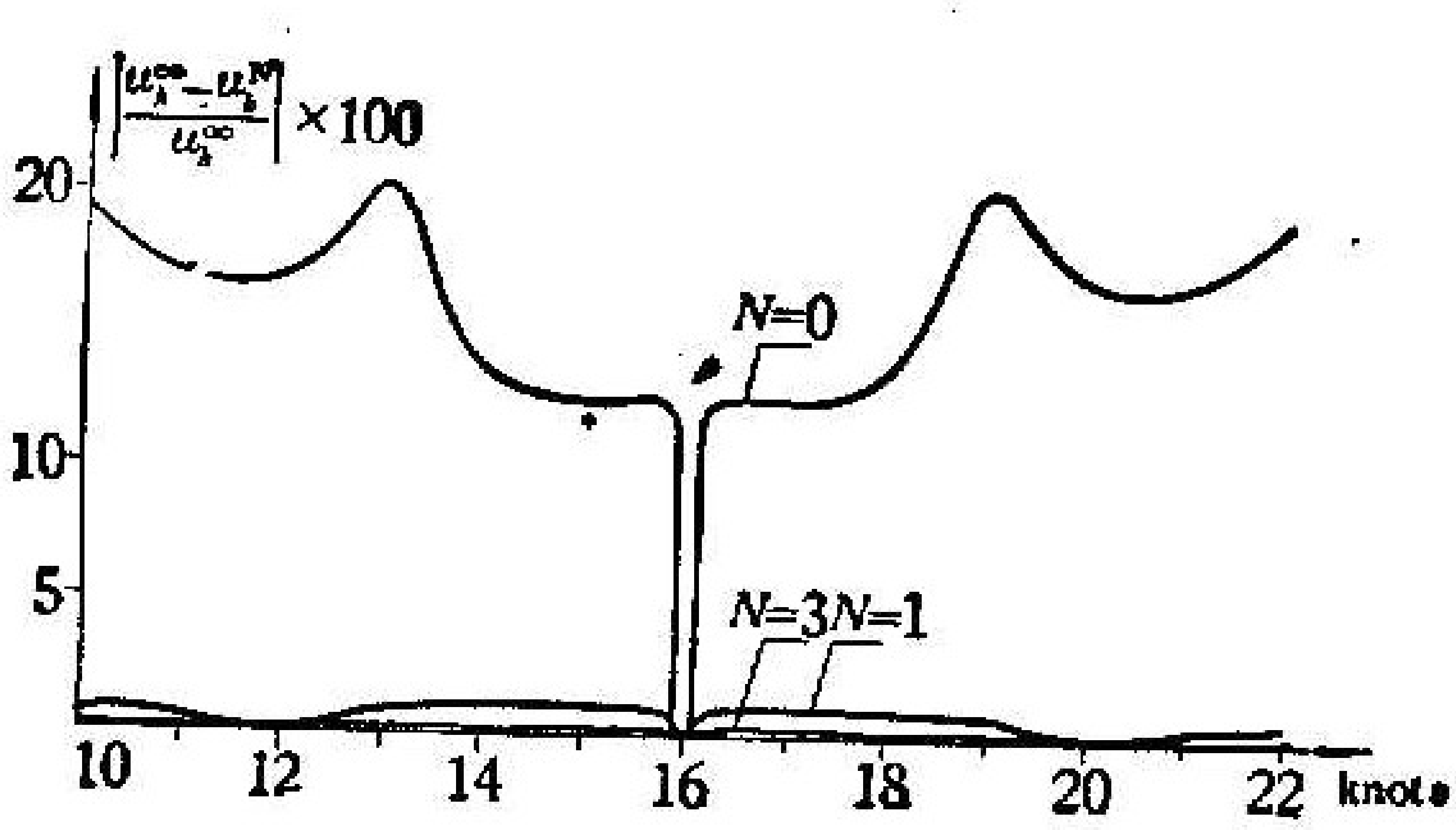

Fig. 9 $h=0.57$ interior points

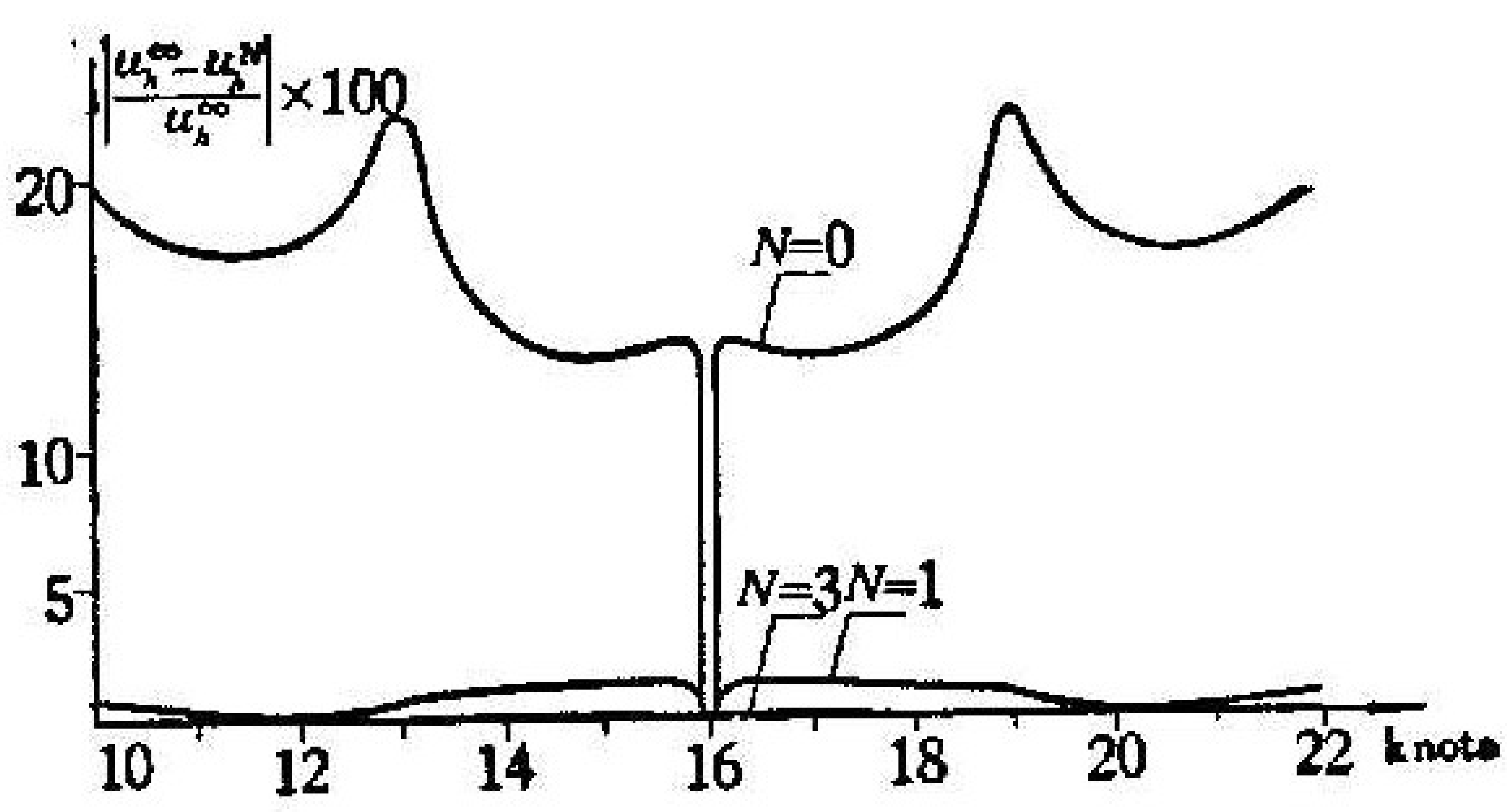

Fig. 10 $h=0.285$ interior points

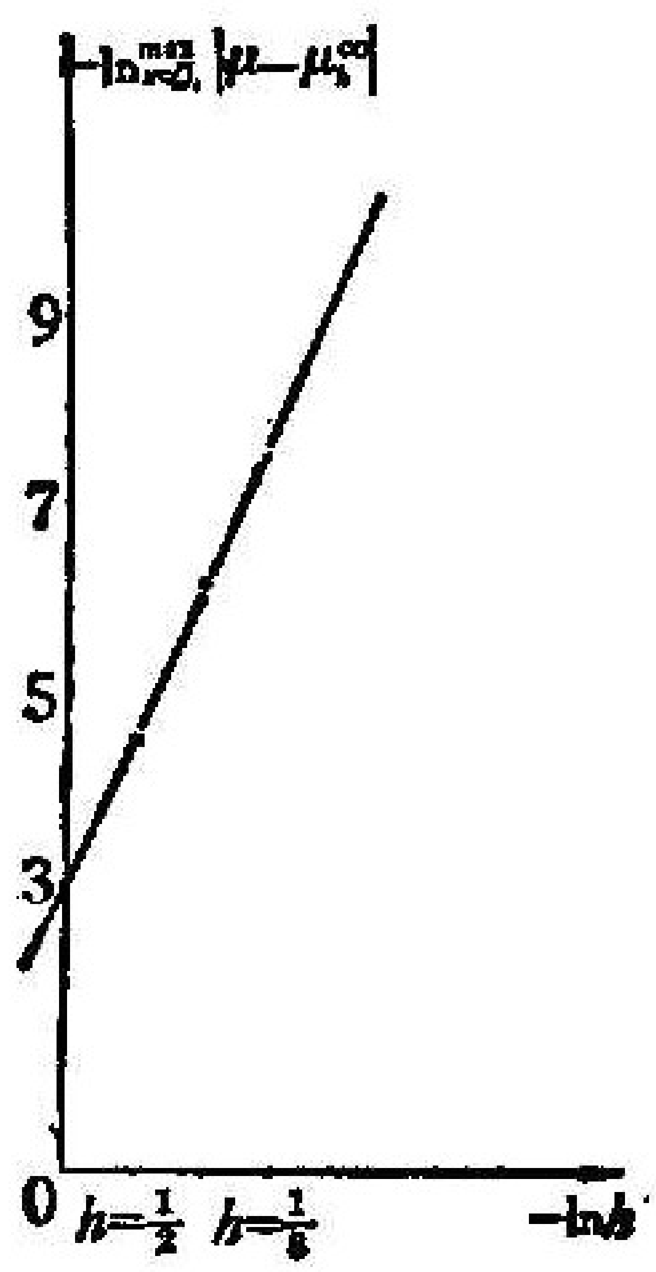

Fig. 11

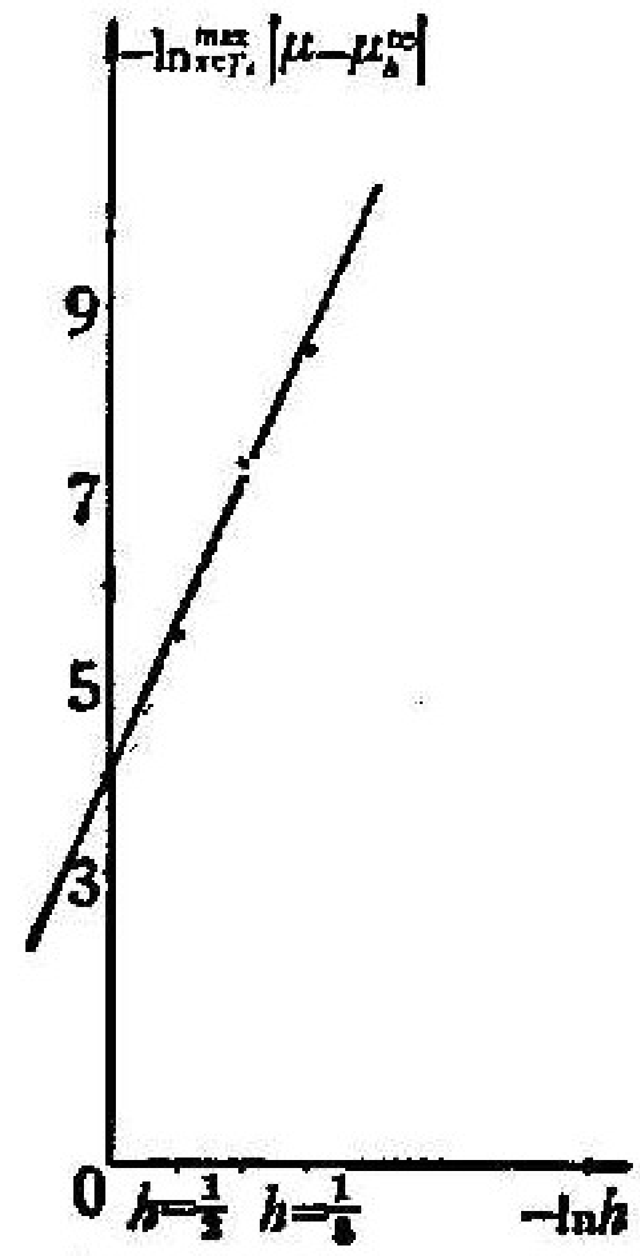

Fig. 12

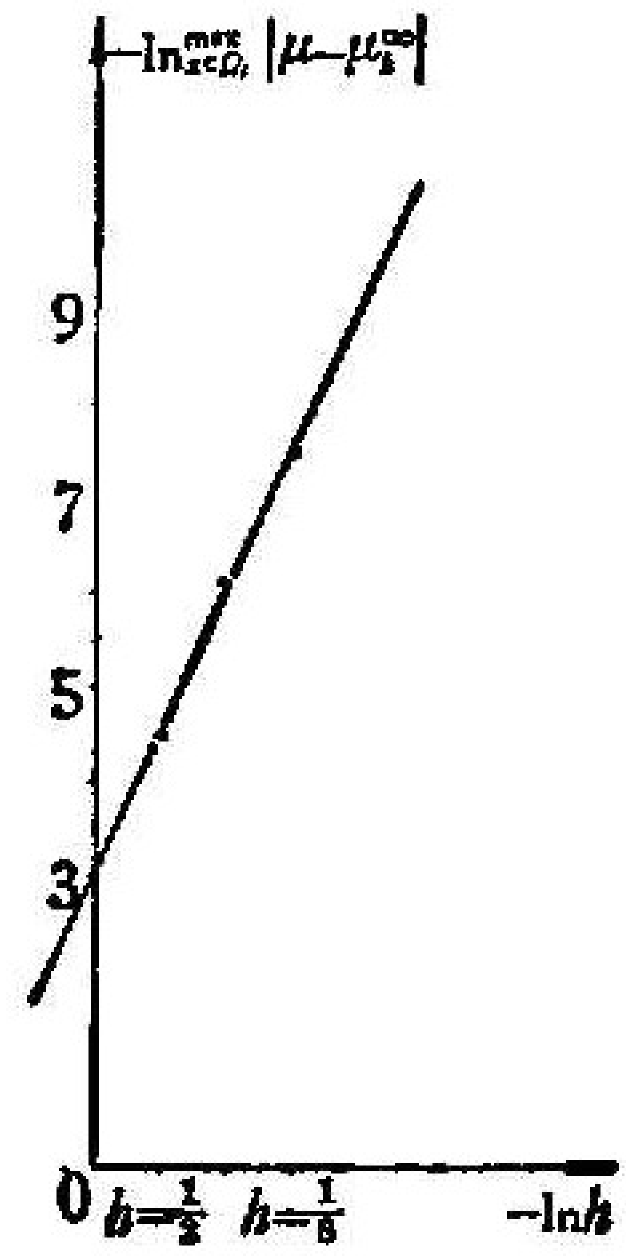

Fig. 13

neglected. Figs. 11—13 show the error in L_∞ norm.

Since there is no singularity in the integral, we need not do any special treatment in the computation. The numerical example shows that this method is very effective in solving the boundary value problem on an unbounded domain.

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