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ON ERROR ESTIMATE OF SPACE-TIME FINITE ELEMENT METHODS FOR PARABOLIC EQUATIONS IN A TIME-DEPENDENT DOMAIN*

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Abstract

We consider linear parabolic equations in a space-time domain with curved boundaries and nonhomogeneous Dirichlet boundary conditions and discuss their approximations with isoparametric space-time finite elements. A general error estimate is proved and applied to some elements of practical interest.

1. Introduction

The front-tracking methods using space-time finite elements are very effective in solving moving boundary problems, as shown by numerical experiments [Bonnerot and Jamet (1974, 1977, 1979, 1981); Li (1982, 1983)]. During the solution process, the original problem is reduced to two coupled problems: determination of the position of the moving boundary and solution of the parabolic equation in a known space-time domain with curved boundaries.

As a first step towards the complete error analysis of the front-tracking method, we want to obtain the error estimate of the approximation of the parabolic equation in a known curved space-time domain. Jamet (1978) considered the case with polygonal boundaries and homogeneous Dirichlet boundary conditions. In solving the moving boundary problems, however, it is not an appropriate approach to transform a problem with nonhomogeneous boundary data into a problem with homogeneous boundary data before discretization, since the position of the moving boundary is not known a priori, and such a transformation will greatly complicate the problem. Moreover, in most cases, the moving boundary is not polygonal. It is, therefore, necessary to consider the direct discretization of the parabolic equation in a curved space-time domain with nonhomogeneous boundary conditions. Such a discretization method will be described in section 2 of this paper. A general error estimate of the approximation will be proved in section 3. It is then applied to some finite elements of practical interest in section 4. We will follow Jamet's technique in the proof of the general error estimate. The results of this paper are extensions of his results.

2. Description of the Discretization Method

Consider a time interval $[0, T]$. Let $\Omega(t)$ be a bounded domain in R^m and

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$\Gamma(t)$ be its boundary. Let $R^T=\{(P, t); P\in\Omega(t), 0<t\leqslant T\}$ be a space-time domain. $\Sigma^T=\{(P, t); P\in\Gamma(t), 0\leqslant t\leqslant T\}$ is its lateral boundary.

We consider the following problem:

$$\frac{\partial u}{\partial t}-\nabla^2 u=f \quad \text{in } R^T, \tag{2.1}$$

$$u=g \quad \text{in } \Sigma^T, \tag{2.2}$$

$$u=u^0 \quad \text{in } \Omega(0), \tag{2.3}$$

where $f\in L^2(R^T)$, $u^0\in L^2(\Omega(0))$ and g is a continuous function on Σ^T. ∇u is the gradient of u with respect to the space variables.

Let $G=G(t_1, t_2)=\{(P, t); P\in\Omega(t), 0\leqslant t_1<t<t_2\leqslant T\}$ be a subdomain of R^T and $\Phi(G)$ be the space of all Lipschitz-continuous functions defined on $\overline{G}$ which vanish on the lateral boundary of G. Then a classical solution u of (2.1)—(2.3) is also the weak solution defined by

$$B_G(u, \phi)=-\left(\left(u, \frac{\partial\phi}{\partial t}\right)\right)_G+((\nabla u, \nabla\phi))_G+(u, \phi)_{\Omega(t_2)}-(u, \phi)_{\Omega(t_1)}=((f, \phi))_G \tag{2.4}$$

for all $\phi\in\Phi(G)$ and for all $0\leqslant t_1<t_2\leqslant T$, together with the initial-boundary conditions (2.2)—(2.3).

Here we have used the notations

$\Omega(t)=$section $\{(P, t); p\in\Omega(t)\}$, $(\cdot,\cdot)_{\Omega(t)}=$inner product in $L^2(\Omega(t))$,

$((\cdot, \cdot))_G=$inner product in $L^2(G)$, $((\nabla u, \nabla\phi))_G=\iint_G \nabla u\cdot\nabla\phi \, d\underset{\sim}{x}\, dt$.

To define the approximate solution, we consider a subdivision of $[0, T]$: $0=t^0<t^1<\cdots<t^n<\cdots<t^N=T$. Let Σ_h^T be a continuous and piecewise smooth approximation to Σ^T. Let G_h^n be the space-time domain bounded by Σ_h^T and the hyperplanes $t=t^n$ and $t=t^{n+1}$. Let $R_h^T=\bigcup_{n=0}^{N-1}\overline{G}_h^n$. We assume that there exists a bounded domain $\tilde{R}^T$ such that $\tilde{R}^T\supset R^T$ and $\tilde{R}^T\supset R_h^T$ for all small enough values of h, the discretization parameter, and for all subdivisions of $[0, T]$. Assume also that the functions f, u^0 and the exact solution u have smooth enough extensions $\tilde{f}$, $\tilde{u}^0$ and $\tilde{u}$ to $\tilde{R}^T$ which also satisfy (2.1) and (2.3).

Let $\Omega_h^n=\Omega_h(t^n)$ be the section of R_h^T on the hyperplane $t=t^n$ and $\underset{\sim}{G}_h^n=\overline{G}_h^n-\overline{\Omega}_h^n$. Let Φ_h^n be a finite dimensional subspace of $\Phi(G_h^n)$, $1\leqslant n\leqslant N-1$, and V_h be the space of all functions v_h defined on R_h^T such that their restriction to each $\underset{\sim}{G}_h^n$ coincides with the restriction of a function $q^{(n)}v_h\in\Phi_h^n$ to $\underset{\sim}{G}_h^n$. Let also $U_h=w_h+V_h$ where w_h is a given function defined to R_h^T, which is Lipschitz-continuous on each $\underset{\sim}{G}_h^n$ and whose restriction to Σ_h^T is an appropriate approximation of the restriction of $\tilde{u}$ to Σ_h^T. Note that the functions $v_h\in V_h$ and $u_h\in U_h$ are in general discontinuous at time $t=t_n$, $0\leqslant n\leqslant N-1$. We denote $v_h(\cdot, t^n)$ and $v_h(\cdot, t^{n+0})$ by v_h^n and v_h^{n+0} respectively.

Now we can define the discrete problem as follows:

Find $u_h\in U_h$ such that $u_h^0=\tilde{u}^0|_{\Omega_h^0}$ and

$$B_{G_h^n}(u_h, \phi_h)=((\tilde{f}, \phi_h))_{G_h^n}, \tag{2.5}$$

for all $\phi_h\in\Phi_h^n$ and for all $0\leqslant n\leqslant N-1$.

The uniqueness of the solution of the discrete problem (2.5) can be easily proved. Suppose (2.5) has two solutions $u_{h,1}$ and $u_{h,2}$. Then $u_{h,1}-u_{h,2}=\mathring{u}_h\in V_h$ and $B_{G\xi}(\mathring{u}_h, \phi_h)=0$. Using Theorem 3.1 in Jamet (1978), we obtain $\mathring{u}_h=0$.

The existence of the solution u_h is a consequence of its uniqueness.

3. General Error Estimate

First, we prove two lemmas.

Lemma 3.1. *Let $\{a^n\}$, $\{b^n\}$, $\{\gamma^n\}$ be three sequences of nonnegative real numbers and α, β, δ be nonnegative real constants. Suppose the inequality*

$$(a^n)^2+(b^n)^2\leqslant\alpha a^n+\beta b^n+\sum_{\nu=1}^{n-1}\gamma^\nu b^\nu+\delta^2 \tag{3.1}$$

is satisfied for all $n\geqslant1$. Note that when $n=1$, the third term of the right-hand side of (3.1) is set zero. Then

$$a^n+b^n\leqslant\sqrt{2}\Big(\alpha+\beta+\sum_{\nu=1}^{n-1}\gamma^\nu+\delta\Big). \tag{3.2}$$

Note. This lemma is an extension of Lemma 4.1 in Jamet (1978).

Proof. Let $c^n=((a^n)^2+(b^n)^2)^{\frac{1}{2}}$. Then (3.1) yields

$$(c^n)^2\leqslant(\alpha+\beta)c^n+\sum_{\nu=1}^{n-1}\gamma^\nu c^\nu+\delta^2. \tag{3.3}$$

Consider a nonnegative sequence $\{d^n\}$ that satisfies

$$(d^n)^2=(\alpha+\beta)d^n+\sum_{\nu=1}^{n-1}\gamma^\nu d^\nu+\delta^2. \tag{3.4}$$

By (3.4), we have $d^n\geqslant\alpha+\beta$ for all n.

Let $g(y)=y^2-(\alpha+\beta)y$. Then $g(y)$ is increasing for $y\geqslant\alpha+\beta$. It is obvious that $g(d^n)\leqslant g(d^{n+1})$. Therefore $d^n\leqslant d^{n+1}$. This yields

$$(d^n)^2-\Big(\alpha+\beta+\sum_{\nu=1}^{n-1}\gamma^\nu\Big)d^n-\delta^2\leqslant0. \tag{3.5}$$

Let $\xi=\alpha+\beta+\sum_{\nu=1}^{n-1}\gamma^\nu$. Then we have

$$d^n\leqslant(\xi+\sqrt{\xi^2+4\delta^2})/2\leqslant\xi+\delta.$$

Since $g(c^1)\leqslant g(d^1)=\delta^2$, we have $c^1\leqslant d^1$. Next, suppose $c^\nu\leqslant d^\nu$ for $1\leqslant\nu\leqslant n-1$. This implies

$$g(c^n)\leqslant g(d^n) \text{ and thus } c^n\leqslant d^n.$$

By induction we conclude that

$$c^n\leqslant d^n\leqslant\xi+\delta \quad \text{for all } n.$$

Therefore, $$a^n+b^n\leqslant\sqrt{2}\,c^n\leqslant\sqrt{2}\Big(\alpha+\beta+\sum_{\nu=1}^{n-1}\gamma^\nu+\delta\Big).$$ Q. E. D.

Lemma 3.2. *Let Ω be a bounded domain in R^m with Lipschitz-continuous boundary Γ. Then*

$$\forall u\in H^1(\Omega),\ |u|_{0,\Omega}\leqslant\frac{d}{\sqrt{m}}|\nabla u|_{0,\Omega}+\sqrt{d/2}\,|u|_{0,\Gamma}, \tag{3.6}$$

where $d=\operatorname{diam}\Omega$, $|\cdot|_{0,\Omega}=$norm *in* $L^2(\Omega)$, *and* $|\nabla u|_{0,\Omega}=\left\{\int_\Omega(\nabla u)^2 d\underset{\sim}{x}\right\}^{\frac{1}{2}}$.

Note. The usual Poincaré inequality is applied to $H_0^1(\Omega)$. This lemma is its extension to $H^1(\Omega)$.

Proof. Without loss of generality, we consider only the case that Ω is contained in the hypercube $[-K, K]^m$ with $2K\leqslant d$. Since

$$\nabla\cdot(\underset{\sim}{x}u^2)=mu^2+2u\underset{\sim}{x}\cdot\nabla u,$$

by Green's theorem we obtain

$$m\int_\Omega u^2 d\underset{\sim}{x}=-2\int_\Omega u\underset{\sim}{x}\cdot\nabla u\,d\underset{\sim}{x}+\int_\Gamma u^2\underset{\sim}{x}\cdot\underset{\sim}{\nu}\,dS. \tag{3.7}$$

This yields

$$m|u|_{0,\Omega}^2\leqslant 2\left|\int_\Omega u^2 d\underset{\sim}{x}\right|^{\frac{1}{2}}\cdot\left|\int_\Omega(\underset{\sim}{x}\cdot\nabla u)^2 d\underset{\sim}{x}\right|^{\frac{1}{2}}+\int_\Omega u^2\underset{\sim}{x}\cdot\underset{\sim}{\nu}\,dS.$$

Since

$$(\underset{\sim}{x}\cdot\nabla u)^2=\left(\sum_{i=1}^m x_i\frac{\partial u}{\partial x_i}\right)^2\leqslant K^2\left(\sum_{i=1}^m\left|\frac{\partial u}{\partial x_i}\right|\right)^2\leqslant mK^2\sum_{i=1}^m\left(\frac{\partial u}{\partial x_i}\right)^2=mK^2(\nabla u)^2$$

and

$$|\underset{\sim}{x}\cdot\underset{\sim}{\nu}|=|\sum_{i=1}^m x_i\nu_i|\leqslant mK,$$

We have

$$m|u|_{0,\Omega}^2\leqslant 2K\sqrt{m}\,|\nabla u|_{0,\Omega}\cdot|u|_{0,\Omega}+mK|u|_{0,\Gamma}^2.$$

Thus

$$|u|_{0,\Omega}^2\leqslant\frac{d}{\sqrt{m}}|\nabla u|_{0,\Omega}\cdot|u|_{0,\Omega}+\frac{d}{2}|u|_{0,\Gamma}^2. \tag{3.8}$$

Let $\alpha=\frac{d}{\sqrt{m}}|\nabla u|_{0,\Omega}$ and $\beta^2=\frac{d}{2}|u|_{0,\Gamma}^2$. Then (3.8) yields

$$|u|_{0,\Omega}\leqslant(\alpha+\sqrt{\alpha^2+4\beta^2})/2\leqslant\alpha+\beta.$$

This is exactly (3.6). Q. E. D.

Now we proceed to prove the general error estimate. We will use the following notations:

$$G_h(0,t^n)=\bigcup_{\nu=0}^{n-1}\overline{G}_h^\nu,\quad \Sigma_h^n=\Sigma_h^T\cap\partial G_h^n,\quad \Sigma_h(0,t^n)=\bigcup_{\nu=0}^{n-1}\Sigma_h^\nu,$$

$|\cdot|_{\Omega(t)}=$norm in $L^2(\Omega(t))$, $\|\cdot\|_G=$norm in $L^2(G)$, $\|\nabla u\|_G^2=((\nabla u,\nabla u))_G$,

$|\cdot|_{0,\Sigma_h(0,t^n)}=$norm in $L^2(\Sigma_h(0,t^n))$, $d=\max(\operatorname{diam}\Omega_h(t),\ 0\leqslant t\leqslant T)$.

Theorem 3.1. *Let u be the solution of problem* (2.1)—(2.3). *Assume that u,u^0, f have smooth enough extensions $\tilde{u}$, $\tilde{u}^0$, $\tilde{f}$ to $\tilde{R}^T$ which also satisfy* (2.1) *and* (2.3) *in $\tilde{R}^T$. Let u_h be the solution of the discrete problem* (2.5). *Then, for all $v_h\in V_h$ and for all $1\leqslant n\leqslant N$,*

$$\begin{aligned}&\|\nabla(\tilde{u}-u_h)\|_{G_h(0,t^n)}+\frac{1}{\sqrt{2}}|\tilde{u}^n-u_h^n|_{\Omega_h^n}\\&\leqslant\sqrt{2/m}\,d\left\{\sum_{\nu=0}^{n-1}\left\{\frac{\partial}{\partial t}(\tilde{u}-v_h)\right\}_{G_h^\nu}^2\right\}^{\frac{1}{2}}+\sqrt{2}\,\|\nabla(\tilde{u}-v_h)\|_{G_h(0,t^n)}\\&\quad+2\max_{1\leqslant\nu\leqslant n}|\tilde{u}^\nu-v_h^\nu|_{\Omega_h^\nu}+2\sum_{\nu=0}^{r-1}|v_h^{\nu+0}-v_h^\nu|_{\Omega_h^\nu}\end{aligned}$$

$$+\sqrt{2}\left\{\sqrt{d/2}\,|\tilde{u}-w_h|_{0,\Sigma_h(0,t^n)}\cdot\left(\sum_{\nu=0}^{n-1}\left\|\frac{\partial}{\partial t}(\tilde{u}-v_h)\right\|^2_{G_h^\nu}\right)^{\frac{1}{2}}+\frac{1}{2}|\tilde{u}-w_h|^2_{0,\Sigma_h(0,t^n)}\right\}^{\frac{1}{2}}. \tag{3.9}$$

Proof. By (2.4) and (2.5), we have for all $\phi_h\in\Phi_h^n$

$$B_{G_h^n}(u_h,\ \phi_h)=((\tilde{f},\ \phi_h))_{G_h^n}$$

and

$$B_{G_h^n}(\tilde{u},\ \phi_h)=((\tilde{f},\ \phi_h))_{G_h^n}.$$

By subtraction we obtain

$$\forall\phi_h\in\Phi_h^n,\quad B_{G_h^n}(\tilde{u}-u_h,\ \phi_h)=0. \tag{3.10}$$

Since $u_h\in U_h=w_h+V_h$, it can be written in the form

$$u_h=w_h+\mathring{u}_h\quad\text{with } \mathring{u}_h\in V_h. \tag{3.11}$$

Let $\widetilde{W}=\tilde{u}-w_h$. Then (3.10) yields

$$\forall\phi_h\in\Phi_h^n,\quad B_{G_h^n}(\widetilde{W}-\mathring{u}_h,\ \phi_h)=0. \tag{3.12}$$

In view of the definition of V_h, we can take in (3.12) $\phi_h=q^{(h)}v_h-q^{(n)}\mathring{u}_h$ and deduce

$$\forall v_h\in V_h,\ B_{G_h^n}(\widetilde{W}-\mathring{u}_h,\ \widetilde{W}-q^{(n)}\mathring{u}_h)=B_{G_h^n}(\widetilde{W}-\mathring{u}_h,\ \widetilde{W}-q^{(n)}v_h). \tag{3.13}$$

It is easy to check that, for any $v_h\in V_h$,

$$\begin{aligned}B_{G_h^n}(\widetilde{W}-\mathring{u}_h,\ \widetilde{W}-q^{(n)}v_h)&=-\left(\left(\widetilde{W}-\mathring{u}_h,\ \frac{\partial}{\partial t}(\widetilde{W}-v_h)\right)\right)_{G_h^n}\\&+((\nabla(\widetilde{W}-\mathring{u}_h),\ \nabla(\widetilde{W}-v_h)))_{G_h^n}+(\widetilde{W}^{n+1}-\mathring{u}_h^{n+1},\ \widetilde{W}^{n+1}-v_h^{n+1})_{\Omega_h^{n+1}}\\&-(\widetilde{W}^n-\mathring{u}_h^n,\ \widetilde{W}^n-v_h^{n+0})_{\Omega_h^n}.\end{aligned} \tag{3.14}$$

An application of Green's theorem gives

$$\begin{aligned}-\left(\left(\widetilde{W}-\mathring{u}_h,\ \frac{\partial}{\partial t}(\widetilde{W}-\mathring{u}_h)\right)\right)_{G_h^n}&=-\frac{1}{2}\iint_{G_h^n}\frac{\partial}{\partial t}(\widetilde{W}-\mathring{u}_h)^2\,d\underset{\sim}{x}\,dt\\&=-\frac{1}{2}\int_{\partial G_h^n}(\widetilde{W}-\mathring{u}_h)^2\nu_t\,d\gamma\\&=-\frac{1}{2}|\widetilde{W}^{n+1}-\mathring{u}_h^{n+1}|^2_{\Omega_h^{n+1}}+\frac{1}{2}|\widetilde{W}^{n+0}-\mathring{u}_h^{n+0}|_{\Omega_h^n}\\&-\frac{1}{2}\int_{\Sigma_h^n}\widetilde{W}^2\nu_t\,d\gamma.\end{aligned} \tag{3.15}$$

We have, also,

$$\begin{aligned}(\widetilde{W}^n-\mathring{u}_h^n,\ \widetilde{W}^n-\mathring{u}_h^{n+0})_{\Omega_h^n}&=(\widetilde{W}^n-\mathring{u}_h^n,\ \widetilde{W}^{n+0}-\mathring{u}_h^{n+0})_{\Omega_h^n}-(\widetilde{W}^n-\mathring{u}_h^n,\ \widetilde{W}^{n+0}-\widetilde{W}^n)_{\Omega_h^n}\\&=\frac{1}{2}|\widetilde{W}^n-\mathring{u}_h^n|^2_{\Omega_h^n}+\frac{1}{2}|\widetilde{W}^{n+0}-\mathring{u}_h^{n+0}|^2_{\Omega_h^n}-\frac{1}{2}|(\widetilde{W}^{n+0}-\mathring{u}_h^{n+0})\\&-(\widetilde{W}^n-\mathring{u}_h^h)|^2_{\Omega_h^n}-(\widetilde{W}^n-\mathring{u}_h^n,\ \widetilde{W}^{n+0}-\widetilde{W}^n)_{\Omega_h^n}.\end{aligned} \tag{3.16}$$

Combination of (3.14)—(3.16) gives

$$\begin{aligned}B_{G_h^n}(\widetilde{W}-\mathring{v}_h,\ \widetilde{W}-q^{(n)}\mathring{u}_h)&=\|\nabla(\widetilde{W}-\mathring{u}_h)\|^2_{G_h^n}+\frac{1}{2}|\widetilde{W}^{n+1}-\mathring{u}_h^{n+1}|^2_{\Omega_h^{n+1}}-\frac{1}{2}|\widetilde{W}^n-\mathring{u}_h^n|^2_{\Omega_h^n}\\&+\frac{1}{2}|(\widetilde{W}^{n+0}-\mathring{u}_h^{n+0})-(\widetilde{W}^n-\mathring{u}_h^n)|^2_{\Omega_h^n}+(\widetilde{W}^n-\mathring{u}_h^n,\ \widetilde{W}^{n+0}-\widetilde{W}^n)_{\Omega_h^n}-\frac{1}{2}\int_{\Sigma_h^n}\widetilde{W}^2\nu_t\,d\gamma.\end{aligned} \tag{3.17}$$

On the other hand, (3.14) yields

$$B_{G_h^n}(\widetilde{W}-\mathring{u}_h,\ \widetilde{W}-q^{(n)}v_h)=-\Big(\Big(\widetilde{W}-\mathring{u}_h,\ \frac{\partial}{\partial t}(\widetilde{W}-v_h)\Big)\Big)_{G_h^n}+((\nabla(\widetilde{W}-\mathring{u}_h),\nabla(\widetilde{W}-v_h)))_{G_h^n}$$
$$+(\widetilde{W}^{n+1}-\mathring{u}_h^{n+1},\ \widetilde{W}^{n+1}-v_h^{n+1})_{\Omega_h^{n+1}}-(\widetilde{W}^n-\mathring{u}_h,\ \widetilde{W}^{n+0}-v_h^{n+0})_{\Omega_h^n}$$
$$+(\widetilde{W}^n-\mathring{u}_h,\ \widetilde{W}^{n+0}-\widetilde{W}^n)_{\Omega_h^n}. \tag{3.18}$$

Combining (3.13), (3.17) and (3.18), we obtain that for all $v_h\in V_h$,

$$\|\nabla(\widetilde{W}-\mathring{u}_h)\|_{G_h^n}^2+\frac{1}{2}|\widetilde{W}^{n+1}-\mathring{u}_h^{n+1}|_{\Omega_h^{n+1}}^2-\frac{1}{2}|\widetilde{W}^n-\mathring{u}_h^n|_{\Omega_h^n}^2$$
$$+\frac{1}{2}|(\widetilde{W}^{n+0}-\mathring{u}_h^{n+0})-(\widetilde{W}^n-\mathring{u}_h^n)|_{\Omega_h^n}^2-\frac{1}{2}\int_{\Sigma_h^n}\widetilde{W}^2\nu_t\,d\gamma$$
$$=-\Big(\Big(\widetilde{W}-\mathring{u}_h,\ \frac{\partial}{\partial t}(\widetilde{W}-v_h)\Big)\Big)_{G_h^n}$$
$$+((\nabla(\widetilde{W}-\mathring{u}_h),\ \nabla(\widetilde{W}-v_h)))_{G_h^n}+(\widetilde{W}^{n+1}-\mathring{u}_h^{n+1},\widetilde{W}^{n+1}-v_h^{n+1})_{\Omega_h^{n+1}}$$
$$-(\widetilde{W}^n-\mathring{u}_h^n,\ \widetilde{W}^n-v_h^n)_{\Omega_h^n}-(\widetilde{W}^n-\mathring{u}_h^n,\ (\widetilde{W}^{n+0}-v_h^{n+0})-(\widetilde{W}^n-v_h^n))_{\Omega_h^n}. \tag{3.19}$$

But $\widetilde{W}-\mathring{u}_h=\widetilde{u}-u_h$, and $\widetilde{W}-v_h=\widetilde{u}-(w_h+v_h)$ with $w_h+v_h\in U_h$. Moreover, $\widetilde{u}^{n+0}=\widetilde{u}^n$, $\widetilde{W}|_{\Sigma_h^n}=(\widetilde{u}-w_h)|_{\Sigma_h^n}$. From (3.19) we deduce that $\forall v_h\in U_h$,

$$\|\nabla(\widetilde{u}-u_h)\|_{G_h^n}^2+\frac{1}{2}|\widetilde{u}^{n+1}-u_h^{n+1}|_{\Omega_h^{n+1}}^2-\frac{1}{2}|\widetilde{u}^n-u_h^n|_{\Omega_h^n}^2+\frac{1}{2}|u_h^{n+0}-u_h^n|_{\Omega_h^n}^2$$
$$=-\Big(\Big(\widetilde{u}-u_h,\frac{\partial}{\partial t}(\widetilde{u}-v_h)\Big)\Big)_{G_h^n}+((\nabla(\widetilde{u}-u_h),\ \nabla(\widetilde{u}-v_h)))_{G_h^n}$$
$$+(\widetilde{u}^{n+1}-u_h^{n+1},\ \widetilde{u}^{n+1}-v_h^{n+1})_{\Omega_h^{n+1}}-(\widetilde{u}^n-u_h^n,\ \widetilde{u}^n-v_h^n)_{\Omega_h^n}$$
$$+(\widetilde{u}^n-u_h^n,\ v_h^{n+0}-v_h^n)_{\Omega_h^n}+\frac{1}{2}\int_{\Sigma_h^n}(\widetilde{u}-w_h)^2\nu_t\,d\gamma. \tag{3.20}$$

Summation of equalities (3.20) from $n=0$ to $n-1$ yields

$$\|\nabla(\widetilde{u}-u_h)\|_{G_h(0,t^n)}^2+\frac{1}{2}|\widetilde{u}^n-u_h^n|_{\Omega_h^n}^2+\frac{1}{2}\sum_{\nu=0}^{n-1}|u_h^{\nu+0}-u_h^\nu|_{\Omega_h^\nu}^2$$
$$=-\sum_{\nu=0}^{n-1}\Big(\Big(\widetilde{u}-u_h,\ \frac{\partial}{\partial t}(\widetilde{u}-v_h)\Big)\Big)_{G_h^\nu}+((\nabla(\widetilde{u}-u_h),\ \nabla(\widetilde{u}-v_h)))_{G_h(0,t^n)}$$
$$+(\widetilde{u}^n-u_h^n,\ \widetilde{u}^n-v_h^n)_{\Omega_h^n}+\sum_{\nu=0}^{n-1}(\widetilde{u}^\nu-u_h^\nu,\ v_h^{\nu+0}-v_h^\nu)_{\Omega_h^\nu}+\frac{1}{2}\int_{\Sigma_h(0,t^n)}(\widetilde{u}-w_h)^2\nu_t\,d\gamma \tag{3.21}$$

for all $v_h\in V_h$ and for all $1\leqslant n\leqslant N$.

Applying Lemma 3.2 and noting the fact that $|\nu_t|\leqslant 1$, we deduce

$$\|\nabla(\widetilde{u}-u_h)\|_{G_h(0,t^n)}^2+\frac{1}{2}|\widetilde{u}^n-u_h^n|_{\Omega_h^n}^2+\frac{1}{2}\sum_{\nu=0}^{n-1}|u_h^{\nu+0}-u_h^\nu|_{\Omega_h^\nu}^2$$
$$\leqslant\|\nabla(\widetilde{u}-u_h)\|_{G_h(0,t^n)}\Big\{\frac{d}{\sqrt{m}}\Big(\sum_{\nu=0}^{n-1}\Big\|\frac{\partial}{\partial t}(\widetilde{u}-v_h)\Big\|_{G_h^\nu}^2\Big)^{\frac{1}{2}}+\Big\|\nabla(\widetilde{u}-v_h)\Big\|_{G_h(0,t^n)}\Big\}$$
$$+|\widetilde{u}^n-u_h^n|_{\Omega_h^n}\cdot|\widetilde{u}^n-v_h^n|_{\Omega_h^n}+\sum_{\nu=0}^{n-1}|\widetilde{u}^\nu-u_h^\nu|_{\Omega_h^\nu}\cdot|v_h^{\nu+0}-v_h^\nu|_{\Omega_h^\nu}$$
$$+\sqrt{d/2}\ |\widetilde{u}-w_h|_{0,\Sigma_h(0,t^n)}\cdot\Big(\sum_{\nu=0}^{n-1}\Big\|\frac{\partial}{\partial t}(\widetilde{u}-v_h)\Big\|_{G_h^\nu}^2\Big)^{\frac{1}{2}}+\frac{1}{2}|\widetilde{u}-w_h|_{0,\Sigma_h(0,t^n)}^2. \tag{3.22}$$

Now let

$$a^n=\|\nabla(\tilde{u}-u_h)\|_{G_h(0,t^n)},\quad b^n=\frac{1}{\sqrt{2}}|\tilde{u}^n-u_h^n|_{\Omega_h^n},$$

$$\alpha^n=\frac{d}{\sqrt{m}}\left\{\sum_{\nu=0}^{n-1}\left\|\frac{\partial}{\partial t}(\tilde{u}-v_h)\right\|_{G_h^\nu}^2\right\}^{\frac{1}{2}}+\|\nabla(\tilde{u}-v_h)\|_{G_h(0,t^n)},$$

$$\beta^n=\sqrt{2}\max_{1<\nu<n}|\tilde{u}^\nu-v_h^\nu|_{\Omega_h^\nu},\quad \gamma^n=\sqrt{2}\,|v_h^{n+0}-v_h^n|_{\Omega_h^n},$$

$$\delta^n=\left\{\sqrt{d/2}\,|\tilde{u}-w_h|_{0,\Sigma_h(0,t^n)}\left(\sum_{\nu=0}^{n-1}\left\|\frac{\partial}{\partial t}(\tilde{u}-v_h)\right\|_{G_h^\nu}^2\right)^{\frac{1}{2}}+\frac{1}{2}|\tilde{u}-w_h|_{0,\Sigma_h(0,t^n)}^2\right\}^{\frac{1}{2}}.$$

Evidently, α^n, β^n and δ^n are all increasing with n. By (3.22), we have

$$(a^n)^2+(b^n)^2\leqslant\alpha^k a^n+\beta^k b^n+\sum_{\nu=1}^{n-1}\gamma^\nu b^\nu+(\delta^k)^2,\quad 1\leqslant n\leqslant k\leqslant N. \tag{3.23}$$

Application of Lemma 3.1 then yields

$$a^n+b^n\leqslant\sqrt{2}\left(\alpha^k+\beta^k+\sum_{\nu=1}^{n-1}\gamma^\nu+\delta^k\right),\quad 1\leqslant n\leqslant k\leqslant N, \tag{3.24}$$

Taking $k=n$ in (3.24), we obtain (3.9). Q. E. D.

Remark. (3.9) generalizes Theorem 4.1 in Jamet (1978), since, in the case of polygonal domain with homogeneous boundary data, if we take $\Sigma_h^T=\Sigma^T$, $\tilde{u}=u$, $w_h=0$, then (3.9) will yield Jamet's estimate.

4. Error of Finite Element Approximations

a) General Lagrange elements (cf. Ciarlet (1978) for the definitions concerned below).

To define the approximation with a family of Lagrange elements, we place a set of nodal points on the lateral boundary Σ^T and define, for each n, a triangulation J_h^n of the subdomain G_h^n such that $\bar{G}_h^n=\{\cup K; K\in J_h^n\}$. Denote by X_h^n the finite element space corresponding to J_h^n. Let $R_h^T=\bigcup_{n=0}^{N-1}\bar{G}_h^n$. Then Σ_h^T is the lateral boundary of R_h^T, which is, in fact, the finite element interpolant of Σ^T.

We choose $\Phi_h^n=\{v_h\in X_h^n;\ v_h=0 \text{ on } \Sigma_h^n\}$ and choose w_h such that its restriction on each G_h^n coincides with a function $q^{(n)}w_h\in X_h^n$ and that $w_h=\tilde{u}$ on the boundary nodes. So $w_h|_{\Sigma_h^T}$ is an interpolant of $\tilde{u}|_{\Sigma_h^T}$.

Denote by J'_{hn} the set of top faces K' of all elements $K\in J_h^{n-1}$, and by J'_{hn+0} the set of bottom faces K' of all elements $K\in J_h^n$. In general the sets J'_{hn} and J'_{hn+0} are not the same, since the sets J_h^{n-1} and J_h^n are independent.

Let $J_h=\{\cup J_h^n,\ 0\leqslant n\leqslant N-1\}$. For each element $K\in J_h$, let Π_K be the finite element interpolation operator. We define the U_h-interpolation operator Π_h such that for each function $u\in C^0(R_h^T)$, $\Pi_h u\in U_h$ and $r_K\Pi_h u$ coincides with $\Pi_K r_K u$, where r_K denotes the operator of restriction to K.

Assume that $\tilde{u}$ is continuous in $\bar{R}_h^T$. Taking $v_h=\Pi_h\tilde{u}$ in (3.9) and using inequality $|v_h^{\nu+0}-v_h^\nu|\leqslant|v_h^{\nu+0}-\tilde{u}^\nu|+|v_h^\nu-\tilde{u}^\nu|$, we obtain

$$\|\nabla(\tilde{u}-u_h)\|_{G_h(0,t^n)}+\frac{1}{\sqrt{2}}|\tilde{u}^n-u_h^n|_{\Omega_h^n}$$

$$\leqslant\sqrt{2/m}\,d\Big(\sum_{K\in J_h}\Big\|\frac{\partial}{\partial t}(\tilde{u}-\Pi_K\tilde{u})\Big\|_K^2\Big)^{\frac{1}{2}}+\sqrt{2}\,\Big(\sum_{K\in J_h}\|\nabla(\tilde{u}-\Pi_K\tilde{u})\|_K^2\Big)^{\frac{1}{2}}$$

$$+2N\max_{1\leqslant\nu\leqslant N}\Big(\sum_{K'\in J'_{h\nu}}|\tilde{u}-\Pi_h\tilde{u}|_{K'}^2\Big)^{\frac{1}{2}}+2(N-1)\max_{1\leqslant\nu\leqslant N}\Big(\sum_{K'\in J'_{h\nu+0}}|\tilde{u}-\Pi_K\tilde{u}|_{K'}^2\Big)^{\frac{1}{2}}$$

$$+\sqrt{2}\Big\{\sqrt{d/2}\,|\tilde{u}-w_h|_{0,\Sigma_h(0,t^n)}\cdot\Big(\sum_{K\in J_h}\Big\|\frac{\partial}{\partial t}(\tilde{u}-\Pi_K\tilde{u})\Big\|_K^2\Big)^{\frac{1}{2}}+\frac{1}{2}|\tilde{u}-w_h|^2_{0,\Sigma_h(0,t^n)}\Big\}^{\frac{1}{2}}. \tag{3.25}$$

Let $\tilde{\Omega}(t)=\{(P,\ t)\in\tilde{R}^T,\ t \text{ fixed}\}$ and $\tilde{d}=\max\{\operatorname{diam}\tilde{\Omega}(t),\ 0\leqslant t\leqslant T\}$.

Theorem 4.1. *Assume that $\tilde{u}$ and Σ^T are continuous and have enough smoothness and that the family of finite elements have the following approximation properties:*

1) $\Big(\sum_{K\in J_h}\Big\|\frac{\partial}{\partial t}(\tilde{u}-\Pi_K\tilde{u})\Big\|_K^2\Big)^{\frac{1}{2}}\leqslant C_1h^k,\quad \Big(\sum_{K\in J_h}\|\nabla(\tilde{u}-\Pi_K\tilde{u})\|_K^2\Big)^{\frac{1}{2}}\leqslant C_1h^k,$

2) $\Big(\sum_{K'\in J'_h}|\tilde{u}-\Pi_K\tilde{u}|_{K'}^2\Big)^{\frac{1}{2}}\leqslant C_2h^{k+1},\quad \Big(\sum_{K'\in J'_{h\nu+0}}|\tilde{u}-\Pi_K\tilde{u}|_{K'}^2\Big)^{\frac{1}{2}}\leqslant C_2h^{k+1},$

3) $|\tilde{u}-w_h|_{\infty,\Sigma_h(0,t^n)}\leqslant C_3h^{k+1},$

4) $\operatorname{meas}(\Sigma_h^T)=\int_{\Sigma_h^T}d\gamma\leqslant C_4$, *where the constants C_i, $1\leqslant i\leqslant 4$, depend only on $\tilde{u}$, Σ^T and $\tilde{R}^T$, not on h,*

5) $Nh\leqslant\lambda<\infty$.

Then,

$$\|\nabla(\tilde{u}-u_h)\|_{G_h(0,t^n)}+\frac{1}{\sqrt{2}}|\tilde{u}^n-u_h^n|_{\Omega_h^n}\leqslant\alpha h^k+\beta h^{k+\frac{1}{2}} \tag{3.26}$$

for all $1\leqslant n\leqslant N$, where

$$\alpha=\sqrt{2}\Big(\frac{\tilde{d}}{\sqrt{m}}+1\Big)C_1+4\lambda C_2,\quad \beta=\Big(\sqrt{2\tilde{d}}\;C_1C_3\sqrt{C_4}+\frac{1}{\sqrt{2}}hC_3^2C_4\Big)^{\frac{1}{2}}.$$

Proof. (3.26) follows immediately from (3.25).

Next, we apply the general error estimate to some particular examples of practical interest.

b) Biquadratic isoparametric space-time finite elements for one-dimensional problems.

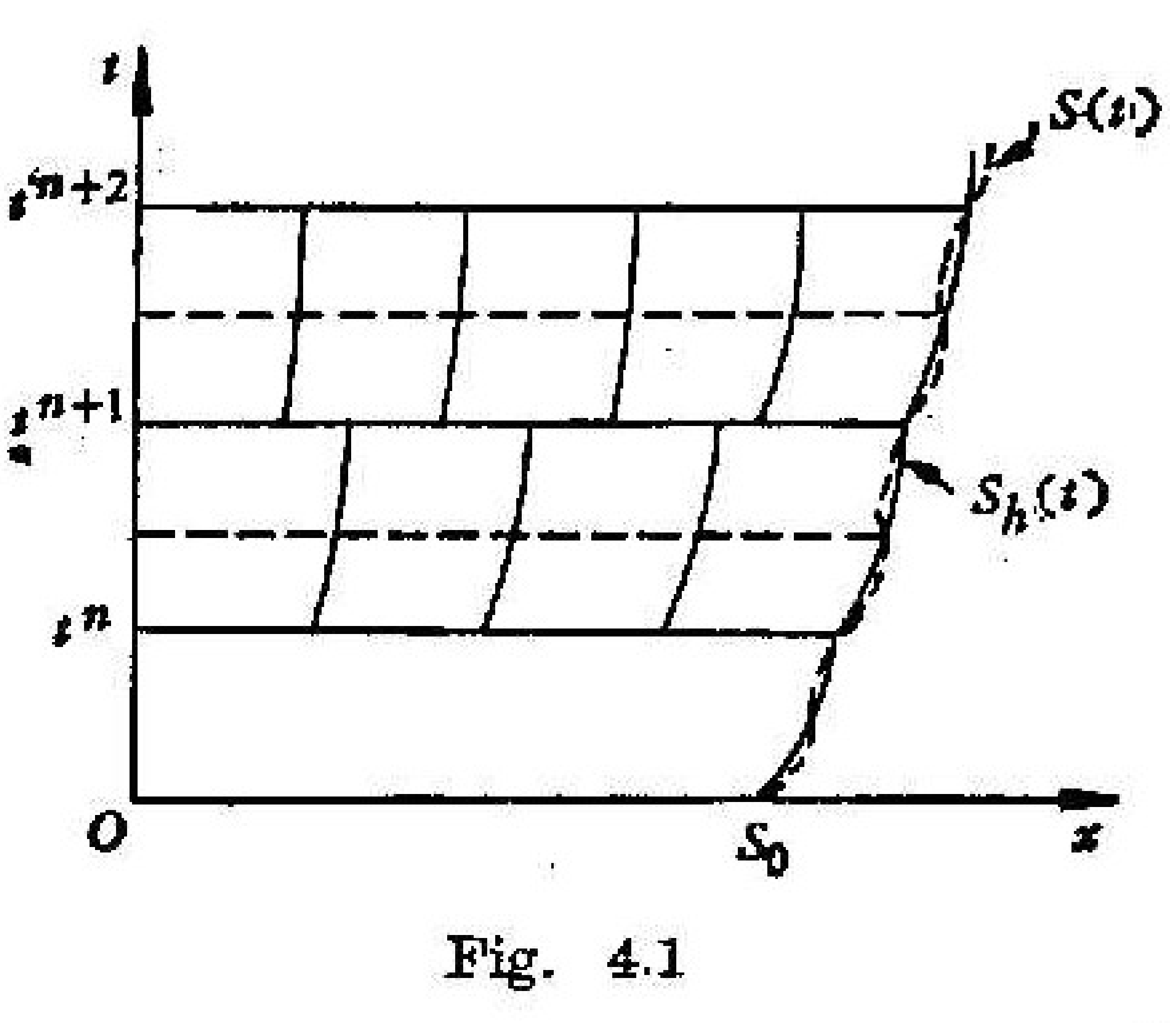


Fig. 4.1

We consider a problem with the boundary conditions

$$u=g(t)\text{ on }x=0\text{ and }u=0\text{ on }x=S(t).$$

As shown in Fig. 4.1, we use a regular family of biquadratic isoparametric space-time elements for discretization (cf. Li (1982, 1983)). In this case the curved boundary $x=S(t)$ is approximated by a piecewise parabola $S_h(t)$, which interpolates the curve $x=S(t)$.

On $x=S_h(t)$, $w_h=0$. Let h be the greatest length of the sides of all elements $K\in J_h$; then we have $\max_{[0,T]}\{|S(t)-S_h(t)|\}\leqslant$

$Ch^3 \max\limits_{[0,T]}(|S'''(t)|)$. This implies $|\tilde{u}-w_h| = |\tilde{u}| \leqslant Ch^3 \max\limits_{\tilde{R}^T}\left|\frac{\partial \tilde{u}}{\partial x}\right| \cdot \max\limits_{[0,T]} |S'''(t)|$. On $x=0$, $|\tilde{u}-w_h| = |g(t)-w_h| \leqslant Ch^3 \max\limits_{[0,T]} |g'''(t)|$.

Since $S_h(t)$ approaches $S(t)$ uniformly as $h \to 0$, it is obvious that meas $(S_h(t)) \leqslant C$ meas $(S(t))$. Therefore the assumptions 3) and 4) of Theorem 4.1 are satisfied with constants

$$C_3 = C \max\left\{\max_{\tilde{R}^T}\left|\frac{\partial \tilde{u}}{\partial x}\right| \cdot \max_{[0,T]}|S'''(t)|,\ \max_{[0,T]}|g'''(t)|\right\} \quad \text{and} \quad C_4 = C \text{ meas } (\Sigma^T).$$

The interpolation errors on an element K are

$$|\tilde{u}-\Pi_K\tilde{u}|_{1,K} \leqslant Ch^2\|\tilde{u}\|_{3,K} \quad \text{and} \quad |\tilde{u}-\Pi_K\tilde{u}|_{K'} \leqslant Ch^3|\tilde{u}|_{3,K'},$$

where $|\cdot|_{i,\Omega}$ and $\|\cdot\|_{i,\Omega}$ are the standard Sobolev seminorm and norm (cf. Ciarlet (1972, 1978) and Li (1982)).

From the above estimates we deduce

$$\left(\sum_{K\in J_h}\left\|\frac{\partial}{\partial t}(\tilde{u}-\Pi_K\tilde{u})\right\|_K^2\right)^{\frac{1}{2}} \leqslant Ch^2\|\tilde{u}\|_{3,R_h^T} \leqslant Ch^2\|\tilde{u}\|_{3,\tilde{R}^T},$$

$$\left(\sum_{K\in J_h}\left\|\frac{\partial}{\partial x}(\tilde{u}-\Pi_K\tilde{u})\right\|_K^2\right)^{\frac{1}{2}} \leqslant Ch^2\|\tilde{u}\|_{3,\tilde{R}^T},$$

$$\left(\sum_{K'\in J_{h'}}|\tilde{u}-\Pi_K\tilde{u}|_{K'}^2\right)^{\frac{1}{2}} \leqslant Ch^3 \max_{0<t<T}|\tilde{u}|_{3,\tilde{\Omega}(t)},$$

$$\left(\sum_{K'\in J_{h_{t+0}}}|\tilde{u}-\Pi_K\tilde{u}|_{K'}^2\right)^{\frac{1}{2}} \leqslant Ch^3 \max_{0<t<T}|\tilde{u}|_{3,\tilde{\Omega}(t)}.$$

Therefore assumptions 1) and 2) are also satisfied with constants

$$C_1 = C\|\tilde{u}\|_{3,\tilde{R}^T} \quad \text{and} \quad C_2 = C \max_{0<t<T}|\tilde{u}|_{3,\tilde{\Omega}(t)}.$$

Application of Theorem 4.1 then yields the following result.

Theorem 4.2. *Assume that*

1) $\tilde{u}\in H^3(\tilde{R}^T)\cap C^0(0, T; H^3(\tilde{\Omega}(t)))$, $S(t)\in C^3([0, T])$, $g(t)\in C^3([0, T])$,

2) *the family of the biquadratic isoparametric space-time finite elements is regular,*

3) $Nh \leqslant \lambda$.

Then

$$\left\|\frac{\partial}{\partial x}(\tilde{u}-u_h)\right\|_{G_h(0,t^n)} + |\tilde{u}^n - u_h^n|_{\Omega_h^n} \leqslant \gamma h^2, \tag{3.27}$$

for $1 \leqslant n \leqslant N$, *where the constant* γ *is independent of* h, n *and* N.

Note. The smoothness assumption 1) is made to ensure the boundness of the constants C_i, $1 \leqslant i \leqslant 4$.

c) Tri-linear isoparametric space-time elements for two-dimensional problems.

We consider a problem where the lateral boundary Σ^T consists of two parts: $\Sigma^T = \Sigma_1^T \cup \Sigma_2^T$ with

$$\Sigma_1^T = \{(P, t);\ P\in\partial_1\Omega(t),\ 0\leqslant t\leqslant T\}$$

and

$$\Sigma_2^T = \{(P, t);\ P\in\partial_2\Omega(t),\ 0\leqslant t\leqslant T\},$$

as shown in Fig. 4.2. The boundary conditions are $u=g_1(P, t)$ on Σ_1^T and $u=0$ on Σ_2^T.

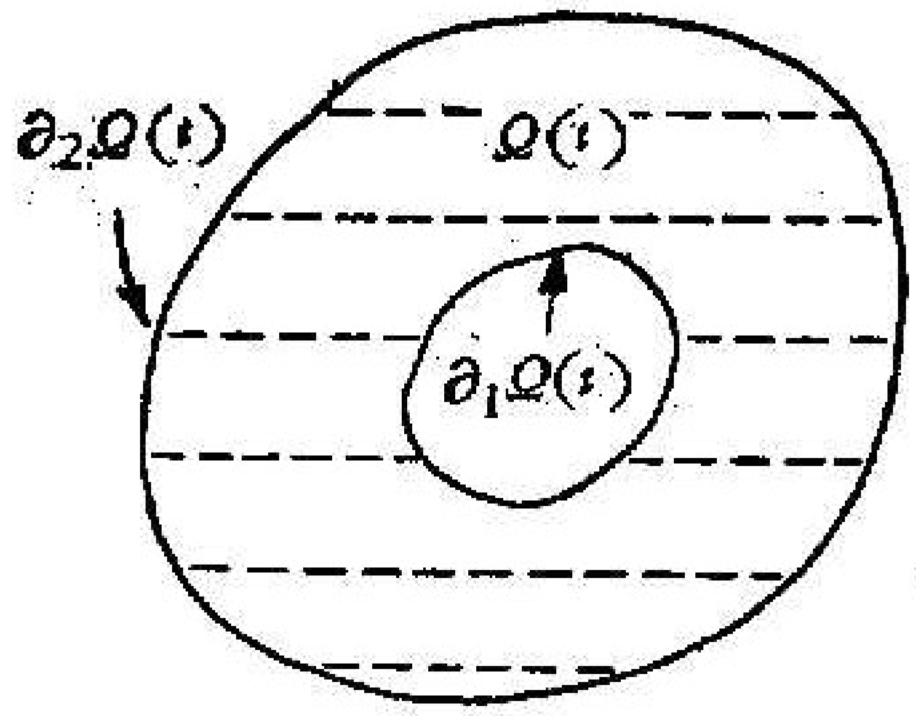


Fig. 4.2

We use a regular family of tri-linear space-time elements for discretization (cf. Li (1982)).

Theorem 4.3. *Assume that*

1) $\tilde{u}\in H^2(\tilde{R}^T)\cap C^0(0, T; H^2(\tilde{\Omega}(t))$ *and* $\tilde{u}$ *has bounded second derivatives,*

2) Σ_i^T, $i=1, 2$, *are* C^2 *surfaces,*

3) *the family of tri-linear isoparametric elements is regular,*

4) $Nh\leqslant\lambda$, *where* h *is the greatest length of the sides of all elements* $K\in J_h$. *Then*

$$\|\nabla(\tilde{u}-u_h)\|_{G_h(0,t^n)}+|\tilde{u}^n-u^n|_{\Omega_h^n}\leqslant\gamma h, \tag{3.28}$$

for $1\leqslant n\leqslant N$, *where the constant* γ *is independent of* h, n *and* N.

Proof. On Σ_{h2}^T, $w_h=0$. It is easy to check that

$$|\tilde{u}-w_h|_{\infty,\Sigma_{h2}^T}=|\tilde{u}|_{\infty,\Sigma_{h2}^T}\leqslant Ch^2 \quad \text{(cf. Fig. 4.3(a))}.$$

On Σ_{h1}^T, w_h is the piecewise bilinear interpolant of $\tilde{u}$, so

$$|\tilde{u}-w_h|_{\infty,\Sigma_{h1}^T}\leqslant Ch^2 \quad \text{(cf. Fig. 4.3(b))}.$$

Therefore assumption 3) of Theorem 4.1 is satisfied. Evidently, assumption 4) is also satisfied.

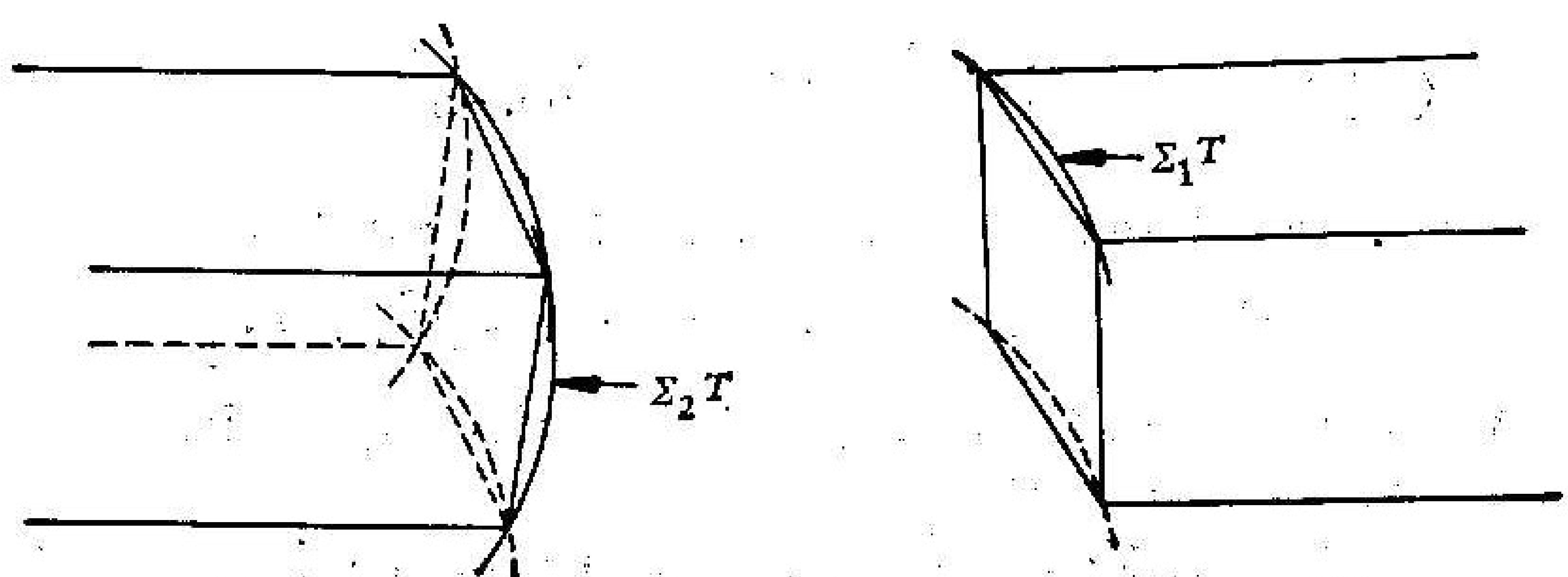


Fig. 4.3

Using the interpolation error estimate (cf. Jamet (1976); Ciarlet (1978))

$$|\tilde{u}-\Pi_K\tilde{u}|_{1,K}\leqslant Ch\|\tilde{u}\|_{2,K} \quad\text{and}\quad |\tilde{u}-\Pi_K\tilde{u}|_{K'}\leqslant Ch^2\|\tilde{u}\|_{2,K'},$$

we can show

$$\Big(\sum_{K\in J_h}\Big\|\frac{\partial}{\partial t}(\tilde{u}-\Pi_K\tilde{u})\Big\|_K^2\Big)^{\frac{1}{2}}\leqslant C_1h, \qquad \Big(\sum_{K\in J_h}\|\nabla(\tilde{u}-\Pi_K\tilde{u})\|_K^2\Big)^{\frac{1}{2}}\leqslant C_1h,$$

$$\Big(\sum_{K'\in J_h^n}|\tilde{u}-\Pi_K\tilde{u}|_{K'}^2\Big)^{\frac{1}{2}}\leqslant C_2h^2, \qquad \Big(\sum_{K'\in J_h^{n+1}}|\tilde{u}-\Pi_K\tilde{u}|_{K'}^2\Big)^{\frac{1}{2}}\leqslant C_2h^2.$$

Application of Theorem 4.1 then gives (3.28).

d) Simplicial elements and prismatic elements.

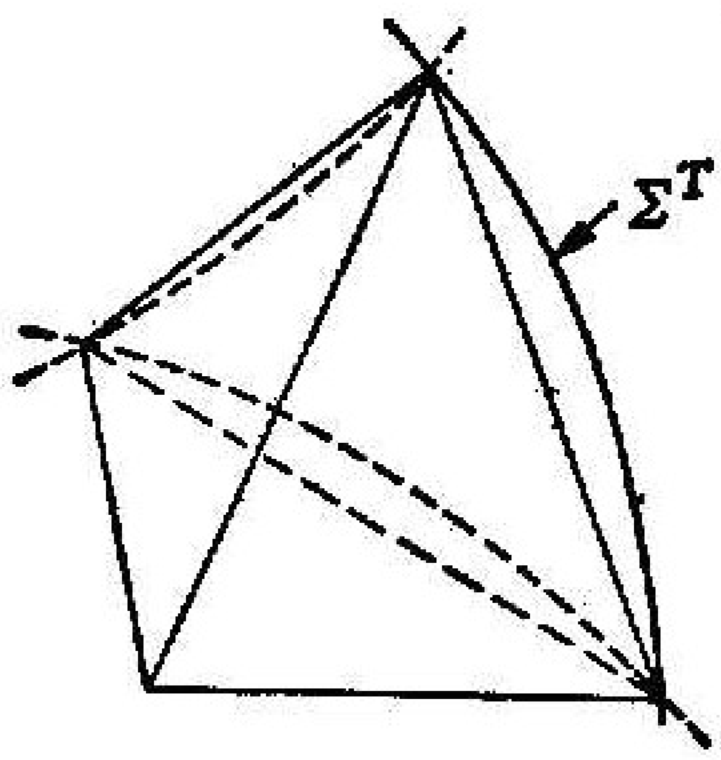
When we use straight elements such as simplicial or prismatic elements (cf.

(1978)) for discretization of a problem in a curved space-time domain with homogeneous Dirichlet boundary conditions, we take $w_h=0$ on Σ_h^T. It is easy to see (cf. Fig. 4.4) that

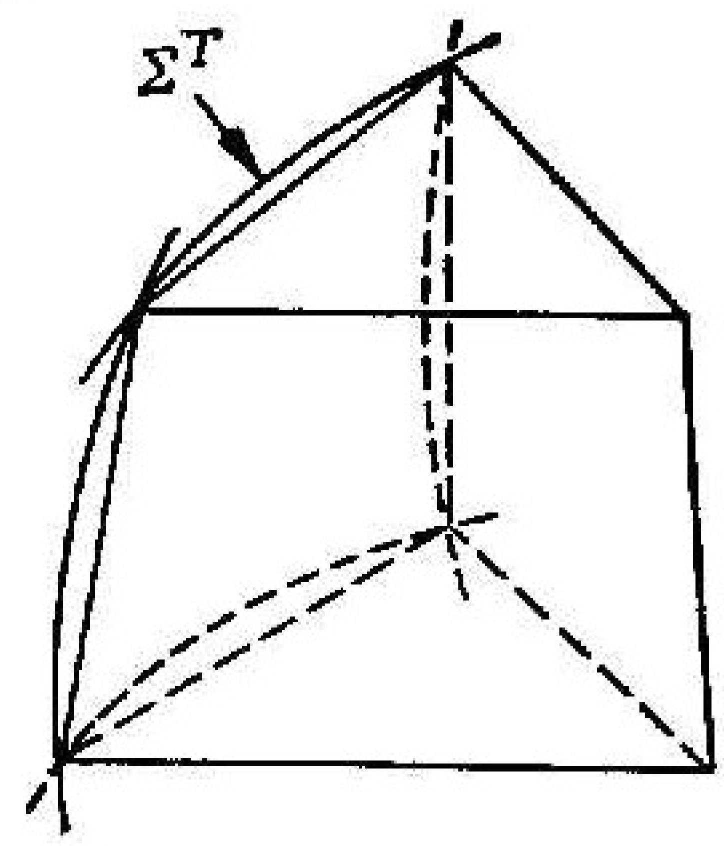
$$\tilde{u}=0(h^2) \quad \text{on } \Sigma_h^T.$$

Then we have in (3.25)

$$|\tilde{u}-w_h|_{0,\Sigma_h(0,t^n)}=0(h^2).$$



(a) Simplicial element



(b) Prismatic element

Fig. 4.4

Due to this effect of boundary approximation, we can expect error estimate of only second order at best, i.e.

$$\|\nabla(\tilde{u}-u_h)\|_{G_h(0,t^n)}+|\tilde{u}^n-u_h^n|_{\Omega_h^n}=O(h^2).$$

In fact, by a more precise analysis similar to that of Strang (1973), § 4.4, we can show, for elements of order $\geqslant 2$,

$$\|\nabla(\tilde{u}-u_h)\|_{G_h(0,t^n)}+|\tilde{u}^n-u_h^n|_{\Omega_h^n}=O(h^{\frac{3}{2}}).$$

Using the maximum principle, we can also show, at best,

$$\tilde{u}-u_h=O(h^2).$$

Therefore, there is no point in using straight space-time elements of order higher than quadratic for problems in a curved space-time domain.

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