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THE GENERAL NONLINEAR MUTUAL BOUNDARY PROBLEMS FOR THE SYSTEMS OF NONLINEAR WAVE EQUATIONS BY FINITE DIFFERENCE METHOD*

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§ 1. Introduction

1. The nonlinear wave equations are often appeared in the physical, chemical, mechanical, biological, geometrical problems and others. For example, the Sine-Gordon equation

$$u_{tt}-u_{xx}=\sin u, \tag{1}$$

the nonlinear vibration equations

$$u_{tt}-u_{xx}+u^3=0, \tag{2}$$

$$u_{tt}-u_{xx}+u^5=0 \tag{3}$$

and the equation

$$u_{tt}-u_{xx}+\sinh u=0 \tag{4}$$

belong to the number of the nonlinear wave equations. A lot of works contributed to the study of the various problems of the nonlinear wave equations[1-17] and the fairly wide systems of nonlinear wave equations[18-20], such as the periodic boundary problem, Cauchy problem, first, second and other boundary problems. In the expressions of the conditions of so-called the absorbing boundary problems for the wave equation, there are the derivatives with respect to the time variable and space variable[21, 22].

Let us consider in the present work the system of nonlinear wave equations of the following form

$$u_{tt}-u_{xx}+\text{grad } F(u)=f(x, t, u, u_x, u_t), \tag{5}$$

which contains the above mentioned nonlinear wave equations as the simple special cases. Here $u(x, t)=(u_1(x, t), \cdots, u_m(x, t))$ is a m-dimensional vector function, $F(u)$ is a scalar function of vector variable $u\in\mathbb{R}^m$ and $f(x, t, u, p, q)$ is a m-dimensional vector function for the scalar variables x, t and the vector variables $u, p, q\in\mathbb{R}^m$. In the rectangular domain $Q_T=\{0\leqslant x<l,\ 0\leqslant t\leqslant T\}$, we take into account of the boundary problem with the nonlinear mutual boundary conditions of the form

$$\begin{aligned}u_t(0, t)&=\Phi_0(u_x(0, t), u_x(l, t), u(0, t), u(l, t), t),\\ -u_t(l, t)&=\Phi_1(u_x(0, t), u_x(l, t), u(0, t), u(l, t), t)\end{aligned} \tag{6}$$

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and the initial conditions

$$u(x, 0)=\varphi(x),\quad u_t(x, 0)=\psi(x), \tag{7}$$

where $\Phi_0(p_0, p_1, u_0, u_1, t)$ and $\Phi_1(p_0, p_1, u_0, u_1, t)$ are two m-dimensional vector functions of $t\in[0, T]$ and u_0, u_1, p_0, $p_1\in\mathbb{R}^m$ and $\varphi(x)$ and $\psi(x)$ are two m-dimensional vector functions of $x\in[0, l]$.

We are going to construct the global generalized solution of the general nonlinear mutual boundary problem (6) and (7) for the system (5) of nonlinear wave equations by the finite difference method. The convergence of the solutions of finite difference scheme is established and the limit is the solution of the problem (6) and (7) for the system (5) of nonlinear wave equations.

By similar way we also consider the mixed problem with the boundary conditions

$$u_t(0, t)=\Phi_0(u_x(0, t), u(0, t), t),\quad u(l, t)=0 \tag{8}$$

and the initial conditions (7) for the system (5) of nonlinear wave equations.

We adopt the similar notations and conventions used in [23—24].

2. Suppose that for the system (5), the general nonlinear mutual boundary conditions (6) and the initial vector functions in (7), the following assumptions are valid.

(I) The scalar non-negative convex function $F(u)\geqslant 0$ of the m-dimensional vector variable $u\in\mathbb{R}^m$ is twice continuously differentiable with respect to $u\in\mathbb{R}^m$.

(II) $f(x, t, u, p, q)$ is a m-dimensional continuous in $(x, t, u, p, q)\in Q_T\times\mathbb{R}^{3m}$ vector function, continuously differentiable with respect to variable x and vector variables $u, p, q\in\mathbb{R}^m$. Further for any $(x, t)\in Q_T$ and $u, p, q\in\mathbb{R}^m$, there are

$$\begin{aligned}&|f(x, t, u, p, q)|^2,\ |f_x(x, t, u, p, q)|\leqslant A\{F(u)+|u|^2+|p|^2+|q|^2+1\},\\ &|f_u(x, t, u, p, q)|^2\leqslant A\{F(u)+|u|^2+1\},\\ &|f_p(x, t, u, p, q)|,\ |f_q(x, t, u, p, q)|\leqslant A,\end{aligned} \tag{9}$$

where A is a constant and for brevity $|\cdot|$ denotes any components of the appropriate vector functions and any elements of the mentioned matrices.

(III) $\Phi_0(p_0, p_1, u_0, u_1, t)$ and $\Phi_1(p_0, p_1, u_0, u_1, t)$ are two m-dimensional continuously differentiable vector functions of the variable $t\in[0, T]$ and the vector variables u_0, u_1, p_0, $p_1\in\mathbb{R}^m$. The $2m\times 2m$ Jacobi derivative matrix $\frac{\partial(\Phi_0, \Phi_1)}{\partial(p_0, p_1)}$ of the $2m$-dimensional vector function (Φ_0, Φ_1) with respect to $2m$-dimensional vector (p_0, p_1) is positively definite, i.e., there is a positive number $\sigma>0$, such that

$$\left(\eta, \frac{\partial(\Phi_0, \Phi_1)}{\partial(p_0, p_1)}\eta\right)\geqslant\sigma|\eta|^2 \tag{10}$$

for any $2m$-dimensional vector $\eta\in\mathbb{R}^{2m}$. Furthermore, there are

$$\begin{gathered}|\Phi_0(0,\,0,\,u_0,\,u_1,\,t)|,\ |\Phi_1(0,\,0,\,u_0,\,u_1,\,t)|\leqslant A\{|u_0|+|u_1|+1\};\\ |\Phi_{0u_0}(p_0,\,p_1,\,u_0,\,u_1,\,t)|,\ |\Phi_{0u_1}(p_0,\,p_1,\,u_0,\,u_1,\,t)|,\ |\Phi_{1u_0}(p_0,\,p_1,\,u_0,\,u_1,\,t)|,\\ |\Phi_{1u_1}(p_0,\,p_1,\,u_0,\,u_1,\,t)|\leqslant B(u_0,\,u_1)\{|p_0|+|p_1|+1\};\\ |\Phi_{0t}(p_0,\,p_1,\,u_0,\,u_1,\,t)|,\ |\Phi_{1t}(p_0,\,p_1,\,u_0,\,u_1,\,t)|\leqslant B(u_0,\,u_1)\{|p_0|^2+|p_1|^2+1\},\end{gathered} \tag{11}$$

where $A\geqslant 0$ is a constant, $B(u_0, u_1)\geqslant 0$ is a continuous function of u_0, $u_1\in\mathbb{R}^m$.

(IV) The m-dimensional initial vector functions $\varphi(x)\in C^2([0, l])$ and $\psi(x)\in C^{(1)}([0, l])$ satisfy the boundary conditions (6), i.e.,

$$\begin{aligned}\psi(0)&=\Phi_0(\varphi'(0),\,\varphi'(l),\,\varphi(0),\,\varphi(l),\,0),\\ -\psi(l)&=\Phi_1(\varphi'(0),\,\varphi'(l),\,\varphi(0),\,\varphi(l),\,0).\end{aligned} \tag{12}$$

3. Suppose that the rectangular domain $Q_T=\{0\leqslant x\leqslant l,\ 0\leqslant t\leqslant T\}$ is divided into small grids by the parallel lines $x=x_j(j=0, 1, \cdots, J)$ and $t=t^n(n=0, 1, \cdots, N)$, where $x_j=jh$, $t^n=n\Delta t$ and $Jh=l$, $N\Delta t=T$. Denote the m-dimensional discrete vector function defined on the grid points (x_j, t^n) by v_j^n $(j=0, 1, \cdots, J;\ n=0, 1, \cdots, N)$.

Let us construct the finite difference system

$$\begin{gathered}\frac{v_j^{n+1}-2v_j^n+v_j^{n-1}}{\Delta t^2}+\frac{\Delta_+\Delta_-v_j^{n+\alpha}}{h^2}+\int_0^1\operatorname{grad}F(v_j^{n+\tau})d\tau\\ =f\Big(x_j,\,t^{n+\alpha},\,\bar v_j^n,\,\frac{\bar\Delta v_j^{n+\alpha}}{h},\,\frac{v_j^{n+\alpha}-v_j^{n+\alpha-1}}{\Delta t}\Big),\\ j=1,\,2,\,\cdots,\,J-1;\ n=1,\,2,\,\cdots,\,N-1\end{gathered} \tag{5$_h$}$$

corresponding to the system (5) of nonlinear wave equations, where

$$\begin{gathered}\bar v_j^n=\alpha a v_j^{n+1}+a'v_j^n+a''v_j^{n-1},\\ \frac{\bar\Delta v_j^{n+\alpha}}{h}=b\,\frac{\Delta_+v_j^{n+\alpha}}{h}+b'\,\frac{\Delta_-v_j^{n+\alpha}}{h},\\ v^{n+\alpha}=\alpha v^{n+1}+(1-\alpha)v^n\end{gathered} \tag{13}$$

and $0\leqslant\alpha\leqslant 1$, $\alpha a+a'+a''=b+b'=1$. As to the nonlinear mutual boundary conditions (6), we have the corresponding finite difference boundary conditions

$$\begin{aligned}\frac{v_0^{n+1}-v_0^n}{\Delta t}&=\Phi_0\Big(\frac{\Delta_+v_0^{n+\alpha}}{h},\ \frac{\Delta_-v_J^{n+\alpha}}{h},\ v_1^{n+\alpha},\ v_{J-1}^{n+\alpha},\ t^{n+\alpha}\Big),\\ -\frac{v_J^{n+1}-v_J^n}{\Delta t}&=\Phi_1\Big(\frac{\Delta_+v_0^{n+\alpha}}{h},\,\frac{\Delta_-v_J^{n+\alpha}}{h},\,v_1^{n+\alpha},\,v_{J-1}^{n+\alpha},\,t^{n+\alpha}\Big),\end{aligned} \tag{6$_h$}$$

where $n=1, 2, \cdots, N-1$. For the initial conditions (7), the corresponding finite difference conditions are of the form

$$\begin{aligned}&v_j^0=\varphi_j, && j=0,\,1,\,\cdots,\,J,\\ &v_j^1=\varphi_j+\Delta t\psi_j, && j=1,\,2,\,\cdots,\,J-1\end{aligned} \tag{7$'_h$}$$

and v_0^1 and v_J^1 are the unique solutions of the system

$$\begin{aligned}\frac{v_0^1-\varphi_0}{\Delta t}&=\Phi_0\Big(\frac{\Delta_+v_0^\alpha}{h},\,\frac{\Delta_-v_J^\alpha}{h},\,\varphi_1+\alpha\Delta t\psi_1,\,\varphi_{J-1}+\alpha\Delta t\psi_{J-1},\,\alpha\Delta t\Big),\\ -\frac{v_J^1-\varphi_J}{\Delta t}&=\Phi_1\Big(\frac{\Delta_+v_0^\alpha}{h},\,\frac{\Delta_-v_J^\alpha}{h},\,\varphi_1+\alpha\Delta t\psi_1,\,\varphi_{J-1}+\alpha\Delta t\,\psi_{J-1},\,\alpha\Delta t\Big),\end{aligned} \tag{7$''_h$}$$

where $\alpha\sigma\Delta t<h$ and $\varphi_j=\varphi(x_j)$, $\psi_j=\psi(x_j)$ $(j=0, 1, \cdots, J)$.

In fact from $(7'')_h$, we have

$$\left(\frac{v_0^1-\varphi_0}{\Delta t}, -\frac{\Delta_+ v_0^\alpha}{h}\right)+\left(\frac{v_J^1-\varphi_J}{\Delta t}, \frac{\Delta_- v_J^\alpha}{h}\right)$$
$$=-\int_0^1\left[\left(\frac{\Delta_+ v_0^\alpha}{h}, \Phi_{0p_0}^\tau \frac{\Delta_+ v_0^\alpha}{h}+\Phi_{0p_1}^\tau \frac{\Delta_- v_J^\alpha}{h}\right)\right.$$
$$\left.+\left(\frac{\Delta_- v_J^\alpha}{h}, \Phi_{1p_0}^\tau \frac{\Delta_+ v_0^\alpha}{h}+\Phi_{1p_1}^\tau \frac{\Delta_- v_J^\alpha}{h}\right)\right]d\tau-\left(\frac{\Delta_+ v_0^\alpha}{h}, \Phi_0^0\right)-\left(\frac{\Delta_- v_J^\alpha}{h}, \Phi_1^0\right)$$
$$\leqslant\frac{1}{4\sigma}\{|\Phi_0^0|^2+|\Phi_1^0|^2\},$$

where the index "τ" means $\Phi^\tau=\Phi\left(\tau\frac{\Delta_+ v_0^\alpha}{h}, \tau\frac{\Delta_- v_J^\alpha}{h}, v_1^\alpha, v_{J-1}^\alpha, \alpha\Delta t\right)(0\leqslant\tau\leqslant 1)$ and similarly for the others. This implices the existence of the solution v_0^1 and v_J^1 for the system $(7'')_h$ by means of ordinary approach of fixed point technique.

Suppose that v_0^1, v_J^1 and $\bar v_0^1$, $\bar v_J^1$ are two solutions of the system $(7'')_h$. By simple verification, we get

$$\frac{v_0^1-\bar v_0^1}{\Delta t}=-\alpha\int_0^1\left(\Phi_{0p_0}^\tau \frac{v_0^1-\bar v_0^1}{h}+\Phi_{0p_1}^\tau \frac{v_J^1-\bar v_J^1}{h}\right)d\tau,$$
$$\frac{v_J^1-\bar v_J^1}{\Delta t}=-\alpha\int_0^1\left(\Phi_{1p_0}^\tau \frac{v_0^1-\bar v_0^1}{h}+\Phi_{1p_1}^\tau \frac{v_J^1-\bar v_J^1}{h}\right)d\tau.$$

Thus there is

$$|v_0^1-\bar v_0^1|^2+|v_J^1-\bar v_J^1|^2\leqslant\frac{\alpha\sigma\Delta t}{h}\{|v_0^1-\bar v_0^1|^2+|v_J^1-\bar v_J^1|^2\}.$$

Hence $v_0^1=\bar v_0^1$, $v_J^1=\bar v_J^1$ as $\frac{\alpha\sigma\Delta t}{h}<1$.

§ 2. Solutions of Difference Scheme

4. Now we turn to consider the existence of the solutions of the finite difference system $(5)_h$, $(6)_h$, $(7')_h$ and $(7'')_h$.

We see that the finite difference scheme is explicit as $\alpha=0$. So v_j^{n+1} $(j=0, 1, \cdots, J)$ can be obtained by direct calculation from the finite difference system $(5)_h$ and $(6)_h$, where v_j^n and v_j^{n-1} $(j=0, 1, \cdots, J)$ are regarded as known vectors.

When $\alpha>0$, the finite difference scheme $(5)_h$, $(6)_h$, $(7')_h$ and $(7'')_h$ is implicit. So the system $(5)_h$ and $(6)_h$ for the unknown vectors v_j^{n+1} $(j=0, 1, \cdots, J)$ is a nonlinear system with given vectors v_j^n and v_j^{n-1} $(j=0, 1, \cdots, J)$.

In order to prove the existence of solution v_j^{n+1} $(j=0, 1, \cdots, J)$ for the nonlinear system $(5)_h$ and $(6)_h$, it is sufficient to obtain the uniform bound of v_j^{n+1} $(j=0, 1, \cdots, J)$ for the nonlinear system

$$\frac{v_j^{n+1}-2v_j^n+v_j^{n-1}}{\Delta t^2}=\lambda\frac{\Delta_+\Delta_- v_j^{n+\alpha}}{h^2}-\lambda\int_0^1 \text{grad } F(v_j^{n+\tau})d\tau$$
$$+\lambda f\left(x_j, t^{n+\alpha}, \bar v_j^n, \frac{\Delta v_j^{n+\alpha}}{h}, \frac{v_j^{n+\alpha}-v_j^{n+\alpha-1}}{\Delta t}\right), \qquad (5)_\lambda$$
$$j=1, 2, \cdots, J-1$$

and

$$\frac{v_0^{n+1}-v_0^n}{\Delta t}=\lambda\Phi_0\left(\frac{\Delta_+v_0^{n+\alpha}}{h},\frac{\Delta_-v_J^{n+\alpha}}{h},v_1^{n+\alpha},v_{J-1}^{n+\alpha},t^{n+\alpha}\right),$$
$$-\frac{v_J^{n+1}-v_J^n}{\Delta t}=\lambda\Phi_1\left(\frac{\Delta_+v_0^{n+\alpha}}{h},\frac{\Delta_-v_J^{n+\alpha}}{h},v_1^{n+\alpha},v_{J-1}^{n+\alpha},t^{n+\alpha}\right) \tag{6$_\lambda$}$$

with respect to the parameter $0\leqslant\lambda\leqslant1$, where v_j^n and v_j^{n-1} $(j=0, 1, \cdots, J)$ are given vectors and Δt, h are given positive constants.

Making the scalar product of the m-dimensional vector $\frac{v_j^{n+1}-v_j^n}{\Delta t}h\Delta t$ and the m-dimensional vector equation $(5)_\lambda$ and summing up the resulting relations for $j=1, 2, \cdots, J-1$, we obtain

$$\sum_{j=1}^{J-1}\left(\frac{v_j^{n+1}-v_j^n}{\Delta t},\frac{v_j^{n+1}-2v_j^n+v_j^{n-1}}{\Delta t^2}\right)h\Delta t+\lambda\sum_{j=1}^{J-1}\left(\frac{v_j^{n+1}-v_j^n}{\Delta t},\frac{\Delta_+\Delta_-v_j^{n+\alpha}}{h^2}\right)h\Delta t$$
$$+\lambda\sum_{j=1}^{J-1}\left(\frac{v_j^{n+1}-v_j^n}{\Delta t},\int_0^1\text{grad }F(v_j^{n+\tau})\,d\tau\right)h\Delta t=\lambda\sum_{j=1}^{J-1}\left(\frac{v_j^{n+1}-v_j^n}{\Delta t},f_j^{n+\alpha}\right)h\Delta t, \tag{14}$$

where $f_j^{n+\alpha}=f\left(x_j,t^{n+\alpha},\bar{v}_j^n,\frac{\bar{\Delta}v_j^{n+\alpha}}{h},\frac{v_j^{n+\alpha}-v_j^{n+\alpha-1}}{\Delta t}\right)$. By simple derivation, there are

$$\sum_{j=1}^{J-1}\left(\frac{v_j^{n+1}-v_j^n}{\Delta t},\frac{v_j^{n+1}-2v_j^n+v_j^{n-1}}{\Delta t^2}\right)h\Delta t\geqslant\frac{1}{2}\sum_{j=1}^{J-1}\left|\frac{v_j^{n+1}-v_j^n}{\Delta t}\right|^2h$$
$$-\frac{1}{2}\sum_{j=1}^{J-1}\left|\frac{v_j^n-v_j^{n-1}}{\Delta t}\right|^2h, \tag{15}$$
$$\sum_{j=1}^{J-1}\left(\frac{v_j^{n+1}-v_j^n}{\Delta t},\int_0^1\text{grad }F(v_j^{n+\tau})d\tau\right)h\Delta t=\sum_{j=1}^{J-1}F(v_j^{n+1})h-\sum_{j=1}^{J-1}F(v_j^n)h.$$

For the second sum of the equality (14), we have

$$\sum_{j=1}^{J-1}\left(\frac{v_j^{n+1}-v_j^n}{\Delta t},\frac{\Delta_+\Delta_-v_j^{n+\alpha}}{h^2}\right)h\Delta t=-\sum_{j=0}^{J-1}\left(\frac{\Delta_+v_j^{n+\alpha}}{h},\frac{\Delta_+v_j^{n+1}}{h}-\frac{\Delta_+v_j^n}{h}\right)h$$
$$-\left(\frac{v_0^{n+1}-v_0^n}{\Delta t},\frac{\Delta_+v_0^{n+\alpha}}{h}\right)\Delta t+\left(\frac{v_J^{n+1}-v_J^n}{\Delta t},\frac{\Delta_-v_J^{n+\alpha}}{h}\right)\Delta t. \tag{16}$$

Here, we see that

$$\sum_{j=0}^{J-1}\left(\frac{\Delta_+v_j^{n+\alpha}}{h},\frac{\Delta_+v_j^{n+1}}{h}-\frac{\Delta_+v_j^n}{h}\right)h\leqslant\frac{\alpha}{2}\|\delta v_h^{n+1}\|_2^2-\frac{1-3\alpha+3\alpha^2}{\alpha}\|\delta v_h^n\|_2^2.$$

And also we see that

$$-\left(\frac{v_0^{n+1}-v_0^n}{\Delta t},\frac{\Delta_+v_0^{n+\alpha}}{h}\right)+\left(\frac{v_J^{n+1}-v_J^n}{\Delta t},\frac{\Delta_-v_J^{n+\alpha}}{h}\right)$$
$$=-\lambda\left(\frac{\Delta_+v_0^{n+\alpha}}{h},\Phi_0^{n+\alpha}\right)-\lambda\left(\frac{\Delta_-v_J^{n+\alpha}}{h},\Phi_1^{n+\alpha}\right)$$
$$\leqslant\frac{\lambda}{4\sigma}\{|\Phi_0(0,0,v_1^{n+\alpha},v_{J-1}^{n+\alpha},t^{n+\alpha})|^2+|\Phi_1(0,0,v_1^{n+\alpha},v_{J-1}^{n+\alpha},t^{n+\alpha})|^2\}$$
$$\leqslant\lambda C_1\{|v_1^{n+\alpha}|^2+|v_{J-1}^{n+\alpha}|^2\}+\lambda C_1\leqslant\lambda C_1\alpha^2\{|v_1^{n+1}|^2+|v_{J-1}^{n+1}|^2\}+\lambda C_2$$
$$\leqslant2\lambda C_1\alpha^2\{\max_{j=1,2,\cdots,J-1}|v_j^{n+1}|\}^2+\lambda C_2$$
$$\leqslant\lambda\frac{\alpha}{8}\|\delta v_h^{n+1}\|_2^2+\lambda C_3\sum_{j=1}^{J-1}|v_j^{n+1}|^2h+\lambda C_2$$
$$\leqslant\lambda\frac{\alpha}{8}\|\delta v_h^{n+1}\|_2^2+\lambda C_4\Delta t^2\sum_{j=1}^{J-1}\left|\frac{v_j^{n+1}-v_j^n}{\Delta t}\right|^2h+\lambda C_5,$$

where C's are constants depending on known values as v_j^n and v_j^{n-1} $(j=0, 1, \cdots, J)$ and others. Thus (16) becomes

$$-\sum_{j=1}^{J-1}\left(\frac{v_j^{n+1}-v_j^n}{\Delta t}, \frac{\Delta_+\Delta_- v_j^{n+\alpha}}{h^2}\right)h\Delta t\leqslant\frac{\alpha}{2}\left(1-\frac{\lambda}{4}\right)\|\delta v_h^{n+1}\|_2^2$$
$$+\lambda C_6\Delta t^2\sum_{j=1}^{J-1}\left|\frac{v_j^{n+1}-v_j^n}{\Delta t^2}\right|^2 h+\lambda C_7.$$

As to the last term of (14), we have

$$\sum_{j=1}^{J-1}\left(\frac{v_j^{n+1}-v_j^n}{\Delta t}, f_j^{n+\alpha}\right)h\Delta t\leqslant\Delta t\left(\sum_{j=1}^{J-1}\left|\frac{v_j^{n+1}-v_j^n}{\Delta t}\right|^2 h\right)^{\frac{1}{2}}\left(\sum_{j=1}^{J-1}|f_j^{n+\alpha}|^2 h\right)^{\frac{1}{2}}.$$

Here

$$\sum_{j=1}^{J-1}|f_j^{n+\alpha}|^2 h\leqslant C_8\left\{\sum_{j=1}^{J-1}F(v_j^{n+1})h+\sum_{j=1}^{J-1}|v_j^{n+1}|^2 h\right.$$
$$\left.+\|\delta v_h^{n+1}\|_2^2+\sum_{j=1}^{J-1}\left|\frac{v_j^{n+1}-v_j^n}{\Delta t}\right|^2 h\right\}+C_9,$$

where C_8 is a constant and C_9 depends on v_j^n and v_j^{n-1} $(j=0, 1, \cdots, J)$. In fact since $F(u)$ is convex, then $F(v_j^{n+\alpha})\leqslant\alpha F(v_j^{n+1})+(1-\alpha)F(v_j^n)$. Hence

$$\sum_{j=1}^{J-1}\left(\frac{v_j^{n+1}-v_j^n}{\Delta t}, f_j^{n+\alpha}\right)h\Delta t\leqslant\Delta tC_{10}\left\{\sum_{j=1}^{J-1}F(v_j^{n+1})h+\|\delta v_h^{n+1}\|_2^2\right.$$
$$\left.+\sum_{j=1}^{J-1}\left|\frac{v_j^{n+1}-v_j^n}{\Delta t^2}\right|^2 h\right\}+\Delta tC_{11}.$$

Combining the above obtained estimations, (14) can be replaced by the following inequality:

$$\sum_{j=1}^{J-1}\left|\frac{v_j^{n+1}-v_j^n}{\Delta t}\right|^2 h+\frac{\alpha\lambda}{2}\|\delta v_h^{n+1}\|_2^2+2\lambda\sum_{j=1}^{J-1}F(v_j^{n+1})h$$
$$\leqslant\Delta tC_{12}\sum_{j=1}^{J-1}\left|\frac{v_j^{n+1}-v_j^n}{\Delta t}\right|^2 h+\lambda\Delta tC_{13}\|\delta v_h^{n+1}\|_2^2$$
$$+\lambda\Delta tC_{14}\sum_{j=1}^{J-1}F(v_j^{n+1})h+\Delta tC_{15}$$

or

$$(1-\Delta tC_{12})\sum_{j=1}^{J-1}\left|\frac{v_j^{n+1}-v_j^n}{\Delta t}\right|^2 h+\lambda\left(\frac{\alpha}{2}-\Delta t\, C_{13}\right)\|\delta v_h^{n+1}\|_2^2$$
$$+\lambda(2-\Delta tC_{14})\sum_{j=1}^{J-1}F(v_j^{n+1})h\leqslant\Delta tC_{15}.$$

When Δt is sufficiently small, there is

$$\sum_{j=1}^{J-1}\left|\frac{v_j^{n+1}-v_j^n}{\Delta t}\right|^2 h\leqslant C_{16},$$

where C_{16} is independent of parameter $0\leqslant\lambda\leqslant1$ and depends on the known vectors v_j^n, v_j^{n-1} $(j=0, 1, \cdots, J)$ and the positive constants h, Δt and α. This implices

$$|v_j^{n+1}|\leqslant C_{17},\quad j=1, 2, \cdots, J-1,$$

where C_{17} is independent of $0\leqslant\lambda\leqslant1$.

From $(6)_\lambda$, we get

$$-\left(\frac{\Delta_+v_0^{n+1}}{h}, \frac{v_0^{n+1}-v_0^n}{\Delta t}\right)+\left(\frac{\Delta_-v_J^{n+1}}{h}, \frac{v_J^{n+1}-v_J^n}{\Delta t}\right)$$

$$=-\left(\frac{\Delta_+v_0^{n+1}}{h}, \lambda\Phi_0^{n+\alpha}\right)-\left(\frac{\Delta_-v_J^{n+1}}{h}, \lambda\Phi_1^{n+\alpha}\right)$$

$$\leqslant\frac{\lambda\alpha^2}{4\sigma}\left\{\left|\Phi_0\left((1-\alpha)\frac{\Delta_+v_0^n}{h}, (1-\alpha)\frac{\Delta_-v_J^n}{h}, v_1^{n+\alpha}, v_{J-1}^{n+\alpha}, t^{n+\alpha}\right)\right|^2\right.$$

$$\left.+\left|\Phi_1\left((1-\alpha)\frac{\Delta_+v_0^n}{h}, (1-\alpha)\frac{\Delta_-v_J^n}{h}, v_1^{n+\alpha}, v_{J-1}^{n+\alpha}, t^{n+\alpha}\right)\right|^2\right\}\leqslant C_{18}.$$

On the other words, we have

$$(v_0^{n+1}-v_1^{n+1}, v_0^{n+1}-v_0^n)+(v_J^{n+1}-v_{J-1}^{n+1}, v_J^{n+1}-v_J^n)\leqslant C_{18}h\Delta t.$$

Hence

$$|v_0^{n+1}|^2+|v_J^{n+1}|^2\leqslant C_{19},$$

where C_{19} is a constant, independent of $0\leqslant\lambda\leqslant1$ and dependent on v_j^n, v_j^{n-1} $(j=0, 1, \cdots, J)$, h, Δt and α.

Hence this completes the proof of the uniform boundedness of all possible solutions v_j^{n+1} $(j=0, 1, \cdots, J)$ for the nonlinear system $(5)_\lambda$ and $(6)_\lambda$ with respect to λ. By the usual argument of fixed point technique, we obtain the existence of the solution v_j^{n+1} for $(5)_\lambda$ and $(6)_\lambda$.

Lemma 1. *Suppose that the conditions* (I), (II), (III) *and* (IV) *are satisfied and suppose that* $\alpha\sigma\Delta t<h$ *and* Δt *is sufficiently small. The solution* v_j^n $(j=0, 1, \cdots, J;$ $n=0, 1, \cdots, N)$ *of the finite difference nonlinear system* $(5)_h$, $(6)_h$, $(7')_h$ *and* $(7'')_h$ *corresponding to the nonlinear mutual boundary problem* (6) *and* (7) *for the system* (5) *of nonlinear wave equations exists for any* $0\leqslant\alpha\leqslant1$.

§ 3. A Priori Estimations

5. In order to establish the convergence to limit of the finite difference scheme mentioned above, we want to get a series of estimates of the finite difference solution v_j^n $(j=0, 1, \cdots, J;\ n=0, 1, \cdots, N)$ for the finite difference nonlinear system $(5)_h$, $(6)_h$, $(7')_h$ and $(7'')_h$ corresponding to the ordinary boundary problem (6) and (7) for the system (5) of the nonlinear wave equations.

Taking the scalar product of the m-dimensional vector $\frac{v_j^{n+1}-v_j^n}{\Delta t}h\Delta t$ with the m-dimensional vector equation $(5)_h$ and summing up the resulting relations for $j=1, 2, \cdots, J-1$, we have

$$\sum_{j=1}^{J-1}\left(\frac{v_j^{n+1}-v_j^n}{\Delta t}, \frac{v_j^{n+1}-2v_j^n+v_j^{n-1}}{\Delta t^2}\right)h\Delta t-\sum_{j=1}^{J-1}\left(\frac{v_j^{n+1}-v_j^n}{\Delta t}, \frac{\Delta_+\Delta_-v_j^{n+\alpha}}{h^2}\right)h\Delta t$$

$$+\sum_{j=1}^{J-1}\left(\frac{v_j^{n+1}-v_j^n}{\Delta t}, \int_0^1\operatorname{grad} F(v_j^{n+\tau})d\tau\right)h\Delta t=\sum_{j=1}^{J-1}\left(\frac{v_j^{n+1}-v_j^n}{\Delta t} f_j^{n+\alpha}\right)h\Delta t, \tag{17}$$

where $n=1, 2, \cdots, N-1$. Here we see that (15) and (16) are valid.

Suppose that $\frac{1}{2}\leqslant\alpha\leqslant1$. Then the first part of the right side of (16) becomes

$$\sum_{j=0}^{J-1}\left(\frac{\Delta_+v_j^{n+\alpha}}{h},\frac{\Delta_+v_j^{n+1}}{h}-\frac{\Delta_+v_j^n}{h}\right)h$$
$$=\alpha\sum_{j=0}^{J-1}\left|\frac{\Delta_+v_j^{n+1}}{h}\right|^2h+(1-2\alpha)\sum_{j=0}^{J-1}\left(\frac{\Delta_+v_j^{n+1}}{h},\frac{\Delta_+v_j^n}{h}\right)h$$
$$-(1-\alpha)\sum_{j=0}^{J-1}\left|\frac{\Delta_+v_j^n}{h}\right|^2h\leqslant\frac{1}{2}\|\delta v_h^{n+1}\|_2^2-\frac{1}{2}\|\delta v_h^n\|_2^2. \tag{18}$$

And for the remaining part of the right hand side of (16), we can derive as follows:

$$-\left(\frac{v_0^{n+1}-v_0^n}{\Delta t},\frac{\Delta_+v_0^{n+\alpha}}{h}\right)+\left(\frac{v_J^{n+1}-v_J^n}{\Delta t},\frac{\Delta_-v_J^{n+\alpha}}{h}\right)$$
$$\leqslant\frac{1}{4\sigma}\{|\Phi_0(0,0,v_1^{n+\alpha},v_{J-1}^{n+\alpha},t^{n+\alpha})|^2+|\Phi_1(0,0,v_1^{n+\alpha},v_{J-1}^{n+\alpha},t^{n+\alpha})|^2\}$$
$$\leqslant C_1\{|v_1^{n+\alpha}|^2+|v_{J-1}^{n+\alpha}|^2\}+C_1\leqslant 2C_1(\max_{j=1,2,\cdots,J-1}|v_j^{n+\alpha}|)^2+C_1$$
$$\leqslant C_2\|\delta v_h^{n+\alpha}\|_2^2+C_2\sum_{j=1}^{J-1}|v_j^{n+\alpha}|^2h+C_1,$$

where C_1 and C_2 are constants independent of h and Δt. Here

$$\|\delta v_h^{n+\alpha}\|_2^2\leqslant 2\alpha^2\|\delta v_h^{n+1}\|_2^2+2(1-\alpha)^2\|\delta v_h^n\|_2^2,$$
$$\sum_{j=1}^{J-1}|v_j^{n+\alpha}|^2h\leqslant 2\alpha^2\sum_{j=1}^{J-1}|v_j^{n+1}|^2h+2(1-\alpha)^2\sum_{j=1}^{J-1}|v_j^n|^2h$$

and

$$\sum_{j=1}^{J-1}|v_j^n|^2h\leqslant 2\sum_{j=1}^{J-1}|\varphi_j|^2h+2t^n\sum_{j=1}^{J-1}\sum_{k=1}^{n}\left|\frac{v_j^k-v_j^{k-1}}{\Delta t}\right|^2h\Delta t.$$

Hence we have

$$-\left(\frac{v_0^{n+1}-v_0^n}{\Delta t},\frac{\Delta_+v_0^{n+\alpha}}{h}\right)+\left(\frac{v_J^{n+1}-v_J^n}{\Delta t},\frac{\Delta_-v_J^{n+\alpha}}{h}\right)$$
$$\leqslant C_3\|\delta v_h^{n+1}\|_2^2+C_3\|\delta v_h^n\|_2^2+C_3t^{n+1}\sum_{j=1}^{J-1}\sum_{k=0}^{n}\left|\frac{v_j^{k+1}-v_j^k}{\Delta t}\right|^2h\Delta t$$
$$+C_3\sum_{j=1}^{J-1}|\varphi_j|^2h+C_3. \tag{19}$$

For the right part of (17), from the condition (II), there is

$$\sum_{j=1}^{J-1}\left(\frac{v_j^{n+1}-v_j^n}{\Delta t},f_j^{n+\alpha}\right)h\leqslant\frac{1}{2}\sum_{j=1}^{J-1}\left|\frac{v_j^{n+1}-v_j^n}{\Delta t}\right|^2h+\frac{1}{2}\sum_{j=1}^{J-1}|f_j^{n+\alpha}|^2h$$
$$\leqslant\frac{1}{2}\sum_{j=1}^{J-1}\left|\frac{v_j^{n+1}-v_j^n}{\Delta t}\right|^2h+C_4\sum_{j=1}^{J-1}\left|\frac{v_j^{n+\alpha}-v_j^{n+\alpha-1}}{\Delta t}\right|^2h$$
$$+C_4\sum_{j=1}^{J-1}\left|\frac{\Delta_+v_j^{n+\alpha}}{h}\right|^2h+C_4\sum_{j=1}^{J-1}\left|\frac{\Delta_-v_j^{n+\alpha}}{h}\right|^2h+C_4\sum_{j=1}^{J-1}|v_j^{n+1}|^2h$$
$$+C_4\sum_{j=1}^{J-1}|v_j^n|^2h+C_4\sum_{j=1}^{J-1}|v_j^{n-1}|^2h+C_4\sum_{j=1}^{J-1}F(v_j^{n+\alpha})h+C_4.$$

Since $F(u)$ is convex, then

$$\sum_{j=1}^{J-1}F(v_j^{n+\alpha})h\leqslant\alpha\sum_{j=1}^{J-1}F(v_j^{n+1})h+(1-\alpha)\sum_{j=1}^{J-1}F(v_j^n)h.$$

By the similar way mentioned above, we have

$$\sum_{j=1}^{J-1}\left(\frac{v_j^{n+1}-v_j^n}{\Delta t}, f_j^{n+\alpha}\right)h \leqslant C_5\left(\sum_{j=1}^{J-1}\left|\frac{v_j^{n+1}-v_j^n}{\Delta t}\right|^2 h+\sum_{j=1}^{J-1}\left|\frac{v_j^n-v_j^{n-1}}{\Delta t}\right|^2 h\right)$$
$$+C_5(\|\delta v_h^{n+1}\|_2^2+\|\delta v_h^n\|_2^2)+C_5\left(\sum_{j=1}^{J-1}F(v_j^{n+1})h+\sum_{j=1}^{J-1}F(v_j^n)h\right)$$
$$+C_5 t^{n+1}\sum_{j=1}^{J-1}\sum_{k=0}^{n}\left|\frac{v_j^{k+1}-v_j^k}{\Delta t}\right|^2 h\Delta t+C_5,$$
$$n=1, 2, \cdots, N-1, \tag{20}$$

where C_5 is a constant independent of Δt and h.

By means of the inequalities (15), (16), (18), (19) and (20), (17) can be replaced by the following inequality:

$$(1-\Delta t C_6)\left\{\sum_{j=1}^{J-1}\left|\frac{v_j^{n+1}-v_j^n}{\Delta t}\right|^2 h+\|\delta v_h^{n+1}\|_2^2+\sum_{j=1}^{J-1}F(v_j^{n+1})h\right\}$$
$$-(1+\Delta t C_6)\left\{\sum_{j=1}^{J-1}\left|\frac{v_j^n-v_j^{n-1}}{\Delta t}\right|^2 h+\|\delta v_h^n\|_2^2+\sum_{j=1}^{J-1}F(v_j^n)h\right\}$$
$$\leqslant C_6\Delta t\sum_{j=1}^{J-1}\sum_{k=0}^{n}\left|\frac{v_j^{k+1}-v_j^k}{\Delta t}\right|^2 h\Delta t+C_6\Delta t\left(\sum_{j=1}^{J-1}|\varphi_j|^2 h\right)+C_6\Delta t. \tag{21}$$

Let us denote

$$W_n=\sum_{j=1}^{J-1}\left|\frac{v_j^n-v_j^{n-1}}{\Delta t}\right|^2 h+\|\delta v_h^n\|_2^2+\sum_{j=1}^{J-1}F(v_j^n)h, \tag{22}$$

where $n=1, 2, \cdots, N$. Then (21) becomes

$$W_{n+1}-W_n\leqslant \Delta t C_6(W_{n+1}+W_n)+\Delta t C_7+C_6\Delta t\sum_{k=1}^{n+1}W_k\Delta t,\ \ n=1, 2, \cdots, N-1,$$

where $C_7=C_6\left(1+\sum_{j=1}^{J-1}|\varphi_j|^2 h\right)$. Hence we have

$$W_{n+1}-W_1\leqslant C_6\sum_{k=1}^{n}(W_{k+1}+W_k)\Delta t+n\Delta t C_7+C_6\sum_{l=2}^{n}\sum_{k=1}^{l}W_k\Delta t^2.$$

This follows that

$$W_{n+1}\leqslant(2C_6+T)\sum_{k=1}^{n+1}W_k\Delta t+(C_7T+W_1).$$

It can be verified from this recurring relation, that

$$W_n\leqslant(C_7T+W_1)(1-2C_6\Delta t-T\Delta t)^{-n}.$$

Therefore we get the following lemma.

Lemma 2. *Suppose that the conditions* (I), (II), (III) *and* (IV) *are fulfilled and suppose that* $\alpha\sigma\Delta t<h$, $\frac{1}{2}\leqslant\alpha\leqslant 1$ *and* Δt *is sufficiently small. For the m-dimensional discrete vector solution* v_j^n ($j=0, 1, \cdots, J$; $n=0, 1, \cdots, N$) *of the nonlinear finite difference system* $(5)_h$, $(6)_h$, $(7')_h$ *and* $(7'')_h$, *there are estimates*

$$\max_{n=1,2,\cdots,N}\sum_{j=1}^{J-1}\left|\frac{v_j^n-v_j^{n-1}}{\Delta t}\right|^2 h+\max_{n=0,1,\cdots,N}\|\delta v_h^n\|_2^2+\max_{n=0,1,\cdots,N}\sum_{j=1}^{J-1}F(v_j^n)h\leqslant K_1, \tag{23}$$

where K_1 *is a constant independent of* Δt *and* h.

6. Lemma 3. *Under the conditions of Lemma* 2, *there is the estimate*

$$\max_{\substack{n=0,1,\cdots,N\\ j=0,1,\cdots,J}}|v_j^n|\leqslant K_2, \tag{24}$$

where K_2 *is a constant independent of* Δt *and* h.

Proof. From the first estimate of (24) and from

$$\sum_{j=1}^{J-1}|v_j^n|^2h\leqslant 2\sum_{j=1}^{J-1}|\varphi_j|^2h+2t^n\sum_{j=1}^{J-1}\sum_{k=1}^{n}\left|\frac{v_j^k-v_j^{k-1}}{\Delta t}\right|^2h\Delta t,$$

there is immediately

$$\max_{n=0,1,\cdots,N}\sum_{j=1}^{J-1}|v_j^n|^2h\leqslant C_8,$$

where C_8 is a constant. From the second estimate of (24), we have

$$\max_{n=0,1,\cdots,N}\sum_{j=1}^{J-2}\left|\frac{\Delta_+v_j^n}{h}\right|^2h\leqslant K_1.$$

Then by use of the interpolation formula for the discrete functions[23, 24], we get

$$\max_{\substack{n=0,1,\cdots,N\\ j=1,2,\cdots,J-1}}|v_j^n|\leqslant C_9.$$

Since

$$|v_0^n|\leqslant|v_1^n|+\|\delta v_h^n\|_2\sqrt{h},$$
$$|v_J^n|\leqslant|v_{J-1}^n|+\|\delta v_h^n\|_2\sqrt{h},$$

then the estimate (24) is valid for any $j=0, 1, \cdots, J$ and $n=0, 1, \cdots, N$. The lemma is proved.

7. Again we make the scalar product of the m-dimensional vector $\dfrac{\Delta_+\Delta_-(v_j^{n+\alpha}-v_j^{n+\alpha-1})}{h^2\Delta t}$ with the m-dimensional vector equation $(5)_h$ and sum up the resulting relations for $j=0, 1, \cdots, J-1$. Then we obtain the equality

$$\begin{aligned}&\sum_{j=1}^{J-1}\left(\frac{\Delta_+\Delta_-(v_j^{n+\alpha}-v_j^{n+\alpha-1})}{h^2\Delta t},\frac{v_j^{n+1}-2v_j^n+v_j^{n-1}}{\Delta t^2}\right)h\Delta t\\&\quad-\sum_{j=1}^{J-1}\left(\frac{\Delta_+\Delta_-(v_j^{n+\alpha}-v_j^{n+\alpha-1})}{h^2\Delta t},\frac{\Delta_+\Delta_-v_j^{n+\alpha}}{h^2}\right)h\Delta t\\&\quad+\sum_{j=1}^{J-1}\left(\frac{\Delta_+\Delta_-(v_j^{n+\alpha}-v_j^{n+\alpha-1})}{h^2\Delta t},\int_0^1\operatorname{grad}F(v_j^{n+\tau})d\tau\right)h\Delta t\\&=\sum_{j=1}^{J-1}\left(\frac{\Delta_+\Delta_-(v_j^{n+\alpha}-v_j^{n+\alpha-1})}{h^2\Delta t},\ f_j^{n+\alpha}\right)h\Delta t,\\&\quad j=1,2,\cdots,J-1;\ n=1,2,\cdots,N-1.\end{aligned}\tag{25}$$

For the first summation of (25), we have

$$\begin{aligned}&\sum_{j=1}^{J-1}\left(\frac{\Delta_+\Delta_-(v_j^{n+\alpha}-v_j^{n+\alpha-1})}{h^2\Delta t},\frac{v_j^{n+1}-2v_j^n+v_j^{n-1}}{\Delta t^2}\right)h\Delta t\\&=-\sum_{j=1}^{J-1}\left(\frac{\Delta_+(v_j^{n+\alpha}-v_j^{n+\alpha-1})}{h\Delta t},\frac{\Delta_+(v_j^{n+1}-2v_j^n+v_j^{n-1})}{h\Delta t^2}\right)h\Delta t\\&\quad-\left(\frac{\Delta_+(v_0^{n+\alpha}-v_0^{n+\alpha-1})}{h\Delta t},\frac{v_0^{n+1}-2v_0^n+v_0^{n-1}}{\Delta t^2}\right)\Delta t\\&\quad+\left(\frac{\Delta_-(v_J^{n+\alpha}-v_J^{n+\alpha-1})}{h\Delta t},\frac{v_J^{n+1}-2v_J^n+v_J^{n-1}}{\Delta t^2}\right)\Delta t,\\&\quad n=1,2,\cdots,N-1.\end{aligned}\tag{26}$$

Here since $\dfrac{1}{2}\leqslant\alpha\leqslant 1$, there is

$$\sum_{j=1}^{J-1}\left(\frac{\Delta_+(v_j^{n+\alpha}-v_j^{n+\alpha-1})}{h\Delta t},\frac{\Delta_+(v_j^{n+1}-2v_j^n+v_j^{n-1})}{h\Delta t^2}\right)h\Delta t$$
$$\leqslant\frac{1}{2}\left\|\delta\left(\frac{v_h^{n+1}-v_h^n}{\Delta t}\right)\right\|_2^2-\frac{1}{2}\left\|\delta\left(\frac{v_h^n-v_h^{n-1}}{\Delta t}\right)\right\|_2^2. \tag{27}$$

For the remaining part, we have

$$\begin{aligned}Z^{n+1}\equiv&-\left(\frac{\Delta_+(v_0^{n+\alpha}-v_0^{n+\alpha-1})}{h\Delta t},\frac{v_0^{n+1}-2v_0^n+v_0^{n-1}}{\Delta t^2}\right)\\&+\left(\frac{\Delta_-(v_J^{n+\alpha}-v_J^{n+\alpha-1})}{h\Delta t},\frac{v_J^{n+1}-2v_J^n+v_J^{n-1}}{\Delta t^2}\right)\\=&-\left(\frac{\Delta_+(v_0^{n+\alpha}-v_0^{n+\alpha-1})}{h\Delta t},\frac{\Phi_0^{n+\alpha}-\Phi_0^{n+\alpha-1}}{\Delta t}\right)\\&+\left(\frac{\Delta_-(v_J^{n+\alpha}-v_J^{n+\alpha-1})}{h\Delta t},\frac{\Phi_1^{n+\alpha}-\Phi_1^{n+\alpha-1}}{\Delta t}\right).\end{aligned} \tag{28}$$

Here we see that

$$\begin{aligned}\frac{\Phi_0^{n+\alpha}-\Phi_0^{n+\alpha-1}}{\Delta t}=&\left(\int_0^1\Phi_{0p_0}^\tau d\tau\right)\frac{\Delta_+(v_0^{n+\alpha}-v_0^{n+\alpha-1})}{h\Delta t}+\left(\int_0^1\Phi_{0p_1}^\tau d\tau\right)\frac{\Delta_-(v_J^{n+\alpha}-v_J^{n+\alpha-1})}{h\Delta t}\\&+\left(\int_0^1\Phi_{0u_0}^\tau d\tau\right)\frac{v_1^{n+\alpha}-v_1^{n+\alpha-1}}{\Delta t}\\&+\left(\int_0^1\Phi_{0u_1}^\tau d\tau\right)\frac{v_{J-1}^{n+\alpha}-v_{J-1}^{n+\alpha-1}}{\Delta t}+\left(\int_0^1\Phi_{0t}^\tau d\tau\right)\end{aligned}$$

and
$$\begin{aligned}\frac{\Phi_1^{n+\alpha}-\Phi_1^{n+\alpha-1}}{\Delta t}=&\left(\int_0^1\Phi_{1p_0}^\tau d\tau\right)\frac{\Delta_+(v_0^{n+\alpha}-v_0^{n+\alpha-1})}{h\Delta t}+\left(\int_0^1\Phi_{1p_1}^\tau d\tau\right)\frac{\Delta_-(v_J^{n+\alpha}-v_J^{n+\alpha-1})}{h\Delta t}\\&+\left(\int_0^1\Phi_{1u_0}^\tau d\tau\right)\frac{v_1^{n+\alpha}-v_1^{n+\alpha-1}}{\Delta t}\\&+\left(\int_0^1\Phi_{1u_1}^\tau d\tau\right)\frac{v_{J-1}^{n+\alpha}-v_{J-1}^{n+\alpha-1}}{\Delta t}+\left(\int_0^1\Phi_{1t}^\tau d\tau\right),\end{aligned}$$

where the abbreviations are as

$$\begin{aligned}\Phi_0^\tau\equiv\Phi_0\Big(&\tau\frac{\Delta_+v_0^{n+\alpha}}{h}+(1-\tau)\frac{\Delta_+v_0^{n+\alpha-1}}{h},\ \tau\frac{\Delta_-v_J^{n+\alpha}}{h}+(1-\tau)\frac{\Delta_-v_J^{n+\alpha-1}}{h},\\&\tau v_1^{n+\alpha}+(1-\tau)v_1^{n+\alpha-1},\ \tau v_{J-1}^{n+\alpha}+(1-\tau)v_{J-1}^{n+\alpha-1},\ t^{n+\alpha+\tau-1}\Big)\end{aligned}$$

and others. Since the $2m\times 2m$ Jacobi derivative matrix $\frac{\partial(\Phi_0,\Phi_1)}{\partial(p_0,p_1)}=\begin{pmatrix}\Phi_{0p_0}&\Phi_{0p_1}\\\Phi_{1p_0}&\Phi_{1p_1}\end{pmatrix}$ of $2m$-dimensional vector functions $\Phi_0(p_0,p_1,u_0,u_1,t)$ and $\Phi_1(p_0,p_1,u_0,u_1,t)$ with respect to $2m$-dimensional vector variable (p_0,p_1) is positively definite for $t\in[0,T]$ and $p_0,p_1,u_0,u_1\in\mathbb{R}^m$, then we have

$$\begin{aligned}Z^{n+1}\leqslant&-\frac{\sigma}{4}\left\{\left|\frac{\Delta_+(v_0^{n+\alpha}-v_0^{n+\alpha-1})}{h\Delta t}\right|^2+\left|\frac{\Delta_-(v_J^{n+\alpha}-v_J^{n+\alpha-1})}{h\Delta t}\right|^2\right\}\\&+\frac{1}{\sigma}\left\{\left|\left(\int_0^1\Phi_{0u_0}^\tau d\tau\right)\frac{v_1^{n+\alpha}-v_1^{n+\alpha-1}}{\Delta t}\right|^2+\left|\left(\int_0^1\Phi_{0u_1}^\tau d\tau\right)\frac{v_{J-1}^{n+\alpha}-v_{J-1}^{n+\alpha-1}}{\Delta t}\right|^2\right.\\&+\left|\left(\int_0^1\Phi_{1u_0}^\tau d\tau\right)\frac{v_1^{n+\alpha}-v_1^{n+\alpha-1}}{\Delta t}\right|^2+\left|\left(\int_0^1\Phi_{1u_0}^\tau d\tau\right)\frac{v_{J-1}^{n+\alpha}-v_{J-1}^{n+\alpha-1}}{\Delta t}\right|^2\\&\left.+\left|\int_0^1\Phi_{0t}^\tau d\tau\right|^2+\left|\int_0^1\Phi_{1t}^\tau d\tau\right|^2\right\}.\end{aligned}$$

For the terms on the right part of above inequality, we can derive for example as follows:

$$\begin{aligned}
&\left|\left(\int_0^1 \Phi^{\tau}_{0u_t}\,d\tau\right)\frac{v_1^{n+\alpha}-v_1^{n+\alpha-1}}{\Delta t}\right|^2\\
&\leqslant C_{10}\left(\left|\frac{\Delta_+ v_0^{n+\alpha}}{h}\right|^2+\left|\frac{\Delta_+ v_0^{n+\alpha-1}}{h}\right|^2+\left|\frac{\Delta_- v_J^{n+\alpha}}{h}\right|^2+\left|\frac{\Delta_- v_J^{n+\alpha-1}}{h}\right|^2+1\right)\\
&\quad\cdot\left(\left|\frac{v_1^{n+1}-v_1^n}{\Delta t}\right|^2+\left|\frac{v_1^n-v_1^{n-1}}{\Delta t}\right|^2\right)\\
&\leqslant C_{10}(\|\delta v_h^{n+\alpha}\|_\infty^2+\|\delta v_h^{n+\alpha-1}\|_\infty^2+1)\\
&\quad\cdot\left(\max_{j=1,2,\cdots,J-1}\left|\frac{v_j^{n+1}-v_j^n}{\Delta t}\right|^2+\max_{j=1,2,\cdots,J-1}\left|\frac{v_j^n-v_j^{n-1}}{\Delta t}\right|^2\right)\\
&\leqslant C_{11}(\|\delta v_h^{n+\alpha}\|_2\|\delta^2 v_h^{n+\alpha}\|_2+\|\delta v_h^{n+\alpha-1}\|_2\|\delta^2 v_h^{n+\alpha-1}\|_2+1)\\
&\quad\cdot\left\{\left(\sum_{j=1}^{J-1}\left|\frac{v_j^{n+1}-v_j^n}{\Delta t}\right|^2 h\right)^{\frac12}\left\|\delta\left(\frac{v_h^{n+1}-v_h^n}{\Delta t}\right)\right\|_2+\sum_{j=1}^{J-1}\left|\frac{v_j^{n+1}-v_j^n}{\Delta t}\right|^2 h\right.\\
&\quad\left.+\left(\sum_{j=1}^{J-1}\left|\frac{v_j^n-v_j^{n-1}}{\Delta t}\right|^2 h\right)^{\frac12}\left\|\delta\left(\frac{v_h^n-v_h^{n-1}}{\Delta t}\right)\right\|_2+\sum_{j=1}^{J-1}\left|\frac{v_j^n-v_j^{n-1}}{\Delta t}\right|^2 h\right\}\\
&\leqslant C_{12}\left\{(\|\delta^2 v_h^{n+\alpha}\|_2+\|\delta^2 v_h^{n+\alpha-1}\|_2+1)\right.\\
&\quad\left.\cdot\left(\left\|\delta\left(\frac{v_h^{n+1}-v_h^n}{\Delta t}\right)\right\|_2+\left\|\delta\left(\frac{v_h^n-v_h^{n-1}}{\Delta t}\right)\right\|_2+1\right)\right\}
\end{aligned}$$

and

$$\begin{aligned}
\left|\int_0^1 \Phi^{\tau}_{0t}\,d\tau\right|^2 &\leqslant C_{13}\left\{\left|\frac{\Delta_+ v_0^{n+\alpha}}{h}\right|^4+\left|\frac{\Delta_+ v_0^{n+\alpha-1}}{h}\right|^4+\left|\frac{\Delta_- v_J^{n+\alpha}}{h}\right|^4+\left|\frac{\Delta_- v_J^{n+\alpha-1}}{h}\right|^4+1\right\}\\
&\leqslant C_{14}\{\|\delta^2 v_h^{n+\alpha}\|_2^2+\|\delta^2 v_h^{n+\alpha-1}\|_2^2+1\}.
\end{aligned}$$

Finally we get the estimate

$$\begin{aligned}
Z^{n+1}\leqslant &-\frac{\sigma}{4}\left\{\left|\frac{\Delta_+(v_0^{n+\alpha}-v_0^{n+\alpha-1})}{h\Delta t}\right|^2+\left|\frac{\Delta_-(v_J^{n+\alpha}-v_J^{n+\alpha-1})}{h\Delta t}\right|^2\right\}\\
&+C_{15}(\|\delta^2 v_h^{n+\alpha}\|_2^2+\|\delta^2 v_h^{n+\alpha-1}\|_2^2)\\
&+C_{15}\left(\left\|\delta\left(\frac{v_h^{n+1}-v_h^n}{\Delta t}\right)\right\|_2^2+\left\|\delta\left(\frac{v_h^n-v_h^{n-1}}{\Delta t}\right)\right\|_2^2\right)+C_{15}. \qquad (29)
\end{aligned}$$

As to the second term of (25), it is easy to see that

$$\sum_{j=1}^{J-1}\left(\frac{\Delta_+\Delta_-(v_j^{n+\alpha}-v_j^{n+\alpha-1})}{h^2\Delta t},\frac{\Delta_+\Delta_- v_j^{n+\alpha}}{h^2}\right)h\Delta t\leqslant\frac12\|\delta^2 v_h^{n+\alpha}\|_2^2-\frac12\|\delta^2 v_h^{n+\alpha-1}\|_2^2. \qquad (30)$$

The third term of (25) can be expressed as

$$\begin{aligned}
&\sum_{j=1}^{J-1}\left(\frac{\Delta_+\Delta_-(v_j^{n+\alpha}-v_j^{n+\alpha-1})}{h^2\Delta t},\int_0^1 \operatorname{grad} F(v_j^{n+\tau})\,d\tau\right)h\Delta t\\
&=-\sum_{j=0}^{J-1}\left(\frac{\Delta_+(v_j^{n+\alpha}-v_j^{n+\alpha-1})}{h\Delta t},\int_0^1\int_0^1 \operatorname{grad}^2 F(v_{j+\xi}^{n+\tau})\,\frac{\Delta_+ v_j^{n+\tau}}{h}\,d\tau\,d\xi\right)h\Delta t\\
&\quad-\left(\frac{\Delta_+(v_0^{n+\alpha}-v_0^{n+\alpha-1})}{h\Delta t},\int_0^1 \operatorname{grad} F(v_0^{n+\tau})\,d\tau\right)\Delta t\\
&\quad+\left(\frac{\Delta_-(v_J^{n+\alpha}-v_J^{n+\alpha-1})}{h\Delta t},\int_0^1 \operatorname{grad} F(v_J^{n+\tau})\,d\tau\right)\Delta t, \qquad (31)
\end{aligned}$$

where

$$v_{j+\xi}^{n+\tau}=\xi\tau v_{j+1}^{n+1}+\xi(1-\tau)v_{j+1}^{n}+(1-\xi)\tau v_j^{n+1}+(1-\xi)(1-\tau)v_j^n.$$

The last two terms of the above equality can be dominated by the expression

$$\frac{\sigma\Delta t}{8}\left\{\left|\frac{\Delta_+(v_0^{n+\alpha}-v_0^{n+\alpha-1})}{h\Delta t}\right|^2+\left|\frac{\Delta_-(v_J^{n+\alpha}-v_J^{n+\alpha-1})}{h\Delta t}\right|^2\right\}+C_{16}\Delta t, \tag{32}$$

where C_{16} is a constant independent of Δt and h. And for the first term of the right part of the above equality, we have

$$\begin{aligned}&\sum_{j=1}^{J-1}\left(\frac{\Delta_+(v_j^{n+\alpha}-v_j^{n+\alpha-1})}{h\Delta t},\int_0^1\int_0^1 \operatorname{grad}^2F(v_{j+\xi}^{n+\tau})\frac{\Delta_+v_j^{n+\tau}}{h}\,d\tau\,d\xi\right)h\Delta t\\ &\leqslant \Delta tC_{17}\left(\left\|\delta\left(\frac{v_h^{n+1}-v_h^n}{\Delta t}\right)\right\|_2^2+\left\|\delta\left(\frac{v_h^n-v_h^{n-1}}{\Delta t}\right)\right\|_2^2\right)\\ &\quad+\Delta tC_{17}(\|\delta v_h^{n+1}\|_2^2+\|\delta v_h^n\|_2^2).\end{aligned} \tag{33}$$

Now it remains only to estimate the right term of (25). Thus we have

$$\begin{aligned}&\sum_{j=1}^{J-1}\left(\frac{\Delta_+\Delta_-(v_j^{n+\alpha}-v_j^{n+\alpha-1})}{h^2\Delta t},f_j^{n+\alpha}\right)h\Delta t\\ &=-\sum_{j=1}^{J-2}\left(\frac{\Delta_+(v_j^{n+\alpha}-v_j^{n+\alpha-1})}{h\Delta t},\frac{\Delta_+f_j^{n+\alpha}}{h}\right)h\Delta t\\ &\quad-\left(\frac{\Delta_+(v_0^{n+\alpha}-v_0^{n+\alpha-1})}{h\Delta t},f_1^{n+\alpha}\right)\Delta t+\left(\frac{\Delta_-(v_J^{n+\alpha}-v_J^{n+\alpha-1})}{h\Delta t},f_{J-1}^{n+\alpha}\right)\Delta t.\end{aligned} \tag{34}$$

The sum of the last two terms can be dominated by the following expression

$$\begin{aligned}&\frac{\sigma\Delta t}{8}\left\{\left|\frac{\Delta_+(v_0^{n+\alpha}-v_0^{n+\alpha-1})}{h\Delta t}\right|^2+\left|\frac{\Delta_-(v_J^{n+\alpha}-v_J^{n+\alpha-1})}{h\Delta t}\right|^2\right\}\\ &\qquad+\frac{2\Delta t}{\sigma}\{|f_1^{n+\alpha}|^2+|f_{J-1}^{n+\alpha}|^2\}.\end{aligned} \tag{35}$$

Here from the condition (II), we have

$$\begin{aligned}|f_1^{n+\alpha}|^2+|f_{J-1}^{n+\alpha}|^2&\leqslant C_{18}\left\{\left|\frac{\Delta_+v_1^{n+\alpha}}{h}\right|^2+\left|\frac{\Delta_-v_1^{n+\alpha}}{h}\right|^2+\left|\frac{\Delta_+v_{J-1}^{n+\alpha}}{h}\right|^2+\left|\frac{\Delta_-v_{J-1}^{n+\alpha}}{h}\right|^2\right.\\ &\qquad\left.+\left|\frac{v_1^{n+\alpha}-v_1^{n+\alpha-1}}{\Delta t}\right|^2+\left|\frac{v_{J-1}^{n+\alpha}-v_{J-1}^{n+\alpha-1}}{\Delta t}\right|^2\right\}+C_{18}\\ &\leqslant 4C_{18}\left\{\|\delta v_h^{n+\alpha}\|_\infty^2+\left\|\frac{v_h^{n+\alpha}-v_h^{n+\alpha-1}}{\Delta t}\right\|_\infty^2\right\}+C_{18}\\ &\leqslant C_{19}\left\{\|\delta^2v_h^{n+\alpha}\|_2^2+\left\|\delta\left(\frac{v_h^{n+1}-v_h^n}{\Delta t}\right)\right\|_2^2+\left\|\delta\left(\frac{v_h^n-v_h^{n-1}}{\Delta t}\right)\right\|_2^2\right\}+C_{19}.\end{aligned}$$

Again we have

$$\begin{aligned}&\sum_{j=1}^{J-2}\left(\frac{\Delta_+(v_j^{n+\alpha}-v_j^{n+\alpha-1})}{h\Delta t},\frac{\Delta_+f_j^{n+\alpha}}{h}\right)h\Delta t\\ &\leqslant\Delta t\left\{\left\|\delta\left(\frac{v_h^{n+1}-v_h^n}{\Delta t}\right)\right\|_2^2+\left\|\delta\left(\frac{v_h^n-v_h^{n-1}}{\Delta t}\right)\right\|_2^2\right\}+\Delta t\sum_{j=1}^{J-2}\left|\frac{\Delta_+f_j^{n+\alpha}}{h}\right|^2h.\end{aligned} \tag{36}$$

Thus from the condition (II) satisfying by the m-dimensional vector function $f(x, t, u, p, q)$, we have

$$\sum_{j=1}^{J-2}\left|\frac{\Delta_+ f_j^{n+\alpha}}{h}\right|^2 h\leqslant \sum_{j=1}^{J-2}\left|\left(\int_0^1 f_q^\tau d\tau\right)\frac{\Delta_+(v_j^{n+\alpha}-v_j^{n+\alpha-1})}{h\Delta t}\right|^2 h$$
$$+\sum_{j=1}^{J-2}\left|\left(\int_0^1 f_p^\tau d\tau\right)\left(b\,\frac{\Delta_+^2 v_j^{n+\alpha}}{h^2}+b'\,\frac{\Delta_+\Delta_- v_j^{n+\alpha}}{h^2}\right)\right|^2 h$$
$$+\sum_{j=1}^{J-2}\left|\left(\int_0^1 f_u^\tau d\tau\right)\left(\alpha a\,\frac{\Delta_+ v_j^{n+1}}{h}+a'\,\frac{\Delta_+ v_j^n}{h}+a''\,\frac{\Delta_+ v_j^{n-1}}{h}\right)\right|^2 h$$
$$+\sum_{j=1}^{J-2}\left|\int_0^1 f_x^\tau d\tau\right| h$$
$$\leqslant C_{20}\left\{\|\delta^2 v_h^{n+\alpha}\|_2^2+\left\|\delta\left(\frac{v_h^{n+1}-v_h^n}{\Delta t}\right)\right\|_2^2+\left\|\delta\left(\frac{v_h^n-v_h^{n-1}}{\Delta t}\right)\right\|_2^2\right\}+C_{20}, \quad (37)$$

where the index "τ" means that for example

$$f^\tau\equiv f\Big(x_{j+\tau},\ t^{n+\alpha},\ \tau\bar v_{j+1}^n+(1-\tau)\bar v_j^n,\ \tau\frac{\bar\Delta v_j^{n+1}}{h}+(1-\tau)\frac{\bar\Delta v_j^n}{h},$$
$$\tau\,\frac{v_{j+1}^{n+\alpha}-v_{j+1}^{n+\alpha-1}}{\Delta t}+(1-\tau)\frac{v_j^{n+\alpha}-v_j^{n+\alpha-1}}{\Delta t}\Big)$$

and so forth.

Substituting the results obtained in (26)—(37) into (25), we obtain the final inequality

$$(1-\Delta tC_{21})W_{n+1}\leqslant(1+\Delta tC_{21})W_n+\Delta tC_{21},\quad n=1,2,\cdots,N-1, \quad (38)$$

where

$$W_n=\left\|\delta\left(\frac{v_h^n-v_h^{n-1}}{\Delta t}\right)\right\|_2^2+\|\delta^2 v_h^{n+\alpha-1}\|_2^2,\quad n=1,2,\cdots,N.$$

This implices immediately

$$W_n=(TC_{21}+W_1)(1-2C_{21}t)^{-n},\quad n=2,3,\cdots,N, \quad (39)$$

where

$$W_1=\|\delta\psi_h\|_2^2+\|\delta^2(\varphi_h+\alpha\Delta t\psi_h)\|_2^2\leqslant 2\|\delta^2\varphi_h\|_2^2+\left(1+\frac{4}{\sigma^2}\right)\|\delta\psi_h\|_2^2 \quad (40)$$

for $\frac{\alpha\sigma\Delta t}{h}<1$. Hence the right part of (39) is bounded with respect to $n=1,2,\cdots,N$ for sufficiently small Δt.

Thus we obtain the following lemma.

Lemma 4. *Under the conditions of Lemma 2, for the solution v_j^n ($j=0,1,\cdots,J$; $n=0,1,\cdots,N$) of the nonlinear finite difference system $(5)_h$, $(6)_h$, $(7')_h$ and $(7'')_h$, there are the estimates*

$$\max_{n=0,1,\cdots,N-1}\left\|\delta\left(\frac{v_h^{n+1}-v_h^n}{\Delta t}\right)\right\|_2+\max_{n=0,1,\cdots,N-1}\|\delta^2 v_h^{n+\alpha}\|_2\leqslant K_3, \quad (41)$$

where K_3 is a constant independent of h and Δt.

8. **Lemma 5.** *Under the conditions of Lemma 2, for v_j^n ($j=0,1,\cdots,J$; $n=0,1,\cdots,N$), there are estimates*

$$\max_{\substack{j=0,1,\cdots,J\\ n=1,2,\cdots,N}}\left|\frac{v_j^n-v_j^{n-1}}{\Delta t}\right|+\max_{\substack{j=0,1,\cdots,J-1\\ n=0,1,\cdots,N-1}}\left|\frac{\Delta_+ v_j^{n+\alpha}}{h}\right|\leqslant K_4, \quad (42)$$

where K_4 is a constant independent of h and Δt.

Proof. From (23), we have

$$\max_{n=0,1,\cdots,N-1} \|\delta v_h^{n+\alpha}\|_2 \leqslant K_1.$$

Then from (41), it follows

$$\max_{n=0,1,\cdots,N-1} \|\delta v_h^{n+\alpha}\|_\infty \leqslant C_{22}.$$

From (23) and (41), we also get

$$\max_{\substack{j=1,2,\cdots,J-1\\ n=1,2,\cdots,N}} \left|\frac{v_j^n - v_j^{n-1}}{\Delta t}\right| \leqslant C_{23}.$$

And the boundedness of $\left|\frac{v_0^n - v_0^{n-1}}{\Delta t}\right|$ and $\left|\frac{v_J^n - v_J^{n-1}}{\Delta t}\right|$ follows directly from the discrete boundary conditions $(6)_h$. This completes the proof of the lemma.

Lemma 6. *Under the conditions of Lemma 2, there is estimate*

$$\max_{n=1,2,\cdots,N-1} \left\|\frac{v_h^{n+1} - 2v_h^n + v_h^{n-1}}{\Delta t^2}\right\|_2 \leqslant K_5, \tag{43}$$

where K_5 is a constant independent of h and Δt.

Proof. From the system $(5)_h$, we see that

$$\max_{n=1,2,\cdots,N-1} \sum_{j=1}^{J-1} \left|\frac{v_j^{n+1} - 2v_j^n + v_j^{n-1}}{\Delta t^2}\right|^2 h \leqslant C_{24}.$$

From the boundary conditions $(6)_h$, it can be verified that $\left|\frac{v_0^{n+1} - 2v_0^n + v_0^{n-1}}{\Delta t^2}\right|^2 h$ and $\left|\frac{v_J^{n+1} - 2v_J^n + v_J^{n-1}}{\Delta t^2}\right|^2 h$ are bounded. In fact for example,

$$\left|\frac{v_0^{n+1} - 2v_0^n + v_0^{n-1}}{\Delta t^2}\right|^2 h = \left|\frac{\Phi_0^{n+\alpha} - \Phi_0^{n+\alpha-1}}{\Delta t}\right|^2 h$$

$$\leqslant C_{25} \left|\frac{\Delta_+(v_0^{n+\alpha} - v^{n+\alpha-1})}{h\Delta t}\right|^2 h + C_{25} \left|\frac{\Delta_-(v_J^{n+\alpha} - v_J^{n+\alpha-1})}{h\Delta t}\right|^2 h + C_{25} h.$$

The right part of the above inequality is bounded. This completes the proof of the lemma.

§ 4. Existence of Solution

9. On the base of the estimates of the m-dimensional discrete vector solution v_j^n ($j=0, 1, \cdots, J$; $n=0, 1, \cdots, N$) of the nonlinear finite difference system $(5)_h$, $(6)_h$, $(7')_h$ and $(7'')_h$ corresponding to the general nonlinear mutual boundary problem (6) and (7) for the system (5) of nonlinear wave equations, obtained in the Lemmas 2—6 of the previous section, we obtain the following estimation relations with the help of the interpolation formulas for the discrete functions[23, 24].

Lemma 7. *Suppose that the conditions* (I), (II), (III) *and* (IV) *are satisfied and suppose that* $\frac{\alpha\sigma\Delta t}{h} < 1$ *and Δt is sufficiently small. Then for the discrete solution v_j^n ($j=0, 1, \cdots, J$; $n=0, 1, \cdots, N$) of the nonlinear system $(5)_h$, $(6)_h$, $(7')_h$ and $(7'')_h$, there are the estimation relations:*

$$|v_j^n| \leqslant K_2, \qquad j=0, 1, \cdots, J;\ n=0, 1, \cdots, N; \tag{44}$$

$$|\Delta_+ v_j^n| \leqslant K_1 h^{\frac{1}{2}}, \qquad j=0, 1, \cdots, J-1;\ n=0, 1, \cdots, N; \tag{45}$$

$$|\Delta_+ v_j^{n+\alpha}| \leqslant K_4 h, \qquad j=0, 1, \cdots, J-1;\ n=0, 1, \cdots, N-1; \tag{46}$$

$$|\Delta_+\Delta_- v_j^{n+\alpha}| \leqslant K_3 h^{\frac{3}{2}}, \qquad j=1, 2, \cdots, J-1;\ n=0, 1, \cdots, N-1; \tag{47}$$

$$|v_j^{n+1} - v_j^n| \leqslant K_4 \Delta t, \qquad j=0, 1, \cdots, J;\ n=0, 1, \cdots, N-1; \tag{48}$$

$$|\Delta_+ v_j^{n+1} - \Delta_+ v_j^n| \leqslant K_3 \Delta t h^{\frac{1}{2}}, \qquad j=0, 1, \cdots, J-1;\ n=0, 1, \cdots, N-1; \tag{49}$$

$$|v_j^{n+1} - 2v_j^n + v_j^{n-1}| \leqslant K_5 \Delta t^{\frac{3}{2}}, \qquad j=0, 1, \cdots, J;\ n=1, 2, \cdots, N-1, \tag{50}$$

where K's are constants independent of h and Δt.

10. For the purpose of the limiting process $h\to 0$ and $\Delta t\to 0$, we make some preparations as follows:

For the m-dimensional discrete vector function v_j^n ($j=0, 1, \cdots, J$; $n=0, 1, \cdots, N$), we construct a set of m-dimensional piecewise constant vector functions as follows: Let $v_{h\Delta t}(x, t) = v_j^{n+1}$ and $\bar{v}_{h\Delta t}(x, t) = \frac{\Delta_+ v_j^{n+1}}{h}$ in $Q_j^{n+1} = \{jh \leqslant x < (j+1)h;\ n\Delta t < t \leqslant (n+1)\Delta t\}$ for $j=0, 1, \cdots, J-1$; $n=0, 1, \cdots, N-1$. Let $\bar{\bar{v}}_{h\Delta t}(x, t) = \frac{\Delta_+\Delta_- v_j^{n+1}}{h^2}$ in Q_j^{n+1} for $j=1, 2, \cdots, J-1$; $n=0, 1, \cdots, N-1$ and $\bar{\bar{v}}_{h\Delta t}(x, t) = \frac{\Delta_+\Delta_- v_0^{n+1}}{h^2}$ in Q_0^{n+1} for $n=0, 1, \cdots, N-1$. Again let $\tilde{v}_{h\Delta t}(x, t) = \frac{v_j^{n+1} - v_j^n}{\Delta t}$ in Q_j^{n+1} for $j=0, 1, \cdots, J-1$; $n=0, 1, \cdots, N-1$. Let $\tilde{\tilde{v}}_{h\Delta t}(x, t) = \frac{v_j^{n+1} - 2v_j^n + v_j^{n-1}}{\Delta t^2}$ in Q_j^{n+1} for $j=0, 1, \cdots, J-1$; $n=1, 2, \cdots, N-1$ and $\tilde{\tilde{v}}_{h\Delta t}(x, t) = \frac{v_j^2 - 2v_j^1 + v_j^0}{\Delta t^2} = \frac{v_j^2 - (\varphi_j + 2\Delta t\psi_j)}{\Delta t^2}$ in Q_j^1 for $j=0, 1, \cdots, J-1$. Similarly we define $\tilde{\bar{v}}_{h\Delta t}(x, t) = \frac{\Delta_+(v_j^{n+1} - v_j^n)}{h\Delta t}$ in Q_j^{n+1} for $j=0, 1, \cdots, J-1$; $n=0, 1, \cdots, N-1$. Also we define $v_{h\Delta t}^\alpha(x, t) = v_j^{n+\alpha}$ in Q_j^{n+1} for $j=0, 1, \cdots, J-1$; $n=0, 1, \cdots, N-1$, then $v_{h\Delta t}^\alpha(x, t) = \alpha v_{h\Delta t}(x, t) + (1-\alpha) v_{h\Delta t}(x, t-\Delta t)$. Similarly we have $\bar{v}_{h\Delta t}^\alpha(x, t)$, $\bar{\bar{v}}_{h\Delta t}^\alpha(x, t)$, $\tilde{\bar{v}}_{h\Delta t}^\alpha(x, t)$ and $\tilde{v}_{h\Delta t}^\alpha(x, t)$.

From the results in Lemmas 2—6, we have the estimation relations for the above constructed vector functions:

$$\max_{0<t<T} \|v_{h\Delta t}(\cdot, t)\|_{L_2(0,l)} + \max_{1<t<T} \|\bar{v}_{h\Delta t}(\cdot, t)\|_{L_2(0,l)} + \max_{0<t<T} \|\bar{\bar{v}}_{h\Delta t}(\cdot, t)\|_{L_2(0,l)}$$
$$+ \max_{0<t<T} \|\tilde{v}_{h\Delta t}(\cdot, t)\|_{L_2(0,l)} + \max_{0<t<T} \|\tilde{\bar{v}}_{h\Delta t}(\cdot, t)\|_{L_2(0,l)} + \max_{0<t<T} \|\tilde{\tilde{v}}_{h\Delta t}(\cdot, t)\|_{L_2(0,l)} \leqslant C_{26}, \tag{51}$$

where C_{26} is a constant independent of h and Δt.

From the estimations (44), (45) and (48), there exists a m-dimensional vector function $u(x, t)$ defined in the rectangular domain Q_T and we can select a sequence of m-dimensional vector functions $\{v_{h_i\Delta t_i}(x, t)\}$ ($i=1, 2, \cdots$) from the set of m-dimensional vector functions $\{v_{h\Delta t}(x, t)\}$, such that $\{v_{h_i\Delta t_i}(x, t)\}$ converges uniformly to $u(x, t)$ in Q_T as $h_i\to 0$ and $\Delta t_i\to 0$, where $\frac{\alpha\sigma\Delta t_i}{h_i} \leqslant \varepsilon \leqslant 1$ is preserved for any i. From (44) and (48), we have

$$|v_j^{n+\alpha}| \leqslant K_2, \quad j=0, 1, \cdots, J;\ n=0, 1, \cdots, N-1 \tag*{(44)$_\alpha$}$$

and

$$|v_j^{n+\alpha} - v_j^{n+\alpha-1}| \leqslant K_4 \Delta t, \quad j=0, 1, \cdots, J-1;\ n=1, 2, \cdots, N-1. \tag*{(48)$_\alpha$}$$

Then from (44), (46) and $(48)_\alpha$, we know that the sequence $\{v_{h_i\Delta t_i}^\alpha(x, t)\}$ is also uniformly convergent to $u^{\alpha}(x, t)=u(x, t)$ in Q_T as $h_i\to 0$ and $\Delta t_i\to 0$.

Here (49) can be rewritten as

$$|\Delta_+v_j^{n+\alpha}-\Delta_+v_j^{n+\alpha-1}|\leqslant K_3\Delta th^{\frac{1}{2}},\quad j=0, 1, \cdots, J-1;\ n=1, 2, \cdots, N-1 \qquad (49)_\alpha$$

and

$$|v_j^{n+\alpha}-2v_j^{n+\alpha-1}+v_j^{n+\alpha-2}|\leqslant K_5\Delta t^{\frac{3}{2}},\quad j=0, 1, \cdots, J;\ n=2, 3, \cdots, N-1. \qquad (50)_\alpha$$

Then we can select from $\{h_i, \Delta t_i\}$ a subsequence still denoted by $\{h_i, \Delta t_i\}$, such that from (46), (47) and $(49)_\alpha$, the subsequence $\{\bar{v}^{\alpha}_{h_i\Delta t_i}(x, t)\}$ converges uniformly to $\bar{u}^{\alpha}(x, t)$ in Q_T, from (48), (49) and (50), the subsequence $\{\tilde{v}_{h_i\Delta t_i}(x, t)\}$ converges uniformly to $\tilde{u}(x, t)$ in Q_T and from $(48)_\alpha$, $(49)_\alpha$ and $(50)_\alpha$, the subsequence $\{\tilde{v}^{\alpha}_{h_i\Delta t_i}(x, t)\}$ converges uniformly to $\tilde{u}^{\alpha}(x, t)$ in Q_T. Similarly we can obtain that the subsequences $\{\bar{\bar{v}}^{\alpha}_{h_i\Delta t_i}(x, t)\}$, $\{\tilde{\bar{v}}^{\alpha}_{h_i\Delta t_i}(x, t)\}$ and $\{\tilde{\tilde{v}}_{h_i\Delta t_i}(x, t)\}$ are weakly convergent to the m-dimensional vector functions $\bar{\bar{u}}^{\alpha}(x, t)$, $\tilde{\bar{u}}^{\alpha}(x, t)$ and $\tilde{\tilde{u}}(x, t)$ respectively in $L_p(0, T; L_2(0, l))$ for any $2\leqslant p<\infty$. Since the norms of $\bar{\bar{u}}^{\alpha}(x, t)$, $\tilde{\bar{u}}^{\alpha}(x, t)$ and $\tilde{\tilde{u}}(x, t)$ in $L_p(0, T; L_2(0, l))$ is uniform with respect to $2\leqslant p<\infty$, then $\bar{\bar{u}}^{\alpha}(x, t)$, $\tilde{\bar{u}}^{\alpha}(x, t)$ and $\tilde{\tilde{u}}(x, t)$ belong to the functional space $L_\infty(0, T; L_2(0, l))$.

It is easy to verify by usual way as in [23—26], that $\bar{u}^{\alpha}(x, t)=u_x(x, t)$ and $\tilde{u}(x, t)=u_t(x, t)$ are the Hölder continuous derivatives of the m-dimensional limiting vector function $u(x, t)$ respectively. Similarly $\bar{\bar{u}}^{\alpha}(x, t)=u_{xx}(x, t)$, $\tilde{\bar{u}}^{\alpha}(x, t)=u_{xt}(x, t)$ and $\tilde{\tilde{u}}(x, t)=u_{tt}(x, t)$ are the appropriate m-dimensional generalized derivations of the m-dimensional limiting vector function $u(x, t)$. Hence the constructed vector function $u(x, t)$ belongs to the functional space $Z\equiv\{u(x, t)\,|\,u\in L_\infty(0, T; H^2(0, l), u_t\in L_\infty(0, T; H^1(0, l)), u_{tt}\in L_\infty(0, T; L_2(0, l))\}$.

11. Let $g(x, t)$ is a smooth test function. Denote by $g_{h\Delta t}(x, t)$ the piecewise constant function constructed as before corresponding to the discrete function $g_j^n=g(x_j, t^n)$ $(j=0, 1, \cdots, J;\ n=0, 1, \cdots, N)$. We define

$$G_{h\Delta t}(x, t)=\int_0^1 \operatorname{grad} F(v_j^{n+\tau})d\tau$$

in Q_j^{n+1} and $H_{h\Delta t}(x, t)=f_j^{n+\alpha}$ in Q_j^{n+1} for $j=0, 1, \cdots, J-1;\ n=0, 1, \cdots, N-1$. As $h_i\to 0$ and $\Delta t_i\to 0$, the sequence $\{G_{h_i\Delta t_i}(x, t)\}$ is uniformly convergent to $F(u(x, t))$. Since $v_{h_i\Delta t_i}(x, t)$, $\bar{v}^{\alpha}_{h_i\Delta t_i}(x, t)$ and $\tilde{v}^{\alpha}_{h_i\Delta t_i}(x, t)$ are uniformly convergent to $u(x, t)$, $u_x(x, t)$ and $u_t(x, t)$ respectively in Q_T, hence $H_{h_i\Delta t_i}(x, t)$ uniformly converges in Q_T to $f(x, t, u(x, t), u_x(x, t), u_t(x, t))$ as $h_i\to 0$ and $\Delta t_i\to 0$.

From the finite difference system $(5)_h$, we have the identity

$$\sum_{n=1}^{N-1}\sum_{j=1}^{J-1} g_j^{n+1}\Big\{\frac{v_j^{n+1}-v_j^n+v_j^{n-1}}{\Delta t^2}-\frac{\Delta_+\Delta_-v_j^{n+\alpha}}{h^2}+\int_0^1 \operatorname{grad} F(v_j^{n+\tau})d\tau$$
$$-f\Big(x_j, t^{n+\alpha}, \alpha a v_j^{n+1}+a'v_j^n+a''v_j^{n-1}, b\frac{\Delta_+v_j^{n+\alpha}}{h}$$
$$+b'\frac{\Delta_-v_j^{n+\alpha}}{h}, \frac{v_j^{n+\alpha}-v_j^{n+\alpha+1}}{\Delta t}\Big)\Big\}h\Delta t=0.$$

This is evidently identical to the integral relation

$$\iint_{Q_T} g_{h\Delta t}(x, t)\,[\tilde{\tilde{v}}_{h\Delta t}(x, t)-\bar{\bar{v}}^{\alpha}_{h\Delta t}(x, t)G_{h\Delta t}(x, t)-H_{h\Delta t}(x, t)]\,dx\,dt=0.$$

When $h_i \to 0$ and $\Delta t_i \to 0$, such that $\frac{\alpha\sigma \Delta t_i}{h_i} \leqslant s < 1$ is preserved for any i, the above integral relation tends to

$$\iint_{Q_T} g(x, t)\,[u_{tt}(x, t) - u_{xx}(x, t) + \text{grad } F(u(x, t)) - f(x, t, u(x, t), u_x(x, t), u_t(x, t)]\,dx\,dt = 0.$$

This shows that the m-dimensional vector function $u(x, t) \in Z$ satisfies the system (5) of nonlinear wave equations in generalized sense.

On the other hand, since the convergences for $v_{h\Delta t}(x, t)$, $\bar{v}^{\alpha}_{h\Delta t}(x, t)$ and $\tilde{v}^{\alpha}_{h\Delta t}(x, t)$ are uniform, then the generalized nonlinear mutual boundary conditions (6) and the initial condition (7) are satisfied by the m-dimensional limiting vector function $u(x, t)$ in classical sense.

Hence the constructed m-dimensional vector function $u(x, t) \in Z$ is the generalized solution of the nonlinear boundary problem (6) and (7) for the system (5) of nonlinear wave equations.

Theorem 1. *Suppose that the conditions* (I), (II), (III) *and* (IV) *are satisfied. Then the generalized nonlinear mutual boundary problem* (6) *and* (7) *for the system* (5) *of nonlinear wave equations has at least one m-dimensional generalized vector solution $u(x, t)$, belonging to the functional space $Z \equiv L_\infty(0, T;\ H^2(0, l)) \cap W^{(1)}_\infty(0, T;\ H^1(0, l)) \cap W^{(2)}_\infty(0, T;\ L_2(0, l))$ and satisfying the system* (5) *in generalized sense and the boundary conditions* (6) *and the initial conditions* (7) *in classical sense.*

Remark. For the existence of generalized global solution $u(x, t)$ of the nonlinear problem (6) and (7) for the system (5), the smoothness assumptions for the initial vector function $\varphi(x)$ and $\psi(x)$ can be weakened that $\varphi(x) \in H^2(0, l)$ and $\psi(x) \in H^1(0, l)$. This can be justified by a simple approaching process.

§ 5. Uniqueness of Solution

12. Suppose that $u(x, t)$ and $\bar{u}(x, t) \in W_2^{(2,2)}(Q_T)$ are two m-dimensional generalized vector solutions of the general nonlinear mutual boundary problem (6) and (7) for the system (5) of nonlinear wave equations. Hence we have in generalized sense

$$u_{tt} - u_{xx} + \text{grad } F(u) = f(x, t, u, u_x, u_t) \tag{5}$$

and

$$\bar{u}_{tt} - \bar{u}_{xx} + \text{grad } F(\bar{u}) = f(x, t, \bar{u}, \bar{u}_x, \bar{u}_t). \tag{5'}$$

$u(x, t)$ and $\bar{u}(x, t)$ satisfy in classical sense the boundary conditions

$$\begin{aligned} u_t(0, t) &= \Phi_0(u_x(0, t), u_x(l, t), u(0, t), u(l, t), t), \\ -u_t(l, t) &= \Phi_1(u_x(0, t), u_x(l, t), u(0, t), u(l, t), t) \end{aligned} \tag{6}$$

and

$$\begin{aligned} \bar{u}_t(0, t) &= \Phi_0(\bar{u}_x(0, t), \bar{u}_x(l, t), \bar{u}(0, t), \bar{u}(l, t), t), \\ -\bar{u}_t(l, t) &= \Phi_1(\bar{u}_x(0, t), \bar{u}_x(l, t), \bar{u}(0, t), \bar{u}(l, t), t) \end{aligned} \tag{6'}$$

and satisfy in classical sense the initial conditions

$$u(x,0)=\varphi(x),\quad u_t(x,0)=\psi(x) \tag{7}$$

and

$$\bar{u}(x,0)=\varphi(x),\quad \bar{u}_t(x,0)=\psi(x). \tag{7'}$$

For the m-dimensional difference vector function $W(x,t)=u(x,t)-\bar{u}(x,t)$, there is

$$W_{tt}-W_{xx}+\tilde{G}^\tau W=\tilde{f}_u^\tau W+\tilde{f}_p^\tau W_x+\tilde{f}_q^\tau W_t, \tag{5}$$

where

$$\tilde{G}^\tau=\int_0^1 \operatorname{grad}^2 F(\tau u+(1-\tau)\bar{u})d\tau,$$

$$\tilde{f}_u^\tau=\int_0^1 f_u(x,t,\tau u+(1-\tau)\bar{u},\tau u_x+(1-\tau)\bar{u}_x,\tau u_t+(1-\tau)\bar{u}_t)d\tau$$

and similar for $\tilde{f}_p^\tau$ and $\tilde{f}_q^\tau$. (5) is a linear system with Hölder continuous coefficients. $W(x,t)$ satisfies the linear homogeneous boundary conditions

$$\begin{aligned} W_t(0,t)&=\tilde{\Phi}_{0p_0}^\tau W_x(0,t)+\tilde{\Phi}_{0p_1}^\tau W_x(l,t)+\tilde{\Phi}_{0u_0}^\tau W(0,t)+\tilde{\Phi}_{0u_1}^\tau W(l,t),\\ -W_t(l,t)&=\tilde{\Phi}_{1p_0}^\tau W_x(0,t)+\tilde{\Phi}_{1p_1}^\tau W_x(l,t)+\tilde{\Phi}_{1u_0}^\tau W(0,t)+\tilde{\Phi}_{1u_1}^\tau W(l,t) \end{aligned} \tag{6}$$

and the homogeneous initial conditions

$$W(x,0)=0,\quad W_t(x,0)=0, \tag{7}$$

where the coefficients in the expressions (6) are bounded.

Making the scalar product of the m-dimensional vector function $W_t(x,t)$ with the vector equation (5) and integrating the resulting relation in rectangular domain Q_T, we get

$$\iint_{Q_T}(W_t,W_{tt})dx\,dt-\iint_{Q_T}(W_t,W_{xx})dx\,dt+\iint_{Q_T}(W_t,\tilde{G}^\tau W)dx\,dt$$
$$=\iint_{Q_T}[(W_t,\tilde{f}_u^\tau W)+(W_t,\tilde{f}_p^\tau W_x)+(W_t,\tilde{f}_q^\tau W_t)]dx\,dt.$$

By simple calculation, this equality transform to the following inequality

$$\begin{aligned}&\|W_t(\cdot,T)\|_{L_2(0,l)}^2+\|W_x(\cdot,T)\|_{L_2(0,l)}^2\\ &\leqslant 2\int_0^T(W_x(l,t),W_t(l,t))dt-2\int_0^T(W_x(0,t),W_t(0,t))dt\\ &\quad+C_{27}\{\|W\|_{L_2(Q_T)}^2+\|W_x\|_{L_2(Q_T)}^2+\|W_t\|_{L_2(Q_T)}^2\},\end{aligned} \tag{52}$$

where the boundary and the initial conditions are used in derivation. Here

$$\begin{aligned}B&=\int_0^T(W_x(l,t),W_t(l,t))dt-\int_0^T(W_x(0,t),W_t(0,t))dt\\ &\leqslant\frac{1}{\sigma}\max_{0<t<}\{|\tilde{\Phi}_{0u_0}^\tau|^2+|\tilde{\Phi}_{0u_1}^\tau|^2+|\tilde{\Phi}_{1u_0}^\tau|^2+|\tilde{\Phi}_{1u_1}^\tau|^2\}\int_0^1(|W(0,t)|^2+|W(l,t)|^2)\,dt,\end{aligned}$$

where $|\tilde{\Phi}_{0u_0}^\tau|$, $|\tilde{\Phi}_{0u}^\tau|$, $|\tilde{\Phi}_{1u_0}^\tau|$ and $|\tilde{\Phi}_{1u_0}^\tau|$ denote the elements of the matrices $\tilde{\Phi}_{0u_0}^\tau$, $\tilde{\Phi}_{0u_1}^\tau$, $\tilde{\Phi}_{1u_0}^\tau$ and $\tilde{\Phi}_{1u_1}^\tau$ respectively. Thus we have

$$B\leqslant C_{28}\int_0^T\|W(\cdot,t)\|_{L_\infty(0,l)}^2dt\leqslant C_{29}\{\|W\|_{L_2(Q_T)}^2+\|W_x\|_{L_2(Q_T)}^2\}.$$

Since the initial value $W(x, 0)$ equals to zero, then

$$\|W\|_{L_2(Q_T)} \leqslant T\|W_t\|_{L_2(Q_T)}.$$

Hence (52) becomes

$$\|W_t(\cdot, T)\|^2_{L_2(0,l)} + \|W_x(\cdot, T)\|^2_{L_2(0,l)} \leqslant C_{30}\{\|W_t\|^2_{L_2(Q_T)} + \|W_x\|^2_{L_2(Q_T)}\}.$$

This implices $W_t(x, t) \equiv W_x(x, t) \equiv 0$, thus $W(x, t) \equiv 0$.

Theorem 2. *The m-dimensional generalized vector solution $u(x, t) \in W_2^{(2,2)}(Q_T)$ of the general nonlinear mutual boundary problem* (6) *and* (7) *for the system* (5) *of nonlinear wave equations is unique.*

§ 6. Convergence of Finite Difference Scheme

13. We can select the subsequence $\{v_{h_i \Delta t_i}(x, t)\}$ from the set of m-dimensional vector functions $\{v_{h\Delta t}(x, t)\}$, defined on the base of m-dimensional finite difference solutions v_j^n ($j=0, 1, \cdots, J$; $n=0, 1, \cdots, N$) of the nonlinear finite difference system $(5)_h$, $(6)_h$, $(7')_h$ and $(7'')_h$, such that as $h_i \to 0$ and $\Delta t_i \to 0$, $v_{h_i \Delta t_i}(x, t)$ tends to a m-dimensional vector function $u(x, t) \in Z$, which is a m-dimensional generalized vector solution of the nonlinear boundary problem (6) and (7) for the system (5) of nonlinear wave equations. Since the generalized global solution of the nonlinear boundary problem (6) and (7) for the system (5) is unique, therefore for any sequences $h_i \to 0$ and $\Delta t_i \to 0$, which preserve $\frac{\alpha\sigma\Delta t_i}{h_i} \leqslant \varepsilon < 1$, the corresponding sequence $\{v_{h_i \Delta t_i}(x, t)\}$ tends to the unique limit $u(x, t) \in Z$. This means that as $h \to 0$ and $\Delta t \to 0$ with $\frac{\alpha\sigma\Delta t}{h} \leqslant \varepsilon < 1$, the sequence $\{v_{h\Delta t}(x, t)\}$ converges to $u(x, t) \in Z$, the generalized global vector solution of the nonlinear boundary problem (6) and (7) for the system (5).

Theorem 3. *Under the conditions* (I), (II), (III) *and* (IV), *the finite difference solution v_j^n ($j=0, 1, \cdots, J$; $n=0, 1, \cdots, N$) of the nonlinear finite difference scheme $(5)_h$, $(6)_h$, $(7')_h$ and $(7'')_h$ with $\frac{1}{2} \leqslant \alpha \leqslant 1$ converges to the m-dimensional vector function $u(x, t) \in Z$, as $h \to 0$ and $\Delta t \to 0$ preserving $\frac{\alpha\sigma\Delta t}{h} \leqslant \varepsilon < 1$ in the following sense: $\{v_j^n\}$, $\left\{\frac{\Delta_+ v_j^n}{h}\right\}$ and $\left\{\frac{v_j^n - v_j^{n-1}}{\Delta t}\right\}$ are uniformly convergent to $u(x, t)$, $u_x(x, t)$ and $u_t(x, t)$ in Q_T respectively and $\left\{\frac{\Delta_+\Delta_- v_j^{n+\alpha}}{h^2}\right\}$, $\left\{\frac{\Delta_+(v_j^{n+\alpha} - v_j^{n+\alpha-1})}{h\Delta t}\right\}$ and $\left\{\frac{v_j^{n+1} - 2v_j^n + v_j^{n-1}}{\Delta t^2}\right\}$ are weakly convergent to $u_{xx}(x, t)$, $u_{xt}(x, t)$ and $u_{tt}(x, t)$ in $L_p(0, T; L_2(0, l))$ for any $2 \leqslant p < \infty$ respectively. Furthermore the limiting vector function $u(x, t) \in Z$ is the unique generalized global solution of the nonlinear boundary problem* (6) *and* (7) *for the system* (5) *of nonlinear wave equations.*

Hence when h and Δt are small, the finite difference solution v_j^n ($j=0, 1, \cdots, J$; $n=0, 1, \cdots, N$) of the nonlinear finite difference system $(5)_h$, $(6)_h$, $(7')_h$ and $(7'')_h$ may be regarded as the approximate solution of the nonlinear boundary problem (6) and (7) for the system (5).

§ 7. Mixed Problem

14. In this section we are going to construct the m-dimensional generalized global vector solution $u(x, t)$ of the problem with the nonlinear mixed boundary conditions

$$\begin{aligned} &u_t(0, t)=\Phi(u_x(0, t), u(0, t), t), \\ &u(l, t)=0 \end{aligned} \tag{8}$$

and the initial conditions

$$\begin{aligned} &u(x, 0)=\varphi(x), \\ &u_t(x, 0)=\psi(x) \end{aligned} \tag{9}$$

for the system

$$u_{tt}-u_{xx}+\operatorname{grad} F(u)=f(x, t, u, u_x, u_t) \tag{5}$$

of nonlinear wave equations in rectangular domain $Q_T=\{0\leqslant x\leqslant l,\ 0\leqslant t\leqslant T\}$, where $u=(u_1, \cdots, u_m)$ is a m-dimensional unknown vector function, $f(x, t, u, p, q)$, $\Phi(p, u, t)$, $\varphi(x)$ and $\psi(x)$ are m-dimensional vector function for $0\leqslant x\leqslant l$, $0\leqslant t\leqslant T$, $u, p, q\in\mathbb{R}^m$, and $F(u)$ is a non-negative convex scalar function of vector variable $u\in\mathbb{R}^m$.

Suppose that the conditions (I) and (II) are satisfied. Further let us assume that the following conditions are fulfilled:

(III′) $\Phi(p, u, t)$ is a m-dimensional continuously differentiable vector functions of the variable $t\in[0, T]$ and the vector variables u, $p\in\mathbb{R}^m$, The $m\times m$ Jacobi derivative matrix $\Phi_p(p, u, t)$ of m-dimensional vector function $\Phi(p, u, t)$ with respect to m-dimensional vector variable $p\in\mathbb{R}^m$ is positively definite, i.e., there is a positive constant $\sigma>0$, such that

$$(\xi, \Phi_p(p, u, t)\xi)\geqslant\sigma|\xi|^2 \tag{53}$$

for any m-dimensional vector $\xi\in\mathbb{R}^m$. Furthermore there are

$$\begin{aligned} &|\Phi(0, u, t)|\leqslant A(|u|+1), \\ &|\Phi_u(p, u, t)|\leqslant B(u)(|p|+1), \\ &|\Phi_t(p, u, t)|\leqslant B(u)(|p|^2+1), \end{aligned} \tag{54}$$

where A is a constant, $B(u)$ is a continuous function of $u\in\mathbb{R}^m$.

(IV′) The m-dimensional initial vector functions $\varphi(x)\in C^2([0, l])$ and $\psi(x)\in C^{(1)}([0, l])$ satisfy the boundary conditions (8), i.e.,

$$\begin{aligned} &\psi(0)=\Phi(\varphi'(0), \varphi(0), 0), \\ &\varphi(l)=\psi(l)=0. \end{aligned} \tag{55}$$

(V′) $\operatorname{grad} F(0)=0$ and $f(l, t, 0, p, 0)\equiv 0$.

The finite difference approximation of the system (5) is also the nonlinear finite difference scheme $(5)_h$. Corresponding to the nonlinear mixed boundary conditions, the finite difference boundary conditions are of the form

$$\begin{aligned} &\frac{v_0^{n+1}-v_0^n}{\Delta t}=\Phi\left(\frac{\Delta_+ v_0^{n+\alpha}}{h},\ v_1^{n+\alpha},\ t^{n+\alpha}\right), \\ &v_J^{n+1}=0, \quad n=0, 1, \cdots, N-1. \end{aligned} \tag{8_h}$$

The corresponding finite difference initial conditions are

$$\begin{aligned} &v_j^0=\varphi_j, && j=0, 1, \cdots, J; \\ &v_j^1=\varphi_j+\Delta t\psi_j, && j=1, 2, \cdots, J \end{aligned} \tag{$7'$}_h$$

and v_0^1 is the unique solution of the system

$$\frac{v_0^1-\varphi_0}{\Delta t}=\Phi\left(\frac{\Delta_+v_0^\alpha}{h}, \varphi_1+\alpha\Delta t\psi_1, \alpha\Delta t\right), \tag{$7''$}_h$$

where $\varphi_j=\varphi(x_j)$ and $\psi_j=\psi(x_j)$ for $j=0, 1, \cdots, J$. The system $(7'')_h$ has a unique solution v_0^1 as $\alpha\sigma\Delta t<h$.

15. In order to establish the existence of the finite difference solution v_j^n ($j=0, 1, \cdots, J$; $n=0, 1, \cdots, N$) for the nonlinear finite difference system $(5)_h$, $(8)_h$, $(7')_h$ and $(7'')_h$ and the existence of the generalized global solution $u(x, t)\in Z$ of the nonlinear boundary problem (8) and (7) for the system (5) of nonlinear wave equations, it needs us to derive a series of a priori estimations for the discrete solution v_j^n ($j=0, 1, \cdots, J$; $n=0, 1, \cdots, N$). In the process of these estimates, the different boundary conditions (6) and (8) may give the different contributions and different influences. The boundary condition in the lateral sides $x=0$ of the rectangular domain Q_T of (8) is the special case of the general nonlinear mutual boundary conditions of (6). So it gives the similar but somewhat simpler contribution in the process of estimation as in the process of estimation for the boundary problem (6) and (7) of the system (5). Hence we ought to notice the influence of the homogeneous boundary condition at $x=l$ in the process of estimation.

In the equality (17) of the proof of Lemma 1 and Lemma 2, the part contributed by the influence of the homogeneous boundary condition $u(l, t)=0$ or the homogeneous discrete boundary condition $v_J^n=0$ ($n=0, 1, \cdots, N$) is contained in the second term of (17) or in the expansion (16). This is the expression

$$\left(\frac{v_J^{n+1}-v_J^n}{\Delta t}, \frac{\Delta_-v_J^{n+\alpha}}{h}\right)\Delta t, \quad n=0, 1, \cdots, N-1,$$

which equals to zero.

In the proof of the Lemma 4, the terms related to the boundary values on the lateral side $x=l$ is

$$\left(\frac{\Delta_-(v_J^{n+\alpha}-v_J^{n+\alpha-1})}{h\Delta t}, \frac{v_J^{n+1}-2v_J^n+v_J^{n-1}}{\Delta t^2}\right)\Delta t$$
$$+\left(\frac{\Delta_-(v_J^{n+\alpha}-v_J^{n+\alpha-1})}{h\Delta t}, \int_0^1 \operatorname{grad} F(v_J^{n+\tau})\,d\tau\right)\Delta t-\left(\frac{\Delta_-(v_J^{n+\alpha}-v_J^{n+\alpha-1})}{h\Delta t}, f_{J-1}^{n+\alpha}\right)\Delta t.$$

The first two parts of the above expression are equal to zero from the homogeneous boundary condition $(8)_h$ and the assumption (V$'$). Since

$$|f_{J-1}^{n+\alpha}|=\left|f\left(l-h, t^{n+\alpha}, \bar{v}_{J-1}^n, \frac{\bar{\Delta}v_{J-1}^{n+\alpha}}{h}, \frac{v_J^{n+\alpha}-v_J^{n+\alpha-1}}{\Delta t}\right)-f\left(l, t^{n+\alpha}, 0, \frac{\bar{\Delta}v_{J-1}^{n+\alpha}}{h}, 0\right)\right|$$
$$\leqslant|\tilde{f}_x|h+|\tilde{f}_u\bar{v}_{J-1}^n|+\left|\tilde{f}_q\left(\frac{v_{J-1}^{n+\alpha}-v_{J-1}^{n+\alpha-1}}{\Delta t}\right)\right|,$$

where

$$\bar{f}_x=\int_0^1 f_x\Big(l-\tau h,\ t^{n+\alpha},\ \tau\bar{v}^n_{J-1},\ \frac{\bar{\Delta}v^{n+\alpha}_{J-1}}{h},\ \tau\Big(\frac{v^{n+\alpha}_{J-1}-v^{n+\alpha-1}_{J-1}}{\Delta t}\Big)\Big)$$

and similar for $\bar{f}_u$ and $\bar{f}_q$, then

$$|f^{n+\alpha}_{J-1}|\leqslant C_{31}h\left\{\left|\frac{\bar{\Delta}v^{n+\alpha}_{J-1}}{h}\right|+\left|\frac{\Delta_+(v^{n+\alpha}_{J-1}-v^{n+\alpha-1}_{J-1})}{h\Delta t}\right|+\left|\frac{\Delta_+\bar{v}^n_{J-1}}{h}\right|\right\}+C_{31}h$$

$$\leqslant C_{31}h^{\frac{1}{2}}\left\{\|\delta v^{n+\alpha}_h\|_2+\left|\frac{\Delta_-(v^{n+\alpha}_J-v^{n+\alpha-1}_J)}{h\Delta t}\right|h^{\frac{1}{2}}+\max_{k=n-1,n,n+1}\|\delta v^k_h\|_2\right\}+C_{31}h.$$

Thus

$$\left|\left(\frac{\Delta_-(v^{n+\alpha}_J-v^{n+\alpha-1}_J)}{h\Delta t},\ f^{n+\alpha}_{J-1}\right)\Delta t\right|\leqslant C_{32}\left(\left\|\delta\Big(\frac{v^{n+1}_h-v^n_h}{\Delta t}\Big)\right\|^2_2+\left\|\delta\Big(\frac{v^n_h-v^{n-1}_h}{\Delta t}\Big)\right\|^2_2\right)+C_{32}.$$

The above discussion shows that for the discrete m-dimensional vector solution v^n_j ($j=0, 1, \cdots, J$; $n=0, 1, \cdots, N$) of the nonlinear finite difference system $(5)_h$, $(6)_h$, $(7')_h$ and $(7'')_h$, the estimation of various difference quotients can be obtained by the similar way used in estimation for the discrete m-dimentional vector solution v^n_j ($j=0, 1, \cdots, J$; $n=0, 1, \cdots, N$) of the finite difference system $(5)_h$, $(6)_h$, $(7')_h$ and $(7'')_h$. Hence we get the following lemmas for the discrete m-dimensional vector solution v^n_j ($j=0, 1, \cdots, J$; $n=0, 1, \cdots, N$) of the nonlinear finite difference system $(5)_h$, $(8)_h$, $(\bar{7}')_h$ and $(\bar{7}'')_h$.

Lemma 8. *Suppose that the conditions* (I), (II), (III′), (IV′) *and* (V′) *are satisfied and suppose that* $a\sigma\Delta t<h$ *and* Δt *is sufficiently small. Then the nonlinear finite difference scheme* $(5)_h$, $(8)_h$, $(\bar{7}')_h$ *and* $(\bar{7}'')_h$, *corresponding to the nonlinear mixed boundary problem* (8) *and* (7) *for the system* (5) *of nonlinear wave equations has at least one* m*-dimensional discrete vector solution* v^n_j ($j=0, 1, \cdots, J$; $n=0, 1, \cdots, N$) *for* $0\leqslant\alpha\leqslant1$.

Lemma 9. *Under the conditions of Lemma 8, for the* m*-dimensional discrete vector solution* v^n_j ($j=0, 1, \cdots, J$; $n=0, 1, \cdots, N$) *of the nonlinear finite difference scheme* $(5)_h$, $(8)_h$, $(\bar{7}')_h$ and $(\bar{7}'')_h$ *with* $\frac{1}{2}\leqslant\alpha\leqslant1$, *there are estimates*

$$\max_{n=0,1,\cdots,N}\|\delta v^n_h\|_2+\max_{n=0,1,\cdots,N-1}\|\delta^2 v^{n+\alpha}_h\|_2+\max_{n=0,1,\cdots,N-1}\left\|\frac{v^{n+1}_h-v^n_h}{\Delta t}\right\|_2$$
$$+\max_{n=0,1,\cdots,N}\|v^n_h\|_2+\max_{n=0,1,\cdots,N-1}\left\|\delta\Big(\frac{v^{n+1}_h-v^n_h}{\Delta t}\Big)\right\|_2$$
$$+\max_{n=1,2,\cdots,N-1}\left\|\frac{v^{n+1}_h-2v^n_h+v^{n-1}_h}{\Delta t^2}\right\|_2\leqslant K_6, \tag{56}$$

where K_6 *is a constant independent of* h *and* Δt.

16. Using the estimations of Lemma 9 and the common argument of limiting process as before, we can get the existence theorem of the generalized global solution for the nonlinear mixed boundary problem (6) and (8) of the system (5) of nonlinear wave equations as follows.

Theorem 4. *Suppose that the conditions* (I), (II), (III′), (IV′) *and* (V′) *are satisfied. Then the nonlinear mixed boundary problem* (8) *and* (7) *for the system* (7) *of nonlinear wave equations has at least one* m*-dimensional generalized vector solution* $u(x, t)$, *belonging to the functional space and satisfying the system* (5) *in generalized sense and the boundary conditions* (8) *and the initial conditions* (7) *in classical sense.*

Remark. In the above theorem, the smoothness assumptions for the initial vector functions $\varphi(x)$ and $\psi(x)$ can be weakened that $\varphi(x)\in H^2(0, l)$ and $\psi(x)\in H^1(0, l)$. This can be justified by the usual approaching process.

The uniqueness of the solution can be established by same method as before.

Theorem 5. *The m-dimensional generalized vector solution $u(x, t)\in W_2^{(2,2)}(Q_T)$ of the nonlinear mixed boundary problem* (8) *and* (7) *for the system* (5) *of nonlinear wave equations is unique.*

Hence we can obtain by the similar method, the following theorem of convergence.

Theorem 6. *Under the conditions* (I), (II), (III′), (IV′) *and* (V′), *the m-dimensional discrete vector solution v_j^n $(j=0, 1, \cdots, J;\ n=0, 1, \cdots, N)$ of the nonlinear finite difference scheme* $(5)_h$, $(8)_h$, $(7')_h$ *and* $(7'')_h$ *with* $\frac{1}{2}\leqslant\alpha\leqslant 1$, *converges to the m-dimensional vector function $u(x, t)\in Z$ as $h\to 0$ and $\Delta t\to 0$ preserving* $\frac{\alpha\sigma\Delta t}{h}\leqslant s<1$ *in the following sense:* $\{v_j^n\}$, $\left\{\frac{\Delta_+ v_j^n}{h}\right\}$ *and* $\left\{\frac{v_j^n-v_j^{n-1}}{\Delta t}\right\}$ *are uniformly convergent to* $u(x, t)$, $u_x(x, t)$ *and* $u_t(x, t)$ *in* Q_T *respectively and* $\left\{\frac{\Delta_+\Delta_- v_j^{n+\alpha}}{h^2}\right\}$, $\left\{\frac{\Delta_+(v_j^{n+\alpha}-v_j^{n+\alpha-1})}{h\Delta t}\right\}$ *and* $\left\{\frac{v_j^{n+1}-2v_j^n+v_j^{n-1}}{\Delta t^2}\right\}$ *are weakly convergent to* $u_{xx}(x, t)$, $u_{xt}(x, t)$ *and* $u_{tt}(x, t)$ *in* $L_\infty(0, T;\ L_2(0, l))$ *respectively. Furthermore the limiting vector function $u(x, t)\in Z$ is unique generalized global solution of the nonlinear mixed boundary problem* (8) *and* (7) *for the system* (5) *of nonlinear wave equations.*

§ 8. Infinite Domain

17. Let us in the infinite domain $Q_T^*=\{x\in\mathbb{R}^+,\ 0\leqslant t\leqslant T\}$ consider the problem with nonlinear condition

$$u_t(0, t)=\Phi(u_x(0, t), u(0, t), t),\quad t\in[0, T] \tag{57}$$

and the initial conditions

$$\begin{aligned} &u(x, 0)=\varphi(x),\\ &u_t(x, 0)=\psi(x),\quad x\in\mathbb{R}^+=[0, \infty) \end{aligned} \tag{58}$$

for the system (5) of nonlinear wave equations by the limiting process $l\to\infty$ in the nonlinear mixed boundary problem (8) and (7) for the system (5) of nonlinear wave equations.

Suppose that the conditions (I) and (III′) are satisfied. And suppose that the following conditions are fulfilled:

(II*) The m-dimensional vector function $f(x, t, u, p, q)$ is continuous in $(x, t, u, p, q)\in Q_T^*\times\mathbb{R}^{3m}$ and is continuously differentiable with respect to $x\in\mathbb{R}^+$ and $u, p, q\in\mathbb{R}^3$. Further for any $(x, t)\in Q_T^*$ and $u, p, q\in\mathbb{R}^m$, there are

$$\begin{aligned} &|f(x, t, u, p, q)|,\ |f_x(x, t, u, p, q)|\leqslant A^*\{F(u)+|u|^2+|p|^2+|q|^2+g^2(x, t)\},\\ &\qquad\qquad |f_u(x, t, u, p, q)|\leqslant A^*\{F(u)+|u|^2+1\},\\ &|f_p(x, t, u, p, q)|,\ |f_q(x, t, u, p, q)|\leqslant A^*, \end{aligned} \tag{59}$$

where A^* is a constant and $g(x, t)$ is a continuous function in Q_T^* with

$$\|g(\cdot,\ t)\|_{L_1(\mathbb{R}^+)}<\infty.$$

(IV*) The m-dimensional initial vector functions $\varphi(x)\in H^2(\mathbb{R}^+)$ and $\psi(x)\in H^1(\mathbb{R}^+)$ satisfy the conditions

$$\begin{aligned}&\psi(0)=\Phi(\varphi'(0),\ \varphi(0),\ 0),\\&\varphi(l)=0.\end{aligned}\tag{60}$$

Further $F(\varphi(x))\in L_1(\mathbb{R}^+)$.

(V*) Also we have grad $F(0)=0$ and $f(x,\ t,\ 0,\ p,\ 0)\equiv 0$.

For each $2\leqslant l<\infty$, we construct a m-dimensional vector function $\varphi_l(x)$ such that $\varphi_l(x)=\varphi(x)$ in $[0,\ l-1]$, $\varphi_l(x)\equiv 0$ in $[l,\ \infty)$ and $\varphi_l(x)\in H^2(\mathbb{R}^+)$ and we also construct a m-dimensional vector function $\psi_l(x)$, such that $\psi_l(x)=\psi(x)$ in $[0,\ l-1]$, $\psi_l(x)\equiv 0$ in $[l,\ \infty)$ and $\psi_l(x)\in H^1(\mathbb{R}^+)$, further $\|\varphi_l\|_{H^2(\mathbb{R}^+)}\leqslant K_7$ and $\|\psi_l\|_{H^1(\mathbb{R}^+)}\leqslant K_7$, where K_7 is a constant independent of $2\leqslant l<\infty$.

Then in the rectangular domain $Q_T^{(l)}=\{0\leqslant x\leqslant l,\ 0\leqslant t\leqslant T\}$ we consider the problem with nonlinear mixed boundary conditions

$$\begin{aligned}&u_t(0,\ t)=\Phi(u_x(0,\ t),\ u(0,\ t),\ t),\\&u(l,\ t)=0,\quad 0\leqslant t\leqslant T\end{aligned}\tag{8$_1$}$$

and the initial conditions

$$\begin{aligned}&u(x,\ 0)=\varphi_l(x),\\&u_t(x,\ 0)=\psi_l(x),\quad 0\leqslant x\leqslant l\end{aligned}\tag{7$_1$}$$

for the system (5) of nonlinear wave equations. Denote by $u_l(x,\ t)\in Z(Q_T^{(l)})$ the unique m-dimensional generalized vector solution of the nonlinear mixed boundary problem $(8)_1$ and $(7)_1$ for the system (5) of nonlinear wave equations.

For the set of m-dimensional vector functions $\{u_l(x,\ t)\}$, we can obtain the following estimations by the analogous method as used in finite difference study:

$$\sup_{0\leqslant t\leqslant T}\|u_l(\cdot,\ t)\|_{H^2(0,l)}+\sup_{0\leqslant t\leqslant T}\|u_{lt}(\cdot,\ t)\|_{H^1(0,l)}+\sup_{0\leqslant t\leqslant T}\|u_{ltt}(\cdot,\ t)\|_{L_2(0,l)}\leqslant K_8,\tag{61}$$

where K_8 is a constant independent of $2\leqslant l<\infty$.

By the method used as in [27—29], it can be proved that there exists a unique m-dimensional vector function

$$u(x,\ t)\in Z(Q_T^*)\equiv L_\infty(0,\ T;\ H^2(\mathbb{R}^+))\cap W_\infty^{(1)}(0,\ T;\ H^1(\mathbb{R}^+))\cap W_\infty^{(2)}(0,\ T;\ L_2(\mathbb{R}^+)),$$

such that for any $0<L<\infty$, the sequence $\{u_l(x,\ t)\}$ converges to $u(x,\ t)$ in the rectangular domain $Q_T^{(L)}$ in the following sense: $\{u_l(x,\ t)\}$, $\{u_{lx}(x,\ t)\}$ and $\{u_{lt}(x,\ t)\}$ uniformly converge to $u(x,\ t)$, $u_x(x,\ t)$ and $u_t(x,\ t)$ respectively in $Q_T^{(L)}$, $\{u_{lxx}(x,\ t)\}$, $\{u_{lxt}(x,\ t)\}$ and $\{u_{ltt}(x,\ t)\}$ are weakly convergent to $u_{xx}(x,\ t)$, $u_{xt}(x,\ t)$ and $u_{tt}(x,\ t)$ respectively in $L_\infty(0,\ T;\ L_2(0,\ L))$. And $u(x,\ t)$ is the unique m-dimensional vector solution of the problem (57) and (58) for the system (5) in the infinite domain Q_T^*.

Theorem 7. *Under the conditions* (I), (II*), (III), (IV*) *and* (V*), *the nonlinear boundary problem* (57) *and* (58) *for the system* (5) *of nonlinear wave equations has a unique m-dimensional vector solution $u(x,\ t)\in Z(Q_T^*)$ in infinite domain Q_T^*, satisfying the system* (5) *in generalized sense and satisfying the nonlinear boundary condition* (57) *and the initial conditions* (58) *in classical sense.*

Theorem 8. *When $l\to\infty$, the solution $u_l(x,\ t)\in Z(Q_T^{(l)})$ of the problem* (5), $(8)_1$ *and* $(7)_1$ *tends to the solution $u(x,\ t)\in Z(Q_T^*)$ in the following way: $u_l(x,\ t)$, $u_{lx}(x,\ t)$*

and $u_{lt}(x, t)$ *converge uniformly to* $u(x, t)$, $u_x(x, t)$ *and* $u_t(x, t)$ *respectively in any finite domain and* $u_{lxx}(x, t)$, $u_{lxt}(v, t)$ *and* $u_{ltt}(x, t)$ *converge weakly to* $u_{xx}(x, t)$, $u_{xt}(x, t)$ *and* $u_{tt}(x, t)$ *respectively in* $L_p(0, T; L_2(0, L))$, $2\leqslant p<\infty$.

Hence $u_l(x, t)\in Q_T^{(l)}$ may be regarded as the approximation of $u(x, t)\in Z(Q_T^*)$.

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