HIGH ORDER APPROXIMATION OF ONE-WAY WAVE EQUATIONS*

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Abstract

In this article the high order approximations of the one-way wave equations are discussed. The approximate dispersion relations are expressed in explicit form of sums of simple fractions. By introducing new functions, the high order approximations of the one-way wave equations are put into the form of systems of lower order equations. The initial-boundary value problem of these systems which corresponds to the migration problem in seismic prospecting is discussed. The energy estimates for their solutions are obtained.

Introduction

The wave equation describes waves propagating in all directions. The equation, which describes only the down-going (or up-coming) waves propagating in the positive (or negative) direction of z, is called the one-way wave equation. In the one-dimensional case the one-way wave equations are the simple wave equations

$$\left(\frac{\partial}{\partial z}\pm\frac{1}{c}\frac{\partial}{\partial t}\right)p=0, \tag{1}$$

the general solutions of which are $f(t\mp z/c)$. The constant c is the velocity of propagation. In the two-dimensional case the one-way wave equations

$$L_{\pm}p=0 \tag{2}$$

describe the waves $f(t-(\alpha z+\beta x)/c)$ for all $\alpha\geqslant 0$ (or $\alpha\leqslant 0$), where α, β satisfy $\alpha^2+\beta^3=1$. The operators $L_{\pm}$ can be defined in terms of the Fourier transforms as pseudo-differential operators. For practical application it is necessary to derive their approximations that have local character. Such approximations are obtained in [1, 2, 3, 4] as the artificial boundary conditions for the wave equation, and also in [5, 6, 7] as the basic equations for migration in seismic prospecting.

The n-th order approximation obtained in the papers mentioned above is the $(n+1)$-th order P. D. E.. It is difficult to apply them for computation when $n\geqslant 2$. One of our purposes is to derive a new form of these approximations which is more convenient for numerical application. First, we derive the explicit expressions of the approximate dispersion relations, based on which the approximations of the one-way wave equations can be obtained. Then we derive systems of lower order P. D. E. as new forms of these approximations. Finally, we discuss the initial-boundary value problem (migration problem in seismic prospecting) of these systems and obtain the energy estimates for their solutions.

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1. Approximate Dispersion Relations

Consider the wave equation

$$\frac{\partial^2 p}{\partial z^2}+\frac{\partial^2 p}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}=0, \tag{1.1}$$

where the constant c is the velocity of propagation. Suppose that the Fourier transform $\hat{p}$ of the solution p exists

$$\hat{p}(z;\ K_x,\ \omega)=\frac{1}{2\pi}\iint \exp(i\omega t+iK_x x)p(z,\ x,\ t)\,dx\,dt;$$

then $\hat{p}$ satisfies the wave equation in frequency domain

$$\left(\frac{d^2}{dz^2}+K_z^2\right)\hat{p}=\left(\frac{d}{dz}-iK_+\right)\left(\frac{d}{dz}-iK_-\right)\hat{p}=0, \tag{1.2}$$

where
$$K_\pm=\pm K_z=\pm K\sqrt{1-K_x^2/K^2}, \tag{1.3}$$

in which $K=\omega/C$. (1.3) is called the dispersion relation of wave equation (1.1). The one-way wave equations in frequency domain are the following

$$\begin{aligned}&\left(\frac{d}{dz}-iK_+\right)\hat{p}=0 \quad \text{for down-going wave,}\\ &\left(\frac{d}{dz}-iK_-\right)\hat{p}=0 \quad \text{for up-coming wave.}\end{aligned} \tag{1.4}$$

From (1.4), we can see that the inverse Fourier transform p of $\hat{p}$ satisfies the equation

$$L_\pm p=\left(\frac{\partial}{\partial z}-\mathscr{K}_\pm\right)p=0 \tag{1.5}$$

with the pseudo-differential operators $\mathscr{K}_\pm$, the symbols of which are $K_\pm$.

The objective of this section is to derive the rational fraction approximations of $K_\pm$.

Let

$$S=(\mp K_\pm+K)/K_x,\quad r=K/K_x. \tag{1.6}$$

Then from (1.3) we see that S satisfies

$$S^2-2rS+1=0. \tag{1.7}$$

The smaller root S_∞ of (1.7) can be approximated by S_n, which are defined by the recursion relation[2]

$$S_0=0,\quad S_{n+1}=1/(2r-S_n). \tag{1.8}$$

Lemma 1. (1) *For any $r>1$, the sequence S_n is monotonically increasing, and*

$$\lim_{n\to\infty}S_n=S_\infty=r-\sqrt{r^2-1}<1, \tag{1.9}$$

$$S_\infty-S_n=0(1/r^{2n+1}). \tag{1.10}$$

(2)
$$S_n(r)=Q_{n-1}(r)/Q_n(r), \tag{1.11}$$

where $Q_n(r)$ is the n-th order Tchebyschev polynomial of the second kind, i.e.

$$Q_n(r)=2^n\prod_{l=1}^{n}(r-\alpha_{n,l}),\quad \alpha_{n,l}=\cos(l\pi/n+1). \tag{1.12}$$

Proof. From (1.8), it is easy to verify, by induction, that (1) holds and that $S_n(r)$ is a rational fraction of r, i.e.

$$S_n(r)=P_n(r)/Q_n(r),$$

where $P_n(r)$ and $Q(r)$ are respectively the $(n-1)$-th and n-th order polynomials of r. Therefore

$$S_{n+1}(r)=\frac{P_{n+1}(r)}{Q_{n+1}(r)}=\frac{1}{2r-S_n(r)}=\frac{1}{2r-P_n(r)/Q_n(r)}=\frac{Q_n(r)}{2rQ_n(r)-P_n(r)},$$

from which we have

$$P_{n+1}(r)=Q_n(r), \tag{1.13}$$

$$Q_{n+1}(r)=2rQ_n(r)-P_n(r)=2rQ_n(r)-Q_{n-1}(r). \tag{1.14}$$

(1.14) with the initial condition $P_1(r)=Q_0(r)=1$, which follows from (1.8), is the recursion relation for Tchebyschev polynomials of the second kind. This yields (2).

From this lemma and (1.6), the rational fraction approximations $K_{\pm}^{(n)}$ of $K_{\pm}$ can be obtained immediately:

$$K_{\pm}^{(n)}=\pm[K-K_xS_n]=\pm[K-K_x^2R_{n-1}(K_x, K)/R_n(K_x, K)], \tag{1.15}$$

where

$$R_n(K_x, K)=K_x^nQ_n(K/K_x). \tag{1.16}$$

Since $S_n(r)$ is a rational fraction of r, it can be decomposed into sums of simple fractions. Thus we have

Lemma 2.

$$S_n(r)=\frac{Q_{n-1}(r)}{Q_n(r)}=\frac{\prod_{j=1}^{n-1}(r-\alpha_{n-1,j})}{2\prod_{l=1}^{n}(r-\alpha_{n,l})}=\frac{1}{2}\sum_{l=1}^{n}\frac{\beta_{n,l}}{r-\alpha_{n,l}}, \tag{1.17}$$

where

$$\beta_{n,l}=\prod_{j=1}^{n-1}(\alpha_{n,l}-\alpha_{n-1,j})\Big/\prod_{j\neq l}^{n}(\alpha_{n,l}-\alpha_{n,j}). \tag{1.18}$$

Moreover

$$\beta_{n,l}=\beta_{n,n+1-l}>0, \tag{1.19}$$

$$\sum_{l=1}^{n}\beta_{n,l}=1. \tag{1.20}$$

Proof. (1.17), (1.18) can be easily verified by multiplying (1.17) by $(r-\alpha_{n,l})$ and substituting $\alpha_{n,l}$ for r. Because of (1.12) we have

$$\begin{aligned}\alpha_{n,1}>\alpha_{n-1,1}>\alpha_{n,2}>\alpha_{n-1,2}>\cdots>\alpha_{n,l-1}>\alpha_{n-1,l-1}\\ >\alpha_{n,l}>\alpha_{n-1,l}>\cdots>\alpha_{n-1,n-1}>\alpha_{n,n}\end{aligned} \tag{1.21}$$

and

$$\alpha_{n,l}=-\alpha_{n,n+1-l}. \tag{1.22}$$

From (1.21), (1.22) it follows that $\beta_{n,l}=\beta_{n,n+1-l}$ and that the numerator and the denominator in (1.18) have the same sign. This gives (1.19).

From (1.10) we have

$$\begin{aligned}0(1/r^{2n+1})&=S_\infty-S_n=(r-\sqrt{r^2-1})-\frac{1}{2}\sum_{l=1}^{n}\frac{\beta_{n-l}}{r-\alpha_{n,l}}\\ &=\left[\frac{1}{2r}+0\left(\frac{1}{r^3}\right)\right]-\left[\frac{1}{2r}\sum_{l=1}^{n}\beta_{n,l}+0\left(\frac{1}{r^3}\right)\right].\end{aligned} \tag{1.23}$$

The coefficient of $1/r$ on the right-hand side is equal to zero. This gives (1.20).

Using (1.17) and (1.6), we can write $K_{\pm}^{(n)}$ in the following form

$$K_{\pm}^{(n)}=\pm(K-K_xS_n)=\pm\left[K-\frac{K_x^2}{2}\sum_{l=1}^{n}\frac{\beta_{n,l}}{K-\alpha_{n,l}K_x}\right]. \tag{1.24}$$

2. Approximation of One-way Wave Equation

In this section we consider only the approximations of the one-way wave equation for up-coming wave. Those for down-going wave can be obtained in the same way.

Substituting expression (1.15) of $K_{-}^{(n)}$ for K_{-} in (1.4), we obtain the approximate one-way wave equation in frequency domain

$$\left(\frac{d}{dz}-iK_{-}^{(n)}\right)\hat{p}=\left\{\frac{d}{dz}+i\left[K-K_x^2R_{n-1}(K_x,\ K)/R_n(K_x,K)\right]\right\}\hat{p}=0. \tag{2.1}$$

Using the correspondence

$$\frac{\partial}{\partial x}\leftrightarrow -iK_x \quad \text{and} \quad \frac{1}{c}\frac{\partial}{\partial t}\leftrightarrow -i\frac{\omega}{c}=-iK, \tag{2.2}$$

we have from (2.1) that the inverse Fourier transform p of $\hat{p}$ satisfies the following approximate one-way wave equation

$$\left[R_n\left(i\frac{\partial}{\partial x},\ \frac{i}{c}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial z}-\frac{1}{c}\frac{\partial}{\partial t}\right)+iR_{n-1}\left(i\frac{\partial}{\partial x},\ \frac{i}{c}\frac{\partial}{\partial t}\right)\frac{\partial^2}{\partial x^2}\right]p=0. \tag{2.3}$$

This equation is obtained in [3] as the radiation boundary condition.

For $n=1$, we have

$$R_0(K_x,\ K)=1,\quad R_1(K_x,\ K)=2K.$$

Thus (2.3) becomes

$$\left[\frac{2}{c}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial z}-\frac{1}{c}\frac{\partial}{\partial t}\right)+\frac{\partial^2}{\partial x^2}\right]p=0, \tag{2.4}$$

which corresponds to the Claerbout equation[5] in the coordinate system

$$z'=z,\quad x'=x,\quad t'=t+z/c. \tag{2.5}$$

For $n=2$, we have

$$\left[\left(\frac{4}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial}{\partial z}-\frac{1}{c}\frac{\partial}{\partial t}\right)+\frac{2}{c}\frac{\partial^3}{\partial t\partial x^2}\right]p=0, \tag{2.6}$$

which corresponds to the so-called 45° equation[6].

For $n=3$, 4, we have

$$\left[\left(\frac{8}{c^3}\frac{\partial^3}{\partial t^3}-\frac{4}{c}\frac{\partial^3}{\partial t\,\partial x^2}\right)\left(\frac{\partial}{\partial z}-\frac{1}{c}\frac{\partial}{\partial t}\right)+\left(\frac{4}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)\frac{\partial^2}{\partial x^2}\right]p=0, \tag{2.7}$$

$$\left[\left(\frac{16}{c^4}\frac{\partial^4}{\partial t^4}-\frac{12}{c^2}\frac{\partial^4}{\partial t^2\partial x^2}+\frac{\partial^4}{\partial x^4}\right)\left(\frac{\partial}{\partial z}-\frac{1}{c}\frac{\partial}{\partial t}\right)+\left(\frac{8}{c^3}\frac{\partial^5}{\partial t^3\partial x^2}-\frac{4}{c}\frac{\partial^5}{\partial t\partial x^4}\right)\right]p=0, \tag{2.8}$$

which correspond to those obtained in [7, 8].

We can see that (2.3) is an $(n+1)$-th order P. D. E., which is difficult to apply in computation of the case $n>2$. In order to overcome this shortcoming, we derive a new form for approximations of the one-way wave equation. For simplicity we discuss only the case of even n. From (1.16) and (1.12), we have for even n

$$R_n(K_x, K)=2^n \prod_{l=1}^{n/2} (K^2-\alpha_{n,l}^2 K_x^2),\ R_{n-1}(K_x, K)=2^{n-1}K \prod_{l=1}^{n/2-1} (K^2-\alpha_{n-1,l}^2 K_x^2). \tag{2.9}$$

Hence (2.3) becomes

$$\Big[2^n \prod_{l=1}^{n/2}\Big(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\alpha_{n,l}^2\frac{\partial^2}{\partial x^2}\Big)\Big(\frac{\partial}{\partial z}-\frac{1}{c}\frac{\partial}{\partial t}\Big) +2^{n-1}\frac{1}{c}\frac{\partial}{\partial t}\prod_{l=1}^{n/2-1}\Big(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\alpha_{n-1,l}^2\frac{\partial^2}{\partial x^2}\Big)\frac{\partial^2}{\partial x^2}\Big]p=0. \tag{2.10}$$

Now we substitute expression (1.24) of $K_-^{(n)}$ in (2.1) and introduce new functions q_l, the Fourier transforms of which are defined by

$$\hat{q}_l(z; K_x, \omega)=\frac{\beta_{n,l}K_x^2}{K^2-\alpha_{n,l}^2 K_x^2}\hat{p}(z; K_x, \omega). \tag{2.11}$$

Then because of (1.19) and (1.22), (2.1) becomes

$$\Big(\frac{d}{dz}+iK\Big)\hat{p}-iK\sum_{l=1}^{n/2}\hat{q}_l=0. \tag{2.12}$$

Using correspondence (2.2), we have from (2.11) and (2.12) that p, q_l satisfy

$$\begin{cases}\Big(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\alpha_{n,l}^2\frac{\partial^2}{\partial x^2}\Big)q_l=\beta_{n,l}\frac{\partial^2 p}{\partial x^2}, \quad l=1, \cdots, n/2, & (2.13)\\ \Big(\frac{\partial}{\partial z}-\frac{1}{c}\frac{\partial}{\partial t}\Big)p=-\frac{1}{c}\sum_{l=1}^{n/2}\frac{\partial q_l}{\partial t}. & (2.14)\end{cases}$$

This system is a new form of approximations of the one-way wave equation.

Theorem 1. *If* $\{p, q_1, \cdots, q_{n/2}\}$ *is a sufficiently smooth solution of system* (2.13—14), *then* p *satisfies the approximate one-way wave equation* (2.10).

Proof. From (1.17) and (1.22), (1.19), we have for even n

$$r\prod_{l=1}^{n/2-1}(K^2-\alpha_{n-1,l}^2K_x^2)=2r\Big[\prod_{l=1}^{n/2}(K^2-\alpha_{n,l}^2K_x^2)\Big]\sum_{l=1}^{n/2}\frac{\beta_{n,l}}{K^2-\alpha_{n,l}^2K_x^2}$$
$$=2r\sum_{l=1}^{n/2}\Big[\beta_{n,l}\prod_{j\neq l}^{n/2}(K^2-\alpha_{n,j}^2K_x^2)\Big], \tag{2.15}$$

which implies

$$\prod_{l=1}^{n/2-1}\Big(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\alpha_{n-1,l}^2\frac{\partial^2}{\partial x^2}\Big)=2\sum_{l=1}^{n/2}\Big[\beta_{n,l}\prod_{j\neq l}^{n/2}\Big(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\alpha_{n,j}^2\frac{\partial^2}{\partial x^2}\Big)\Big]. \tag{2.16}$$

Applying the operator $2^n\prod_{l=1}^{n/2}\Big(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\alpha_{n,l}^2\frac{\partial^2}{\partial x^2}\Big)$ to (2.14) and using (2.13), (2.16), we verify immediately (2.10). The theorem is thus proved.

Applying $\frac{1}{c}\frac{\partial}{\partial t}$ to (2.14) and using (2.13), (1.20), we obtain

$$\frac{1}{c}\frac{\partial}{\partial t}\Big(\frac{\partial}{\partial z}-\frac{1}{c}\frac{\partial}{\partial t}\Big)p=-\frac{1}{2}\frac{\partial^2 p}{\partial x^2}-\sum_{l=1}^{n/2}\alpha_{n,l}^2\frac{\partial^2 q_l}{\partial x^2},$$

which can be considered as a corrected equation of the first order approximate one-way wave equation (2.4). The second term on the right-hand side is the correction term.

System (2.13—14) is more convenient for numerical computation than equation (2.10). Equation (2.13) is of second order with the one-dimensional wave operator $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\alpha_{n,l}^2\frac{\partial^2}{\partial x^2}$. Equation (2.14) is of first order with the directional derivatives $\frac{\partial}{\partial z}-\frac{1}{c}\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial t}$. It is not difficult to construct finite difference schemes for these equations. In comparison with equation (2.10), system (2.13—14) has another advantage in that it is uniform for different orders of approximation n. Therefore this system for different n can be treated by a unified computer program in numerical application.

3. Energy Estimation

In this section we discuss the initial-boundary value problem of system (2.13—14), which corresponds to the migration problem in seismic data processing. This problem consists in extrapolating downward the up-coming wave, recorded at the surface of earth. The mathematical problem in coordinate system (2.5) is the following:

$$\begin{cases}\frac{\partial p}{\partial z'}=\frac{-1}{c}\sum_{l=1}^{n/2}\frac{\partial q_l}{\partial t'}, \quad \text{for } z'>0,\ t'<T_{\max}, & (3.1)\\ \left(\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}-\alpha_{n,l}^2\frac{\partial^2}{\partial x'^2}\right)q_l=\beta_{n,l}\frac{\partial^2 p}{\partial x'^2}, \quad l=1,\cdots,n/2, & (3.2)\\ p(z',x',t')|_{z'=0}=\psi(x',t'), & (3.3)\\ p(z',x',t')=q_l(z',x',t')\equiv 0, \quad \text{for } t'\geqslant T_{\max}, & (3.4)\end{cases}$$

where p is the up-coming wave field, the known function $\psi(x',t')$ is the record at the surface of the earth. (3.4) means that reflection waves can be neglected for sufficiently large time t'. Now we transform these equations in the new coordinate system $z''=z'$, $x''=x'$, $t''=c(T_{\max}-t')$.

For simplicity we use the notation z, x, t for z'', x'', t''. Then (3.1—4) becomes

$$\begin{cases}\frac{\partial p}{\partial z}=\sum_{l=1}^{n/2}\frac{\partial q_l}{\partial t}, \quad \text{for } z>0,\ t>0, & (3.5)\\ \left(\frac{\partial^2}{\partial t^2}-\alpha_{n,l}^2\frac{\partial^2}{\partial x^2}\right)q_l=\beta_{n,l}\frac{\partial^2 p}{\partial x^2}, \quad l=1,\cdots,n/2, & (3.6)\\ p(z,x,t)|_{z=0}=\psi(x,t), & (3.7)\\ p(z,x,t)=q_l(z,x,t)\equiv 0, \quad \text{for } t\leqslant 0. & (3.8)\end{cases}$$

Theorem 2. *Let $\{p, q_1, \cdots, q_{n/2}\}$ be the solution of problem (3.5—8). Assume that $p, q_1, \cdots, q_{n/2}, \psi$ and their first derivatives are quadratically integrable with respect to x. Then the following estimates hold*

$$\int_0^\tau \int \left[\left(\frac{\partial p(\bar{z}, x, t)}{\partial x} \right)^2 + \left(\frac{\partial p(\bar{z}, x, t)}{\partial t} \right)^2 \right] dx\, dt$$

$$+ \int_0^{\bar{z}} \int \sum_{l=1}^{n/2} \frac{1}{\beta_{n,l}} \left[\alpha_{n,l}^2 \left(\frac{\partial q_l(z, x, \tau)}{\partial x} \right)^2 + (1+\alpha_{n,l}^2) \left(\frac{\partial q_l(z, x, \tau)}{\partial t} \right)^2 \right.$$

$$\left. + \left(\beta_{n,l} \frac{\partial p(z, x, \tau)}{\partial x} + \alpha_{n,l}^2 \frac{\partial q_l(z, x, \tau)}{\partial x} \right)^2 \right] dx\, dz \leqslant \int_0^\tau \int (\psi_x^2 + \psi_t^2)\, dx\, dt, \tag{3.9}$$

$$\int_0^{\bar{z}} \int \left\{ \left(\frac{\partial p(z, x, \tau)}{\partial x} \right)^2 + \sum_{l=1}^{n/2} \left[\left(\frac{\partial q_l(z, x, \tau)}{\partial x} \right)^2 + \left(\frac{\partial q_l(z, x, \tau)}{\partial t} \right)^2 \right] \right\} dx\, dz$$

$$\leqslant \text{const} \int_0^\tau \int (\psi_x^2 + \psi_t^2)\, dx\, dt, \tag{3.10}$$

where the interval of integration with respect to x is $(-\infty, \infty)$.

Proof. Applying $\frac{\partial}{\partial x}$ to (3.5), multiplying by $2\frac{\partial p}{\partial x}$ and integrating with respect to x, we have

$$\frac{\partial}{\partial z} \int \left(\frac{\partial p}{\partial x} \right)^2 dx = 2 \int \frac{\partial p}{\partial x} \left(\sum_{l=1}^{n/2} \frac{\partial^2 q_l}{\partial t\, \partial x} \right) dx. \tag{3.11}$$

Multiplying (3.6) by $\frac{2}{\beta_{n,l}} \frac{\partial q_l}{\partial t}$ and integrating with respect to x, we have

$$\frac{\partial}{\partial t} \int \frac{1}{\beta_{n,l}} \left[\left(\frac{\partial q_l}{\partial t} \right)^2 + \left(\alpha_{n,l} \frac{\partial q_l}{\partial x} \right)^2 \right] dx = -2 \int \frac{\partial p}{\partial x} \frac{\partial^2 q_l}{\partial t\, \partial x} dx. \tag{3.12}$$

From (3.11) and (3.12), we get

$$\frac{\partial}{\partial z} \int \left(\frac{\partial p}{\partial x} \right)^2 dx + \frac{\partial}{\partial t} \int \sum_{l=1}^{n/2} \frac{1}{\beta_{n,l}} \left[\left(\frac{\partial q_l}{\partial t} \right)^2 + \left(\alpha_{n,l} \frac{\partial q_l}{\partial x} \right)^2 \right] dx = 0. \tag{3.13}$$

Using (3.12), we obtain

$$-2\alpha_{n,l}^2 \int \frac{\partial p}{\partial t} \frac{\partial^2 q_l}{\partial x^2} dx = 2\alpha_{n,l}^2 \int \frac{\partial^2 p}{\partial t\, \partial x} \frac{\partial q_l}{\partial x} dx$$

$$= 2\alpha_{n,l}^2 \left\{ \frac{\partial}{\partial t} \int \frac{\partial p}{\partial x} \frac{\partial q_l}{\partial x} dx - \int \frac{\partial p}{\partial x} \frac{\partial^2 q_l}{\partial t\, \partial x} dx \right\}$$

$$= \alpha_{n,l}^2 \frac{\partial}{\partial t} \int \left\{ 2 \frac{\partial p}{\partial x} \frac{\partial q_l}{\partial x} + \frac{1}{\beta_{n,l}} \left[\left(\frac{\partial q_l}{\partial t} \right)^2 + \left(\alpha_{n,l} \frac{\partial q_l}{\partial x} \right)^2 \right] \right\} dx.$$

Hence

$$2 \int \frac{\partial p}{\partial t} \sum_{l=1}^{n/2} \frac{\partial^2 q_l}{\partial t^2} dx = 2 \int \frac{\partial p}{\partial t} \sum_{l=1}^{n/2} \left[\beta_{n,l} \frac{\partial^2 p}{\partial x^2} + \alpha_{n,l}^2 \frac{\partial^2 q_l}{\partial x^2} \right] dx$$

$$= -\frac{\partial}{\partial t} \int \sum_{l=1}^{n/2} \left\{ \beta_{n,l} \left(\frac{\partial p}{\partial x} \right)^2 + 2\alpha_{n,l}^2 \frac{\partial p}{\partial x} \frac{\partial q_l}{\partial x} \right.$$

$$\left. + \frac{\alpha_{n,l}^2}{\beta_{n,l}} \left[\left(\frac{\partial q_l}{\partial t} \right)^2 + \left(\alpha_{n,l} \frac{\partial q_l}{\partial x} \right)^2 \right] \right\} dx$$

$$= -\frac{\partial}{\partial t} \int \sum_{l=1}^{n/2} \frac{1}{\beta_{n,l}} \left[\left(\beta_{n,l} \frac{\partial p}{\partial x} + \alpha_{n,l}^2 \frac{\partial q_l}{\partial x} \right)^2 + \left(\alpha_{n,l} \frac{\partial q_l}{\partial t} \right)^2 \right] dx. \tag{3.14}$$

Applying $\frac{\partial}{\partial t}$ to (3.5), multiplying $2\frac{\partial p}{\partial t}$ and integrating with respect to x, we have

$$\int 2\,\frac{\partial p}{\partial t}\Big(\frac{\partial^2 p}{\partial t\partial z}-\sum_{l=1}^{n/2}\frac{\partial^2 q_l}{\partial t^2}\Big)dx$$

$$=\frac{\partial}{\partial z}\int\Big(\frac{\partial p}{\partial t}\Big)^2dx+\frac{\partial}{\partial t}\int\sum_{l=1}^{n/2}\frac{1}{\beta_{n,l}}\Big[\Big(\beta_{n,l}\frac{\partial p}{\partial x}+\alpha_{n,l}^2\frac{\partial q_l}{\partial x}\Big)^2$$

$$+\Big(\alpha_{n,l}\frac{\partial q_l}{\partial t}\Big)^2\Big]dx=0. \tag{3.15}$$

Combining (3.13) with (3.15), we get

$$\frac{\partial}{\partial z}\int\Big[\Big(\frac{\partial p}{\partial x}\Big)^2+\Big(\frac{\partial p}{\partial t}\Big)^2\Big]dx+\frac{\partial}{\partial t}\int\sum_{l=1}^{n/2}\frac{1}{\beta_{n,l}}\Big[\alpha_{n,l}^2\Big(\frac{\partial q_l}{\partial x}\Big)^2$$

$$+(1+\alpha_{n,l}^2)\Big(\frac{\partial q_l}{\partial t}\Big)^2+\Big(\beta_{n,l}\frac{\partial p}{\partial x}+\alpha_{n,l}^2\frac{\partial q_l}{\partial x}\Big)^2\Big]dx=0. \tag{3.16}$$

Integrating (3.16) with respect to z, t in the domain $(0,\bar{z})\times(0,\tau)$, we obtain immediately (3.9).

Using conditions (3.7) and (3.8), we obtain by integrating (3.13)

$$\int_0^{\bar z}\int\sum_{l=1}^{n/2}\frac{1}{\beta_{n,l}}\Big[\Big(\frac{\partial q_l(z,x,\tau)}{\partial t}\Big)^2+\Big(\alpha_{n,l}\frac{\partial q_l(z,x,\tau)}{\partial x}\Big)^2\Big]dx\,dz\leqslant\int_0^{\tau}\int\psi_x^2\,dx\,dt. \tag{3.17}$$

In the same way, from (3.15) we obtain

$$\int_0^{\bar z}\int\sum_{l=1}^{n/2}\Big\{\beta_{n,l}\Big(\frac{\partial p(z,x,\tau)}{\partial x}\Big)^2+2\alpha_{n,l}^2\frac{\partial p(z,x,\tau)}{\partial x}\,\frac{\partial q_l(z,x,\tau)}{\partial x}+\frac{\alpha_{n,l}^4}{\beta_{n,l}}\Big(\frac{\partial q_l(z,x,\tau)}{\partial x}\Big)^2$$

$$+\frac{\alpha_{n,l}^2}{\beta_{n,l}}\Big(\frac{\partial q_l(z,x,\tau)}{\partial t}\Big)^2\Big\}dx\,dz\leqslant\int_0^{\tau}\int\psi_t^2\,dx\,dt. \tag{3.18}$$

Because of (1.19) and (1.12), it is easy to obtain (3.10) by multiplying (3.17) by a sufficiently large constant and adding (3.18). The theorem is thus proved.

Using system (3.5—8), we can perform steep dip migration by the finite difference method. This will be discussed in another article.

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