SOME NONLINEAR BOUNDARY PROBLEMS FOR THE SYSTEMS OF NONLINEAR WAVE EQUATIONS BY FINITE SLICE METHOD*

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§1. Introduction

1. The purpose of this work is to study some nonlinear boundary problems for the system

$$u_{tt}-u_{xx}+\text{grad } F(u)=B(x, t, u)u_t+f(x, t, u, u_x, u_t) \tag{1}$$

of the nonlinear wave equations in the rectangular domain $Q_T=\{0\leqslant x\leqslant l, 0\leqslant t\leqslant T\}$, where $u(x, t)=(u_1(x, t), \cdots, u_m(x, t))$ is the m-dimensional unknown vector function, $f(x, t, u, p, q)$ is given m-dimensional vector function of $(x, t)\in Q_T$ and $u, p, q\in\mathbb{R}^m$, $B(x, t, u)$ is a $m\times m$ matrix of $(x, t)\in Q_T$ and $u\in\mathbb{R}^m$, $F(u)$ is a non-negative scalar function of $u\in\mathbb{R}^m$ and "grad" is the gradient operator with respect to $u\in\mathbb{R}^m$. The well-known Sine-Gordon equation

$$u_{tt}-u_{xx}=\sin u,$$

the nonlinear forced vibration equation

$$u_{tt}-u_{xx}+u^3=0$$

and the nonlinear wave equation

$$u_{tt}-u_{xx}+\sinh u=0$$

are the simple cases of the above mentioned system (1) of nonlinear wave equations. Many authors have paid great attention to the study of the various problems for these special nonlinear wave equations[1–17]. Some general systems of this type have been considered in [18—20].

At first we are going to consider the boundary problem for the system (1) with the fairly wide nonlinear mutual boundary conditions

$$\begin{aligned} u_x(0, t)&=\text{grad}_0\ \Phi(u(0, t), u(l, t), t), \\ -u_x(l, t)&=\text{grad}_1\ \Phi(u(0, t), u(l, t), t) \end{aligned} \tag{2}$$

and the initial conditions

$$\begin{aligned} u(x, 0)&=\varphi(x), \\ u_t(x, 0)&=\psi(x), \end{aligned} \tag{3}$$

where $\Phi(u_0, u_1, t)$ is a non-negative scalar function of $t\in[0, T]$ and $u_0, u_1\in\mathbb{R}^m$, "grad_0" and "grad_1" are the gradient operators with respect to the vector variables

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u_0 and u_1 respectively and $\varphi(x)$ and $\psi(x)$ are two m-dimensional initial vector functions.

When $\Phi(u_0, u_1, t)=\Phi_0(u_0, t)+\Phi_1(u_1, t)$, the boundary conditions (2) become the ordinary nonlinear (non-mutual) boundary conditions

$$\begin{aligned} u_x(0, t) &= \text{grad } \Phi_0(u(0, t), t), \\ -u_x(l, t) &= \text{grad } \Phi_1(u(l, t), t). \end{aligned} \tag{4}$$

If $\Phi(u_0, u_1, t)$ is a polynomial of $u_0, u_1 \in \mathbb{R}^m$ of the form

$$\begin{aligned} \Phi(u_0, u_1, t) = (u_0, B_{00}(t)u_0) + (u_0, B_{01}(t)u_1) + (u_1, B_{10}(t)u_0) \\ + (u_1, B_{11}(t)u_1) + (g_0(t), u_0) + (g_1(t), u_1), \end{aligned} \tag{5}$$

then (2) are simplified to a linear symmetric boundary conditions

$$\begin{aligned} u_x(0, t) &= (B_{00}(t)+B^*_{00}(t))u(0, t) + (B_{01}(t)+B^*_{10}(t))u(l, t)+g_0(t), \\ -u_x(l, t) &= (B^*_{01}(t)+B_{10}(t))u(0, t) + (B_{11}(t)+B^*_{11}(t))u(l, t)+g_1(t), \end{aligned} \tag{6}$$

where $B(t)$'s are the $m\times m$ matrices, "$*$" denotes the transpose of matrix and $g_0(t)$ and $g_1(t)$ are two m-dimensional vector functions of $t\in[0, T]$.

For the nonlinear partial differential equations and systems, it is natural to take into consideration of the nonlinear boundary problems both in theoretical and in pratical studies[21,22]. Hence the nonlinear boundary problems are of the number of the fundamental problems as the classical linear boundary problems.

In § 2 of the present work, we will give a series of a priori estimations for the solution $v_j(t)$ $(j=0, 1, \cdots, J)$ of the nonlinear finite slice system. Then we will establish the existence of the solution $v_j(t)$ $(j=0, 1, \cdots, J)$ for the nonlinear finite slice system by the fixed point method on the base of these estimations. By the limit process as $h\to 0$, we will obtain the generalized solution $u(x, t)$ of the ordinary boundary problem (2) and (3) for the system (1) of nonlinear wave equations. By this way the convergence behaviors of the solution $v_j(t)$ $(j=0, 1, \cdots, J)$ of the nonlinear finite slice system are studied.

At the end of this work, we will take in consideration of some more general nonlinear boundary problems with the mixed conditions[23,24]

$$\begin{aligned} u_x(0, t) &= \text{grad } \Phi(u(0, t), t), \\ -u_t(l, t) &= \Phi_1(u_x(l, t), u(0, t), u(l, t), t) \end{aligned} \tag{7}$$

by the finite slice method.

We adopt the similar notations and conventions as used in [18—20, 25, 26].

2. Suppose that for the system (1) of nonlinear wave equations, the nonlinear boundary conditions and the initial vector functions, $\varphi(x)$ and $\psi(x)$ the following conditions are satisfied:

(I) $F(u)\geqslant 0$ is a non-negative twice continuously differentiable scalar function of vector variables $u\in\mathbb{R}^m$.

(II) $B(x, t, u)$ is a $m\times m$ matrix, continuous for $(x, t)\in Q_T$ and $u\in\mathbb{R}^m$ and continuously differentiable with respect to x and u. $B(x, t, u)$ is semibound, i. e., for any $\xi\in\mathbb{R}^m$, $(\xi, B(x, t, u)\xi)\leqslant b|\xi|^2$, where b is a constant.

(III) $f(x, t, u, p, q)$ is a m-dimensional vector function of lower degree, continuous for $(x, t)\in Q_T$ and $u, p, q\in R^m$ and continuously differentiable with respect to x, u, p and q, i.e., for any $(x, t)\in Q_T$ and $u, p, q\in\mathbb{R}^m$, there are

$$\begin{aligned}&|f(x, t, u, p, q)|^2,\ |f_x(x, t, u, p, q)|^2\leqslant A\{F(u)+|u|^2+|p|^2+|q|^2+1\},\\&|f_u(x, t, u, p, q)|^2\leqslant A\{F(u)+|u|^2+1\},\\&|f_p(x, t, u, p, q)|,\ |f_q(x, t, u, p, q)|\leqslant A,\end{aligned} \tag{8}$$

where A is a constant and $|\cdot|$ denotes any component of appropriate vectors and any element of appropriate matrices.

(IV) $\Phi(u_0, u_1, t)\geqslant 0$ is a non-negative scalar function of $t\in[0, T]$ and u_0, $u_1\in\mathbb{R}^m$. $\Phi(u_0, u_1, t)$, $\Phi_t(u_0, u_1, t)$ and $\Phi_{tt}(u_0, u_1, t)$ are three times, twice and once continuously differentiable with respect to $u_0, u_1\in\mathbb{R}^m$. And

$$|\Phi_t(u_0, u_1, t)|\leqslant A\{|u_0|^2+|u_1|^2+1\}, \tag{9}$$

where A is a constant. Furthermore, the Hessian matrix

$$H(u_0, u_1, t)=\begin{pmatrix}\mathrm{grad}_0^2\Phi(u_0, u_1, t), & \mathrm{grad}_0\mathrm{grad}_1\Phi(u_0, u_1, t)\\ \mathrm{grad}_1\,\mathrm{grad}_0\,\Phi(u_0, u_1, t), & \mathrm{grad}_1^2\Phi(u_0, u_1, t)\end{pmatrix}$$

of $\Phi(u_0, u_1, t)$ with respect to u_0 and u_1 is non-negatively definite. Hence $\Phi(u_0, u_1, t)$ is convex with respect to $u_0, u_1\in\mathbb{R}^m$, i.e., for any $\tau\in[0, 1]$,

$$\Phi(\tau u_0+(1-\tau)\bar{u}_0,\ \tau u_1+(1-\tau)\bar{u}_1,\ t)\leqslant\tau\Phi(u_0, u_1, t)+(1-\tau)\Phi(\bar{u}_0, \bar{u}_1, t),$$

where $u_0, u_1, \bar{u}_0, \bar{u}_1\in\mathbb{R}^m$.

(V) The m-dimensional initial vector function $\varphi(x)\in H^2(0, l)$ satisfies the boundary conditions (2), i.e.,

$$\begin{aligned}\varphi'(0)&=\mathrm{grad}_0\Phi(\varphi(0), \varphi(l), 0),\\-\varphi'(l)&=\mathrm{grad}_1\Phi(\varphi(0), \varphi(l), 0).\end{aligned} \tag{2$_0$}$$

And $\psi(x)\in H^1(0, l)$.

3. Let us divided the rectangular domain Q_T into thin slice by the parallel lines $x=x_j$ $(j=0, 1, \cdots, J)$, where $x_j=jh$ $(j=0, 1, \cdots, J)$ and $Jh=l$. Denote the m-dimensional discrete vector function on the slice lines $x=x_j$ by $v_j(t)$ $(j=0, 1, \cdots, J)$. We take the finite slice system

$$v_j''-\frac{\Delta_+\Delta_-v_j}{h^2}+\mathrm{grad}\,F(v_j)=B(x_j, t, v_j)v_j'+f\left(x_j, t, v_j, \frac{\bar{\Delta}v_j}{h}, v_j'\right), \tag{1$_h$}$$

$$j=1, 2, \cdots, J-1$$

corresponding to the system (1) of nonlinear wave equations, where $\frac{\bar{\Delta}v_j}{h}=b\,\frac{\Delta_+v_j}{h}+b'\frac{\Delta_-v_j}{h}$ with $b+b'=1$. The finite slice boundary conditions corresponding to the nonlinear mutual boundary conditions (2) are as follows:

$$\begin{aligned}\frac{\Delta_+v_0}{h}&=\mathrm{grad}_0\,\Phi(v_0, v_J, t),\\-\frac{\Delta_-v_J}{h}&=\mathrm{grad}_1\,\Phi(v_0, v_J, t).\end{aligned} \tag{2$_h$}$$

The finite slice initial conditions

$$\begin{aligned}v_j(0)&=\varphi_j, & j&=1, 2, \cdots, J-1\\v_j'(0)&=\psi_j, & j&=0, 1, \cdots, J\end{aligned} \tag{3$'_h$}$$

and $v_0(0)$ and $v_J(0)$ are the unique solution of the system

$$v_0(0)=\varphi_1-h\,\mathrm{grad}_0\,\Phi(v_0(0),\ v_J(0),\ 0),$$
$$v_J(0)=\varphi_{J-1}-h\,\mathrm{grad}_1\,\Phi(v_0(0),\ v_J(0),\ 0), \tag{3''$_h$}$$

where $\varphi_j=\varphi(x_j)$ and $\psi_j=\psi(x_j)$ $(j=0,\ 1,\ \cdots,\ J)$.

It can be verified that the system $(3'')_h$ has a unique solution $v_0(0)$ and $v_J(0)$, when the Hessian matrix of $\Phi(u_0,\ u_1,\ t)$ with respect to $u_0,\ u_1\in\mathbb{R}^m$ is assumed to be non-negatively definite.

§ 2. A Priori Estimations

4. In this section we want to get a series of estimates for the m-dimensional discrete finite slice function $v_j(t)$ $(j=0,\ 1,\ \cdots,\ J)$ of the nonlinear system $(1)_h$, $(2)_h$ and $(3)_h$.

Making the scalar product of the m-dimensional vector $v_j'(t)h$ with the m-dimensional vector equation $(1)_h$ and summing up for $j=1,\ 2,\ \cdots,\ J-1$ the resulting relations, we get

$$\sum_{j=1}^{J-1}(v_j',\ v_j'')h-\sum_{j=1}^{J-1}\Big(v_j',\ \frac{\Delta_+\Delta_-v_j}{h^2}\Big)h+\sum_{j=1}^{J-1}(v_j',\ \mathrm{grad}\ F(v_j))h$$
$$=\sum_{j=1}^{J-1}(v_j',\ B(x_j,\ t,\ v_j)v_j')h+\sum_{j=1}^{J-1}\Big(v_j',\ f\Big(x_j,\ t,\ v_j,\ \frac{\overline{\Delta}v_j}{h},\ v_j'\Big)\Big)h. \tag{10}$$

For the first, third and fourth terms of the above equality, we have

$$\sum_{j=1}^{J-1}(v_j',\ v_j'')h=\frac{1}{2}\ \frac{d}{dt}\Big(\sum_{j=1}^{J-1}|v_j'|^2h\Big),$$
$$\sum_{j=1}^{J-1}(v_j',\ \mathrm{grad}F(v_j))h=\frac{d}{dt}\Big(\sum_{j=1}^{J-1}F(v_j)h\Big)$$

and

$$\sum_{j=1}^{J-1}(v_j',\ B(x_j,\ t,\ v_j)v_j')h\leqslant b\Big(\sum_{j=1}^{J-1}|v_j'|^2h\Big).$$

As to the second term, we can derive as follows:

$$-\sum_{j=1}^{J-1}\Big(v_j',\ \frac{\Delta_+\Delta_-v_j}{h^2}\Big)h=\sum_{j=0}^{J-1}\Big(\frac{\Delta_+v_j'}{h},\ \frac{\Delta_+v_j}{h}\Big)h+\Big(v_0',\ \frac{\Delta_+v_0}{h}\Big)-\Big(v_J',\ \frac{\Delta_-v_J}{h}\Big)$$
$$=\frac{1}{2}\ \frac{d}{dt}\Big(\sum_{j=0}^{J-1}\Big|\frac{\Delta_+v_j}{h}\Big|^2h\Big)+(v_0',\ \mathrm{grad}_0\,\Phi(v_0,\ v_J,\ t))+(v_J',\ \mathrm{grad}_1\,\Phi(v_0,\ v_J,\ t))$$
$$=\frac{1}{2}\ \frac{d}{dt}\|\delta v_h(t)\|_2^2+\frac{d}{dt}\ \Phi(v_0,\ v_J,\ t)-\Phi_t(v_0,\ v_J,\ t). \tag{11}$$

From the condition (IV), we have

$$|\Phi_t(v_0,\ v_J,\ t)|\leqslant A\{|v_0|^2+|v_J|^2+1\}$$
$$\leqslant C_1\{|v_1|^2+|v_{J-1}|^2+h\|\delta v_h\|_2^2+1\}$$
$$\leqslant C_2\{(\max_{j=1,2,\cdots,J-1}|v_j|)^2+\|\delta v_h\|_2^2+1\},$$

where

$$|v_0|\leqslant|v_1|+\Big|\frac{\Delta_+v_0}{h}\Big|h\leqslant|v_1|+h^{\frac{1}{2}}\|\delta v_h\|_2,$$
$$|v_J|=|v_{J-1}|+\Big|\frac{\Delta_-v_J}{h}\Big|h\leqslant|v_{J-1}|+h^{\frac{1}{2}}\|\delta v_h\|_2 \tag{12}$$

and C_1 and C_2 are constants independent of h. Since

$$\max_{j=1,2,\cdots,J-1}|v_j|\leqslant\|\delta v_h\|_2+C_3\Big(\sum_{j=1}^{J-1}|v_j|^2h\Big)^{\frac{1}{2}} \tag{13}$$

and

$$\sum_{j=1}^{J-1}|v_j|^2h\leqslant 2\sum_{j=1}^{J-1}|\varphi_j|^2h+C_4\int_0^t\Big(\sum_{j=1}^{J-1}|v_j'|^2h\Big)dt. \tag{14}$$

We have

$$|\Phi_t(v_0,\ v_J,\ t)|\leqslant C_5\Big\{\|\delta v_h\|_2^2+\int_0^t\Big(\sum_{j=1}^{J-1}|v_j'|^2h\Big)dt+1\Big\}.$$

Hence (11) becomes

$$-\sum_{j=1}^{J-1}\Big(v_j',\ \frac{\Delta_+\Delta_-v_j}{h^2}\Big)h\geqslant\frac{1}{2}\frac{d}{dt}\|\delta v_h(t)\|_2^2+\frac{d}{dt}\Phi(v_0,\ v_J,\ t)$$
$$-C_6\Big\{\|\delta v_h(t)\|_2^2+\int_0^t\Big(\sum_{j=1}^{J-1}|v_j'|^2h\Big)dt\Big\}-C_6,$$

where C_6 is a constant independent of h. As convention, we denote by C with different index the constant independent of $h>0$.

For the last term of the equality (10), we see that

$$\Big|\sum_{j=1}^{J-1}(v_j',\ f_j)h\Big|\leqslant\frac{1}{2}\sum_{j=1}^{J-1}|v_j'|^2h+\frac{1}{2}\sum_{j=1}^{J-1}|f_j|^2h$$
$$\leqslant C_7\Big\{\sum_{j=1}^{J-1}|v_j'|^2h+\sum_{j=1}^{J-1}F(v_j)h+\sum_{j=1}^{J-1}\Big|\frac{\bar{\Delta}v_j}{h}\Big|^2h+\sum_{j=1}^{J-1}|v_j|^2h\Big\}+C_7$$
$$\leqslant C_8\Big\{\sum_{j=1}^{J-1}|v_j'|^2h+\|\delta v_h\|_2^2+\sum_{j=1}^{J-1}F(v_j)h+\int_0^t\sum_{j=1}^{J-1}|v_j'|^2h\Big\}+C_8,$$

where

$$f_j=f\Big(x_j,\ t,\ v_j,\ \frac{\bar{\Delta}v_j}{h},\ v_j'\Big),\qquad j=1,\ 2,\ \cdots,J-1.$$

Thus (10) becomes

$$\frac{d}{dt}\Big\{\sum_{j=1}^{J-1}|v_j'(t)|^2h+\|\delta v_h(t)\|_2^2+2\sum_{j=1}^{J-1}F(v_j(t))h+2\Phi(v_0(t),\ v_J(t),\ t)\Big\}$$
$$\leqslant C_9\Big\{\sum_{j=1}^{J-1}|v_j'(t)|^2h+\|\delta v_h(t)\|_2^2+\sum_{j=1}^{J-1}F(v_j(t))h+\int_0^t\Big(\sum_{j=1}^{J-1}|v_j'(t)|^2h\Big)dt\Big\}+C_9.$$

Let us denote

$$w(t)=\sum_{j=1}^{J-1}|v_j'(t)|^2h+\|\delta v_h(t)\|_2^2+2\sum_{j=1}^{J-1}F(v_j(t))h+2\Phi(v_0(t),\ v_J(t),\ t). \tag{15}$$

Since $F(u)$ and $\Phi(u_0,\ u_1,\ t)$ are two non-negative scalar functions, we have

$$\frac{d}{dt}w(t)\leqslant C_9w(t)+C_9+C_9\int_0^t w(\tau)d\tau,$$

where

$$w(0)=\sum_{j=1}^{J-1}|\psi_j|^2h+\|\delta\varphi_h\|_2^2+2\sum_{j=1}^{J-1}F(\varphi_j)h+2\Phi(\varphi_0,\ \varphi_J,\ t)$$

is clearly bounded. This follows that

$$w(t)\leqslant C_{10},\quad t\in[0,\ T], \tag{16}$$

where C_{10} is a constant independent of h and $t\in[0,\ T]$.

Lemma 1. *Under the conditions* (I), (II), (III), (IV) *and* (V) *for the solution* $v_j(t)$ $(j=0,\ 1,\ \cdots,\ J)$ *of the nonlinear finite slice system* $(1)_h$, $(2)_h$ *and* $(3)_h$ *and for* *sufficiently small Δt, there are estimates*

$$\sup_{0<t<T}\|v_h(t)\|_\infty+\sup_{0<t<T}\|v_h'(t)\|_2+\sup_{0<t<T}\|\delta v_h(t)\|_2\leqslant K_1, \tag{17}$$

where K_1 is a constant independent of h and $t\in[0, T]$.

Proof. From (15) and (16), we see that the third estimate of (17) is valid. Again from (15) and (16), we have for any $t\in[0, T]$

$$\sum_{j=1}^{J-1}|v_j'(t)|^2h\leqslant C_{10}.$$

Then from (13) and (14), we get

$$\max_{j=1,2,\cdots,J-1}|v_j(t)|\leqslant C_{11}.$$

Combining this result and (12), it follows directly the first estimate of (17). For the second estimate of (17) we have from (15) and (16)

$$\sum_{j=1}^{J-1}|v_j'(t)|^2h\leqslant C_{10}.$$

Differentiating the finite slice boundary conditions $(2)_h$, we obtain

$$(E+h\,\mathrm{grad}_0^2\,\Phi(v_0, v_J, t))v_0'+h\,\mathrm{grad}_1\,\mathrm{grad}_0\,\Phi(v_0, v_J, t)v_J'=v_1'-h\,\mathrm{grad}_0\,\Phi_t(v_0, v_J, t),$$
$$h\,\mathrm{grad}_0\,\mathrm{grad}_1\,\Phi(v_0, v_J, t)v_0'+(E+h\,\mathrm{grad}_1^2\,\Phi(v_0, v_J, t))v_J'=v_{J-1}'-h\,\mathrm{grad}_1\,\Phi_t(v_0, v_J, t),$$

where E is a $m\times m$ unit matrix. This shows that v_0' and v_J' can be expressed as the linear functions of v_1' and v_{J-1}'. Thus

$$\|v_h'(t)\|_2^2\leqslant C_{12}\sum_{j=1}^{J-1}|v_j'(t)|^2h+C_{12}.$$

Hence the second estimate of (17) is proved. The lemma is proved.

5. Now we take the scalar product of the m-dimensional vector $\frac{\Delta_+\Delta_-v_j'}{h^2}h$ and the m-dimensional vector equation $(1)_h$ and sum up the resulting relations for $j=1, 2, \cdots, J-1$. Then we obtain

$$\sum_{j=1}^{J-1}\Big(\frac{\Delta_+\Delta_-v_j'}{h^2}, v_j''\Big)h-\sum_{j=1}^{J-1}\Big(\frac{\Delta_+\Delta_-v_j'}{h^2}, \frac{\Delta_+\Delta_-v_j}{h^2}\Big)h+\sum_{j=1}^{J-1}\Big(\frac{\Delta_+\Delta_-v_j'}{h^2}, \mathrm{grad}\,F(v_j)\Big)h$$
$$=\sum_{j=1}^{J-1}\Big(\frac{\Delta_+\Delta_-v_j'}{h^2}, B(x_j, t, v_j)v_j'\Big)h+\sum_{j=1}^{J-1}\Big(\frac{\Delta_+\Delta_-v_j'}{h^2}, f_j\Big)h. \tag{18}$$

Here we have

$$\sum_{j=1}^{J-1}\Big(\frac{\Delta_+\Delta_-v_j'}{h^2}, \frac{\Delta_+\Delta_-v_j}{h^2}\Big)h=\frac{1}{2}\frac{d}{dt}\|\delta^2v_h(t)\|_2^2. \tag{19}$$

Let us now simplify the first term of the equality (18). It is clear that

$$\sum_{j=1}^{J-1}\Big(\frac{\Delta_+\Delta_-v_j'}{h^2}, v_j''\Big)h=-\sum_{j=0}^{J-1}\Big(\frac{\Delta_+v_j'}{h}, \frac{\Delta_+v_j''}{h}\Big)h-\Big(\frac{\Delta_+v_0'}{h}, v_0''\Big)+\Big(\frac{\Delta_-v_J'}{h}, v_J''\Big)$$
$$=-\frac{1}{2}\frac{d}{dt}\|\delta v_h'(t)\|_2^2-\Big(\frac{d}{dt}\,\mathrm{grad}_0\,\Phi(v_0, v_J, t), v_0''\Big)-\Big(\frac{d}{dt}\,\mathrm{grad}_1\,\Phi(v_0, v_J, t), v_J''\Big)$$
$$=-\frac{1}{2}\frac{d}{dt}\|\delta v_h'(t)\|_2^2-J_1, \tag{20}$$

where J_1 denotes the sum of the last two parenthesis. Here we have

$$
\begin{aligned}
J_1 = & (\mathrm{grad}_0^2\, \Phi v_0',\ v_0'') + (\mathrm{grad}_1\, \mathrm{grad}_0\, \Phi v_J',\ v_0'') + (\mathrm{grad}_0\, \mathrm{grad}_1\, \Phi v_0',\ v_J'') \\
& + (\mathrm{grad}_1^2\, \Phi v_J',\ v_J'') + (\mathrm{grad}_0\, \Phi_t,\ v_0'') + (\mathrm{grad}_1 \Phi_t,\ v_J'') \\
= & \frac{1}{2} \frac{d}{dt} \{ (\mathrm{grad}_0^2\, \Phi v_0',\ v_0') + (\mathrm{grad}_1\, \mathrm{grad}_0\, \Phi v_J',\ v_0') + (\mathrm{grad}_0\, \mathrm{grad}_1\, \Phi v_0',\ v_J') \\
& + (\mathrm{grad}_1^2\, \Phi v_J',\ v_J') \} - \frac{1}{2} \{ ((\mathrm{grad}_0^2\, \Phi)_t v_0',\ v_0') + ((\mathrm{grad}_1\, \mathrm{grad}_0\, \Phi)_t v_J',\ v_0') \\
& + ((\mathrm{grad}_0\, \mathrm{grad}_1\, \Phi)_t v_0',\ v_J') + ((\mathrm{grad}_1^2\, \Phi)_t v_J',\ v_J') \} \\
& + \frac{d}{dt} \{ (\mathrm{grad}_0 \Phi_t,\ v_0') + (\mathrm{grad}_1 \Phi_t,\ v_J') \} - \{ ((\mathrm{grad}_0\, \Phi_t)_t,\ v_0') + ((\mathrm{grad}_1\, \Phi_t)_t,\ v_J') \} \\
= & \frac{1}{2} \frac{d}{dt} J_2(t) - J_3(t) + \frac{d}{dt} J_4(t) - J_5(t),
\end{aligned} \tag{21}
$$

where the expressions in the curved brackets are denoted by $J_2(t)$, $J_3(t)$, $J_4(t)$ and $J_5(t)$ in order. In fact the $2m \times 2m$ matrix $\begin{pmatrix} \mathrm{grad}_0^2\, \Phi, & \mathrm{grad}_1 \mathrm{grad}_0\, \Phi \\ \mathrm{grad}_0\, \mathrm{grad}_1\, \Phi, & \mathrm{grad}_1^2\, \Phi \end{pmatrix}$ is the Hessian matrix of the scalar function $\Phi(v_0,\ v_J,\ t)$ with respect to the $2m$-dimensional vector $(v_0,\ v_J)$, thus it is symmetric. It is easy to see that

$$
\begin{aligned}
& |J_2(t)| \leqslant C_{13}\{ |v_0'(t)|^2 + |v_J'(t)|^2 \}, \\
& |J_3(t)| \leqslant C_{13}\{ |v_0'(t)|^3 + |v_J'(t)|^3 + |v_0'(t)|^2 + |v_J'(t)|^2 \}, \\
& |J_4(t)| \leqslant C_{13}\{ |v_0'(t)| + |v_J'(t)| \}, \\
& |J_5(t)| \leqslant C_{13}\{ |v_0'(t)|^2 + |v_J'(t)|^2 + |v_0'(t)| + |v_J'(t)| \}.
\end{aligned}
$$

Since
$$
|v_0'(t)| + |v_J'(t)| \leqslant 2\| v_h'(t) \|_\infty \leqslant C_{14} \| v_h'(t) \|_2^{\frac{1}{2}} \| \delta v_h'(t) \|_2^{\frac{1}{2}} + C_{14} \| v_h'(t) \|_2,
$$
then
$$
|J_i(t)| \leqslant \frac{1}{4} \| \delta v_h'(t) \|_2^2 + C_{15}, \quad i = 2,\ 3,\ 4,\ 5. \tag{22}
$$

Hence we have
$$
\sum_{j=1}^{J-1} \Big(\frac{\Delta_+ \Delta_- v_j'}{h^2},\ v_j'' \Big) h = -\frac{1}{2} \frac{d}{dt} (\| \delta v_h'(t) \|_2^2 + J_2(t) + 2J_4(t)) + J_3(t) + J_5(t). \tag{23}
$$

For the third term of (18), we have
$$
\begin{aligned}
\sum_{j=1}^{J-1} \Big(\frac{\Delta_+ \Delta_- v_j'}{h^2}, \mathrm{grad}\ F(v_j) \Big) h = & - \sum_{j=0}^{J-1} \Big(\frac{\Delta_+ v_j'}{h}, \frac{\Delta_+}{h} (\mathrm{grad}\ F(v_j)) \Big) h \\
& - \Big(\frac{\Delta_+ v_0'}{h},\ \mathrm{grad}\ F(v_0) \Big) + \Big(\frac{\Delta_- v_J'}{h},\ \mathrm{grad}\, F(v_J) \Big).
\end{aligned}
$$

From
$$
\frac{\Delta_+}{h} (\mathrm{grad}\ F(v_j)) = \Big(\int_0^1 \mathrm{grad}^2\ F(\tau v_{j+1} + (1-\tau) v_j) d\tau \Big) \frac{\Delta_+ v_j}{h},
$$
we get
$$
\Big| \sum_{j=0}^{J-1} \Big(\frac{\Delta_+ v_j'}{h},\ \frac{\Delta_+}{h}\ \mathrm{grad}\ F(v_j) \Big) h \Big| \leqslant \| \delta v_h'(t) \|_2^2 + C_{16}.
$$

On the other hand, we have
$$
\begin{aligned}
\Big| \Big(\frac{\Delta_+ v_0'}{h},\ \mathrm{grad}\ F(v_0) \Big) \Big| & = | ((\mathrm{grad}_0\ \Phi)_t,\ \mathrm{grad}\ F(v_J)) | \\
& \leqslant C_{17}\{ |v_0'(t)| + |v_J'(t)| + 1 \} \leqslant \| \delta v_h'(t) \|_2^2 + C_{18}.
\end{aligned}
$$

Similarly
$$
\Big| \Big(\frac{\Delta_- v_J'}{h},\ \mathrm{grad}\ F(v_J) \Big) \Big| \leqslant \| \delta v_h'(t) \|_2^2 + C_{18}.
$$

Hence we obtain

$$\left|\sum_{j=1}^{J-1}\left(\frac{\Delta_+\Delta_- v'_j}{h^2}, \operatorname{grad} F(v_j)\right)h\right| \leqslant C_{19}\{\|\delta v'_h(t)\|_2^2+1\}. \tag{24}$$

Now we turn to consider the first term on the right hand side of the equality (18). At first we have

$$\sum_{j=1}^{J-1}\left(\frac{\Delta_+\Delta_- v'_j}{h^2}, B(x_j, t, v_j)v'_j\right)h = -\sum_{j=0}^{J-1}\left(\frac{\Delta_+ v'_j}{h}, \frac{\Delta_+}{h}(B(x_j, t, v_j)v'_j)\right)$$
$$-\left(\frac{\Delta_+ v'_0}{h}, B(0, t, v_0)v'_0\right)+\left(\frac{\Delta_- v'_J}{h}, B(l, t, v_J)v'_J\right).$$

We can expanded the expression containing in the first term on the right part of the above equality as follows:

$$\frac{\Delta_+}{h}(B(x_j, t, v_j)v'_j) = B(x_j, t, v_j)\frac{\Delta_+ v'_j}{h} + \int_0^1 (B_x(x_{j+\tau}, t, \tau v_{j+1}+(1-\tau)v_j)v'_{j+1})d\tau$$
$$+\int_0^1\left(B_u(x_{j+\tau}, t, \tau v_{j+1}+(1-\tau)v_j)\frac{\Delta_+ v'_j}{h}v'_{j+1}\right)d\tau,$$

where $B_u(x, t, u)$ is a $m\times m\times m$ tensor of three-dimension and then $B_u(x, t, u)v$ is a $m\times m$ matrix and $B_u(x, t, u)vw$ is a m-dimensional vector for $u, v, w\in\mathbb{R}^m$. Hence we have

$$-\sum_{j=1}^{J-1}\left(\frac{\Delta_+\Delta_- v'_j}{h^2}, B(x_j, t, v_j)v'_j\right)h \leqslant b\|\delta v'_h(t)\|_2^2+|J_6(t)|+|J_7(t)|+|J_8(t)|, \tag{25}$$

since from assumption (II), $B(x, t, u)$ is semibounded, where

$$J_6(t)=\sum_{j=0}^{J-1}\left(\int_0^1 B_x(x_{j+\tau}, t, \tau v_{j+1}+(1-\tau)v_j)v'_{j+1}\,d\tau, \frac{\Delta_+ v'_j}{h}\right),$$
$$J_7(t)=\sum_{j=0}^{J-1}\left(\int_0^1 B_u(x_{j+\tau}, t, \tau v_{j+1}+(1-\tau)v_j)\frac{\Delta_+ v_j}{h}v'_{j+1}\,d\tau, \frac{\Delta_+ v'_j}{h}\right)$$

and $$J_8(t)=((\operatorname{grad}\Phi)_t, B(0, t, v_0)v'_0)+((\operatorname{grad}\Phi)_t, B(l, t, v_J)v'_J).$$

Here we have

$$|J_6(t)| \leqslant C_{20}\|\delta v'_h(t)\|_2\|v'_h(t)\|_2 \leqslant C_{21}\|\delta v'_h(t)\|_2^2+C_{21}$$

and $$|J_8(t)| \leqslant C_{22}\{|v'_0(t)|^2+|v'_J(t)|^2+1\} \leqslant C_{23}\|\delta v'_h(t)\|_2^2+C_{23}.$$

From the expression of $J_7(t)$, we see

$$|J_7(t)| \leqslant C_{24}\|\delta v'_h(t)\|_2\|\delta v_h(t)\|_4^{\frac{1}{4}}\|v'_h(t)\|_4^{\frac{1}{4}}.$$

By means of the interpolation formulas for the discrete functions,

$$\|\delta v_h(t)\|_4 \leqslant C_{25}\|\delta v_h(t)\|_2^{\frac{3}{4}}\|\delta^2 v_h(t)\|_2^{\frac{1}{4}}+C_{25}\|\delta v_h(t)\|_2,$$
$$\|v'_h(t)\|_4 \leqslant C_{25}\|v'_h(t)\|_2^{\frac{3}{4}}\|\delta v'_h(t)\|_2^{\frac{1}{4}}+C_{25}\|v'_h(t)\|_2.$$

Then we obtain

$$|J_7(t)| \leqslant C_{26}\{\|\delta v'_h(t)\|_2^2+\|\delta^2 v_h(t)\|_2^2+1\}.$$

It remains to estimate the last term of the equality (18). This term can be rewritten

$$\sum_{j=1}^{J-1}\left(\frac{\Delta_+\Delta_- v'_j}{h^2}, f_j\right)h = -\sum_{j=1}^{J-2}\left(\frac{\Delta_+ v'_j}{h}, \frac{\Delta_+ f_j}{h}\right)h-\left(\frac{\Delta_+ v'_0}{h}, f_1\right)+\left(\frac{\Delta_- v'_J}{h}, f_{J-1}\right).$$

Here we have

$$\frac{\Delta f_j}{h}=\left(\int_0^1 \tilde f_x d\tau\right)+\left(\int_0^1 \tilde f_u d\tau\right)\frac{\Delta_+ v_j}{h}+\left(\int_0^1 \tilde f_p d\tau\right)\left(b\frac{\Delta_+^2 v_j}{h^2}+b'\frac{\Delta_+\Delta_- v_j}{h^2}\right)$$
$$+\left(\int_0^1 \tilde f_q d\tau\right)\frac{\Delta_+ v_j'}{h},\quad j=1,2,\cdots,J-2,$$

where $\tilde f_x=f_x\left(x_{j+\tau},\ t,\ \tau v_{j+1}+(1-\tau)v_j,\ \tau\frac{\bar\Delta v_{j+1}}{h}+(1-\tau)\frac{\bar\Delta v_j}{h},\ \tau v_{j+1}'+(1-\tau)v_j'\right)$ and similar for $\tilde f_u$, $\tilde f_p$ and $\tilde f_q$. Since from Lemma 1 and the assumption (III), $\tilde f_u$, $\tilde f_p$ and $\tilde f_q$ are bounded and $\tilde f_x$ is bounded in $L_\infty(0, T;\ L_2(0, l))$, we get the estimation relation

$$\left|\sum_{j=1}^{J-2}\left(\frac{\Delta_+ v_j'}{h},\ \frac{\Delta_+ f_j}{h}\right)h\right|\le C_{27}\{\|\delta v_h'(t)\|_2^2+\|\delta^2 v_h(t)\|_2^2+1\}.$$

On the other hand, we have

$$\left|\left(\frac{\Delta_+ v_0'}{h}, f_1\right)\right|=|((\mathrm{grad}_0\Phi)_t, f_1)|$$
$$\le C_{28}(|v_0'|+|v_J'|+1)\left(|v_1'|+\left|\frac{\Delta_+ v_1}{h}\right|+\left|\frac{\Delta_+ v_0}{h}\right|+1\right)$$
$$\le C_{29}\{\|\delta v_h'(t)\|_2^2+\|\delta^2 v_h(t)\|_2^2+1\}$$

and similarly

$$\left|\left(\frac{\Delta_- v_J'}{h},\ f_{J-1}\right)\right|\le C_{29}\{\|\delta v_h'(t)\|_2^2+\|\delta^2 v_h(t)\|_2^2+1\}.$$

Hence we finally have the estimate

$$\left|\sum_{j=1}^{J-1}\left(\frac{\Delta_+\Delta_- v_j'}{h^2}, f_j\right)h\right|\le C_{30}\{\|\delta v_h'(t)\|_2^2+\|\delta^2 v_h(t)\|_2^2+1\}. \tag{26}$$

Substituting the obtained estimations (19)—(26) altogether into the equality (18), we obtain the equality

$$\frac{d}{dt}\{\|\delta v_h'(t)\|_2^2+\|\delta^2 v_h(t)\|_2^2+J_2(t)+2J_4(t)\}\le C_{31}\{\|\delta v_h'(t)\|_2^2+\|\delta^2 v_h(t)\|_2^2+1\}.$$

Integrating this inequality with respect to t, we have

$$\|\delta v_h'(t)\|_2^2+\|\delta^2 v_h(t)\|_2^2+J_2(t)+2J_4(t)\le C_{31}\int_0^t(\|\delta v_h'(\tau)\|_2^2+\|\delta^2 v_h(\tau)\|_2^2)d\tau+C_{32},$$

where
$$C_{32}=C_{31}+\|\delta\psi_h\|_2^2+\|\delta^2\varphi_h\|_2^2+J_2(0)+2J_4(0).$$

We see that

$$|J_2(0)|\le C_{33}\{|\psi_0|^2+|\psi_J|^2+1\},$$
$$|J_4(0)|\le C_{33}\{|\psi_0|+|\psi_J|+1\},$$

where C_{33} is a constant depending on $\|\varphi_h\|_\infty$ or $\|\varphi\|_\infty$. Thus we have

$$w(t)\le C_{31}\int_0^t w(\tau)d\tau+C_{34},$$

where
$$w(t)=\|\delta v_h'(t)\|_2^2+\|\delta^2 v_h(t)\|_2^2.$$

This implies that

$$w(t)\le C_{35},$$

where C_{35} is a constant independent of h and $t\in[0, T]$.

Lemma 2. *Under the conditions of Lemma 1, for the solution $v_j(t)$ $(j=0, 1, \cdots, J)$ of the nonlinear finite slice system* $(1)_h$, $(2)_h$ and $(3)_h$, *there are estimates*

$$\sup_{0\le t\le T}\|v''_h(t)\|_2+\sup_{0\le t\le T}\|\delta v'(t)\|_2+\sup_{0\le t\le T}\|\delta^2 v_h(t)\|_2\le K_2, \tag{27}$$

where K_2 is a constant independent of h and $t\in[0, T]$.

The first estimate of (27) follows immediately from the later two estimates and the vector equation $(1)_h$.

Corollary. Under the conditions of Lemma 1, there are

$$\sup_{0\le t\le T}\|\delta v_h(t)\|_\infty+\sup_{0\le t\le T}\|v'_h(t)\|_\infty\le K_3, \tag{28}$$

where K_3 is a constant independent of h and $t\in[0, T]$.

§ 3. Solution of Finite Slice System

6. In this section we are going to prove the existence and the uniqueness of the solution $v_j(t)$ $(j=0, 1, \cdots, J)$ for the nonlinear finite slice system $(1)_h$, $(2)_h$ and $(3)_h$ by the fixed point technique, where $h>0$ is regarded as a given constant.

Denote $G^{(k)}\equiv C^{(k)}([0, T])$ for $k=1, 2$. For any m-dimensional vector functions $z_j(t)\in G^{(1)}$ $(j=0, 1, \cdots, J)$, we define the m-dimensional vector functions $v_j(t)$ $(j=0, 1, \cdots, J)$ by the following way:

$$v''_j(t)=\lambda\frac{\Delta_+\Delta_-z_j}{h^2}-\lambda\ \mathrm{grad}\,F(z_j)+\lambda B(x_j, t, z_j)z'_j+\lambda f\left(x_j, t, z_j, \frac{\Delta z_j}{h}, z'_j\right), \tag{$(1)_\lambda$}$$
$$j=1, 2, \cdots, J-1$$

and

$$\begin{aligned} v_0(t)&=v_1(t)-\lambda h\,\mathrm{grad}_0\,\Phi(v_0, v_J, t),\\ v_J(t)&=v_{J-1}(t)-\lambda h\,\mathrm{grad}_1\,\Phi(v_0, v_J, t)\end{aligned} \tag{$(2)_\lambda$}$$

with the initial conditions

$$\begin{aligned} v_j(0)&=\lambda\varphi_j, \qquad j=1, 2, \cdots, J-1;\\ v'_j(0)&=\lambda\psi_j, \qquad j=0, 1, \cdots, J\end{aligned} \tag{$(3')_\lambda$}$$

and

$$\begin{aligned} v_0(0)&=\lambda\varphi_1-\lambda h\,\mathrm{grad}_0\,\Phi(v_0(0), v_J(0), 0),\\ v_J(0)&=\lambda\varphi_{J-1}-\lambda h\,\mathrm{grad}_1\,\Phi(v_0(0), v_J(0), 0),\end{aligned} \tag{$(3'')_\lambda$}$$

where $\lambda\in[0, 1]$ is a parameter.

From $(1)_\lambda$ we see that $v_j(t)\in G^{(2)}$ for $j=1, 2, \cdots, J-1$.

Let us now turn to consider the solution $v_0(t)$ and $v_J(t)$ of the system $(2)_\lambda$ for $t\in[0, T]$.

Making the scalar product of the m-dimensional vectors $v_0(t)$ and $v_J(t)$ with the m-dimensional vector equations of $(2)_\lambda$ respectively and summing up the resulting relations, we get

$$\begin{aligned}|v_0|^2+|v_J|^2=(v_0, v_1)+(v_J, v_{J-1})\\ -\lambda h(v_0, \mathrm{grad}_0\,\Phi(v_0, v_J, t))-\lambda h(v_J, \mathrm{grad}_1\,\Phi(v_0, v_J, t)).\end{aligned}$$

This can be rewritten in the form

$$|v_0|^2+|v_J|^2=(v_0,\ v_1)+(v_J,\ v_{J-1})$$
$$-\lambda h\{(v_0,\ \tilde{\Phi}^\tau_{00}v_0)+(v_0,\ \tilde{\Phi}^\tau_{10}v_J)+(v_J,\ \tilde{\Phi}^\tau_{01}v_0)+(v_J,\ \tilde{\Phi}^\tau_{11}v_J)\}$$
$$-\lambda h\{(v_0,\ \mathrm{grad}_0\,\Phi(0,\ 0,\ t))+(v_J,\ \mathrm{grad}_1\,\Phi(0,\ 0,\ t))\},$$

where $\tilde{\Phi}^\tau_{00}=\int_0^1 \mathrm{grad}_0^2\,\Phi\ (\tau v_0,\ \ \tau v_J,\ \ t)d\tau$ and similar for $\tilde{\Phi}^\tau_{10}$, $\tilde{\Phi}^\tau_{01}$ and $\tilde{\Phi}^\tau_{11}$. Since the Hessian matrix of $\Phi(u_0,\ u_1,\ t)$ with respect to $(u_0,\ u_1)$ is non-negatively definite, the expression in curved bracket is non-negative. Hence we get

$$|v_0|^2+|v_J|^2\leqslant(v_0,\ v_1-\lambda h\,\mathrm{grad}_0\,\Phi(0,\ 0,\ t))+(v_J,\ v_{J-1}-\lambda h\,\mathrm{grad}_1\,\Phi(0,\ 0,\ t)).$$

This shows that $|v_0|$ and $|v_J|$ are bounded, i.e.,

$$|v_0|^2+|v_J|^2\leqslant K_4,$$

where $1\geqslant h>0$ and

$$K_4=\max_{0<t<T}2\{|v_1|^2+|v_{J-1}|^2+|\mathrm{grad}_0\,\Phi(0,\ 0,\ t)|^2+|\mathrm{grad}_1\,\Phi(0,\ 0,\ t)|^2\}$$

is a constant independent of h and $t\in[0,\ T]$.

Let us regard (2) as a mapping

$$\bar{v}_0=A_0(v_0,\ v_J),\qquad \bar{v}_J=A(v_0,\ v_J)$$

of $2m$-dimensional ball $B=\{v_0,\ v_J\,|\,|v_0|^2+|v_J|^2\leqslant K_4\}$. Here

$$|\bar{v}_0|^2+|\bar{v}_J|^2\leqslant|v_1|^2+|v_{J-1}|^2+\lambda^2h^2\{|\mathrm{grad}_0\,\Phi(v_0,\ v_J,\ t)|^2+|\mathrm{grad}_1\,\Phi(v_0,\ v_J,\ t)|^2\}\leqslant K_4$$

for sufficiently small $h>0$. Hence $(2)_\lambda$ has solution for any $\lambda\in[0,\ 1]$.

Suppose that v_0, v_J and $\bar{v}_0$, $\bar{v}_J$ are two solutions of $(2)_\lambda$. Then $w_0=v_0-\bar{v}_0$ and $w_J=v_J-\bar{v}_J$ satisfy

$$w_0=-\lambda h\bar{\Phi}^\tau_{00}w_0-\lambda h\bar{\Phi}^\tau_{10}w_J,$$
$$w_J=-\lambda h\bar{\Phi}^\tau_{01}w_0-\lambda h\bar{\Phi}^\tau_{11}w_J,$$

where $\bar{\Phi}^\tau_{00}=\int_0^1\mathrm{grad}_0^2\,\Phi(\tau v_0+(1-\tau)\bar{v}_0,\ \ \tau v_J+(1-\tau)\bar{v}_J,\ t)d\tau$ and similar for $\bar{\Phi}^\tau_{10}$, $\bar{\Phi}^\tau_{01}$ and $\bar{\Phi}^\tau_{11}$. This implies $w_0=w_J=0$.

Hence $(2)_\lambda$ has a unique solution $v_0(t)$ and $v_J(t)$. From the twice continuous differentiability of functions $v_1(t)$, $v_{J-1}(t)$ and $\mathrm{grad}_0\,\Phi(u_0,\ u_1,\ t)$ and $\mathrm{grad}_1\,\Phi(u_0,\ u_1,\ t)$, $v_0(t)$ and $v_J(t)$ are also twice continuously differentiable. Thus we have $v_j(t)\in G^{(2)}$ for $j=0,\ 1,\ \cdots,\ J$. Therefore the mapping defined by $(1)_\lambda$, $(2)_\lambda$, $(3')_\lambda$ and $(3'')_\lambda$ is completely continuous.

As $\lambda=0$, $v_j(t)\equiv0$ for $j=0,\ 1,\ \cdots J$.

In order to complete the proof of the existence of solution $v_j(t)$ for the nonlinear finite slice system $(1)_h$, $(2)_h$ and $(3)_h$, it is sufficient to prove the uniform boundedness of all possible solutions in $G^{(1)}$ for the system

$$v_j''-\lambda\frac{\Delta_+\Delta_-v_j}{h^2}+\lambda\,\mathrm{grad}\ F(v_j)=\lambda B(x_j,\ t,\ v_j)v_j'+\lambda f\left(x_j,\ t,\ v_j,\frac{\bar{\Delta}v_j}{h},\ v_j'\right),\qquad (\bar{1})_\lambda$$
$$j=1,\ 2,\ \cdots,\ J-1$$

and $(2)_\lambda$, $(3')_\lambda$, $(3'')_\lambda$. By similar method of estimation as used in § 2, no. 4, we can also obtain

$$\sup_{0<t<T}\sum_{j=0}^{J}|v_j'(t)|^2h\leqslant K_5,$$

where K_5 is a constant independent of h and $t\in[0, T]$. For the constant h, we have

$$\sup_{0\le t\le T}\max_{j=0,1,\cdots,J}|v_j'(t)|\le(K_5h^{-1})^{\frac{1}{2}}.$$

This proves $v_j(t)\,(j=0,\ 1,\ \cdots,\ J)$ uniformly bounded in $G^{(1)}$ with respect to $0\le\lambda\le1$.

7. It remains to establish the uniqueness of the solution. Suppose that there are two solutions $v_j(t)$ and $\bar v_j(t)\,(j=0,\ 1,\ \cdots,\ J)$ for the nonlinear finite slice system $(1)_h, (2)_h$ and $(3)_h$. Denote $w_j(t)=v_j(t)-\bar v_j(t)\,(j=0,\ 1,\ \cdots,\ J)$. Then $w_j(t)\,(j=0,\ 1,\ \cdots,\ J)$ satisfies the system

$$w_j''-\frac{\Delta_+\Delta_-w_j}{h^2}+\bar F^\tau_{uu}w_j=\bar B^\tau_u v_j'w_j+\bar B^0w_j'+\bar f^\tau_u w_j+\bar f^\tau_p\Big(b\,\frac{\Delta_+w_j}{h}+b'\,\frac{\Delta_-w_j}{h}\Big)+\bar f^\tau_q w_j'$$

and

$$w_0=w_1-h\bar\Phi^\tau_{00}w_0-h\bar\Phi^\tau_{10}w_J,$$
$$w_J=w_{J-1}-h\bar\Phi^\tau_{01}w_0-h\bar\Phi^\tau_{11}w_J$$

with homogeneous initial conditions

$$w_j(0)=0,\qquad j=1,\ 2,\ \cdots,\ J-1,$$
$$w_j'(0)=0,\qquad j=0,\ 1,\ \cdots,\ J$$

and

$$w_0(0)=w_1(0)-h\bar\Phi^\tau_{00}(0)w_0(0)-h\bar\Phi^\tau_{10}(0)w_J(0),$$
$$w_J(0)=w_{J-1}(0)-h\bar\Phi^\tau_{10}(0)w_0(0)-h\bar\Phi^\tau_{11}(0)w_J(0),$$

where $\bar F^\tau_{uu}=\int_0^1\mathrm{grad}^2\,F(\tau v_j+(1-\tau)\bar v_j)d\tau$, $\bar B^\tau_u=\int_0^1B_u(x_j,\ t,\ \tau v_j+(1-\tau)\bar v_j)d\tau$, $\bar B^0=B(x_j,\ t,\ \bar v_j)$ and similar for $\bar f^\tau_u$, $\bar f^\tau_p$, $\bar f^\tau_q$, $\bar\Phi^\tau_{00}$, $\bar\Phi^\tau_{10}$, $\bar\Phi^\tau_{01}$, $\bar\Phi^\tau_{11}$, also $\bar\Phi^\tau_{00}(0)=\int_0^1\mathrm{grad}^2_0\,\Phi\ (\tau v_0(0)+(1-\tau)\bar v_0(0),\ \tau v_J(0)+(1-\tau)\bar v_J(0),\ 0)d\tau$ and similar for $\bar\Phi^\tau_{10}(0)$, $\bar\Phi^\tau_{01}(0)$ and $\bar\Phi^\tau_{11}(0)$. From this homogeneous equations and homogeneous initial conditions, it is easy to verify that $w_j(t)\equiv0\ (j=0,\ 1,\ \cdots,\ J)$. Hence the solution $v_j(t)\ (j=0,\ 1,\ \cdots,\ J)$ for the nonlinear finite slice system $(1)_h$, $(2)_h$ and $(3)_h$ is unique.

Lemma 3. *Under the conditions* (I), (II), (III), (IV) *and* (V), *for the sufficiently small h the nonlinear finite slice system $(1)_h$, $(2)_h$ and $(3)_h$ has a unique solution $v_j(t)\in C^{(2)}([0,\ T])$, $j=0,\ 1,\ \cdots,\ J$.*

§ 4. Existence of Solution

8. From the Lemma 1 and Lemma 2 on the estimations of the finite slice solution $v_j(t)\,(j=0,\ 1,\ \cdots,\ J)$ for the nonlinear system $(1)_h$, $(2)_h$ and $(3)_h$, we have the following lemma by use of the interpolation formulas for the discrete functions[25,26] and the functions of Sobolev's spaces.

Lemma 4. *Under the conditions* (I), (II), (III), (IV) *and* (V), *for the sufficiently small $h>0$, the solutions $v_j(t)\,(j=0,\ 1,\ \cdots,\ J)$ of the nonlinear finite slice system $(1)_h$, $(2)_h$ and $(3)_h$ corresponding to the nonlinear boundary problem* (2) *and* (3) *for the system* (1) *of nonlinear wave equations have the following estimation relations:*

$$\max_{j=0,1,\cdots,J}|v_j(t)|\le K_1,\tag{29}$$

$$\max_{j=0,1,\cdots,J-1}|\Delta_+v_j(t)|\le K_3h,\tag{30}$$

$$\max_{j=1,2,\cdots,J-1}|\Delta_+\Delta_-v_j(t)|\le K_2h^{\frac{3}{2}},\tag{31}$$

$$\max_{j=0,1,\cdots,J} |v'_j(t)| \leqslant K_3, \tag{32}$$

$$\max_{j=0,1,\cdots,J-1} |\Delta_+ v'_j(t)| \leqslant K_2 h^{\frac{1}{2}}, \tag{33}$$

$$\max_{j=0,1,\cdots,J} |v'_j(t+\Delta t) - v'_j(t)| \leqslant K_6 \Delta t^{\frac{1}{2}}, \tag{34}$$

where t, $t+\Delta t \in [0, T]$, $\Delta t > 0$ and K's are constants independent of h, t and Δt.

Proof. (29) follows directly from the first estimate of (17). (30) and (32) are the immediate consequences of (28).

From the third estimate of (27), we have

$$\left|\frac{\Delta_+\Delta_- v_j(t)}{h^2}\right|^2 h \leqslant \sum_{i=1}^{J-1} \left|\frac{\Delta_+\Delta_- v_i(t)}{h^2}\right|^2 h \leqslant K_2^2, \quad j=1, 2, \cdots, J-1.$$

Hence (31) is valid. Similarly from the second estimate of (29), we get

$$\left|\frac{\Delta_+ v'_j(t)}{h}\right|^2 h \leqslant \sum_{i=0}^{J-1} \left|\frac{\Delta_+ v'_i(t)}{h}\right|^2 h \leqslant K_2^2, \qquad j=0, 1, \cdots, J-1.$$

This shows (33).

By the interpolation formula of the discrete functions, we obtain

$$|v'_j(t+\Delta t) - v'_j(t)| \leqslant \|v'_h(t+\Delta t) - v'_h(t)\|_\infty$$
$$\leqslant C_{36} \|v'_h(t+\Delta t) - v'_h(t)\|_2^{\frac{1}{2}} \|\delta v'_h(t+\Delta t) - \delta v'_h(t)\|_2^{\frac{1}{2}} + C_{36} \|v'_h(t+\Delta t) - v'_h(t)\|_2.$$

Here we have

$$\|v'_h(t+\Delta t) - v'_h(t)\|_2 \leqslant \Delta t \max_{t\leqslant\tau\leqslant t+\Delta t} \|v''_h(\tau)\|_2$$

and

$$\|\delta v'_h(t+\Delta t) - \delta v'_h(t)\|_2 \leqslant 2 \max_{t\leqslant\tau\leqslant t+\Delta t} \|\delta v'_h(t)\|_2.$$

Thus (34) is valid. Then the lemma is proved.

9. Let us denote $u_h(x, t) = v_j(t)$ for $(x, t) \in S_j = \{jh \leqslant x \leqslant (j+1)h,\ 0 \leqslant t \leqslant T\}$ $(j=0, 1, \cdots, J-1)$. Then $u_h(x, t)$ is a m-dimensional vector function in slice form defined on the rectangular domain $Q_T = \{0 \leqslant x \leqslant l,\ 0 \leqslant t \leqslant T\}$. By similar way we define $\bar{u}_h(x, t) = \frac{\Delta_+ v_j(t)}{h}$, $\tilde{u}_h(x, t) = v'_j(t)$, $\bar{\bar{u}}_h(x, t) = \frac{\Delta_+ v'_j(t)}{h}$ and $\tilde{\tilde{u}}_h(x, t) = v''_j(t)$ in S_j $(j=0, 1, \cdots, J-1)$ the m-dimensional vector functions on Q_T corresponding to the appropriate discrete finite slice vector functions respectively. Let us put $\bar{\bar{\bar{u}}}_h(x, t) = \frac{\Delta_+\Delta_- v_j(t)}{h^2}$ in S_j $(j=1, 2, \cdots, J-1)$ and $\bar{\bar{\bar{u}}}_h(x, t) = \frac{\Delta_+\Delta_- v_1(t)}{h^2}$ in S_0.

It follows directly from the Lemma 1 and Lemma 2 with its corollary for the estimations of the finite slice solutions $v_j(t)$ $(j=0, 1, \cdots, J)$ of the system $(1)_h$, $(2)_h$ and $(3)_h$, that the above constructed m-dimensional vector functions $u_h(x, t)$, $\bar{u}_h(x, t)$, $\tilde{u}_h(x, t)$, $\bar{\bar{u}}_h(x, t)$, $\tilde{\bar{u}}_h(x, t)$ and $\tilde{\tilde{u}}_h(x, t)$ have the following estimates

$$\|u_h\|_{L_\infty(Q_T)} + \|\bar{u}_h\|_{L_\infty(Q_T)} + \|\tilde{u}_h\|_{L_\infty(Q_T)} + \sup_{0\leqslant t\leqslant T} \|\bar{\bar{u}}_h(\cdot, t)\|_{L_2(0,l)}$$
$$+ \sup_{0\leqslant t\leqslant T} \|\tilde{\bar{u}}_h(\cdot, t)\|_{L_2(0,l)} + \sup_{0\leqslant t\leqslant T} \|\tilde{\tilde{u}}_h(\cdot, t)\|_{L_2(0,l)} \leqslant K_7, \tag{35}$$

where K_7 is a constant independent of h.

We can select a sequence $\{h_i\}$, such that as $i \to \infty$, $h_i \to 0$ and $u_h(x, t)$, $\bar{u}_h(x, t)$ and $\tilde{u}_h(x, t)$ weakly converge to $u(x, t)$, $\bar{u}(x, t)$ and $\tilde{u}(x, t)$ in $L_p(Q_T)$ respectively.

Since the norm of the weak limiting functions of the sequence not exceeds the lower limit of norms of the functions of the sequence, thus the norms of the functions $u_h(x, t)$, $\bar{u}(x, t)$ and $\tilde{u}(x, t)$ in $L_p(Q_T)$ are uniformly bounded with respect to $2 \leqslant p < \infty$. Hence they are bounded in $L_\infty(Q_T)$, i.e., $u(x, t)$, $\bar{u}(x, t)$ and $\tilde{u}(x, t)$ belong to $L_\infty(Q_T)$. We can also select $\{h_i\}$ such that as $h_i \to 0$, $\bar{\bar{u}}_h(x, t)$, $\tilde{\bar{u}}_h(x, t)$ and $\tilde{\tilde{u}}_h(x, t)$ weakly converge to $\bar{\bar{u}}(x, t)$, $\tilde{\bar{u}}(x, t)$ and $\tilde{\tilde{u}}(x, t)$ in $L_p(0, T; L_2(0, l))$ respectively, where $2 \leqslant p < \infty$. The norms of $\bar{\bar{u}}(x, t)$, $\tilde{\bar{u}}(x, t)$ and $\tilde{\tilde{u}}(x, t)$ in $L_p(0, T; L_2(0, l))$ are also uniformly bounded with respect to $2 \leqslant p < \infty$. Hence they belong to the functional space $L_\infty(0, T; L_2(0, l))$. So we have

$$\|u\|_{L_\infty(Q_T)} + \|\bar{u}\|_{L_\infty(Q_T)} + \|\tilde{u}\|_{L_\infty(Q_T)} + \sup_{0 \leqslant t \leqslant T} \|\bar{\bar{u}}(\cdot, t)\|_{L_2(0,l)} + \sup_{0 \leqslant t \leqslant T} \|\tilde{\bar{u}}(\cdot, t)\|_{L_2(0,l)} + \sup_{0 \leqslant t \leqslant T} \|\tilde{\tilde{u}}(\cdot, t)\|_{L_2(0,l)} \leqslant K_7. \tag{36}$$

Let $g(x, t)$ be a smooth function with finite support in rectangular domain $\{0 < x < l,\ 0 < t < T\}$. We can write the relation

$$\int_0^T \sum_{j=0}^{J-1} \left(g_j(t) \frac{\Delta_+ v_j(t)}{h} + \frac{\Delta_+ g_j(t)}{h} v_{j+1}(t) \right) h\, dt = \int_0^T [-g_0(t) v_0(t) + g_J(t) v_J(t)]\, dt = 0,$$

where $g_j(t) = g(x_j, t)$, $j = 0, 1, \cdots, J$ and $g_0(t) = g_J(t) = 0$. We define $g_h(x, t) = g_j(t)$ and $\bar{g}_h(x, t) = \frac{\Delta_+ g_j(t)}{h}$ in $S_j (j = 0, 1, \cdots, J-1)$. The above equality can be written in the integral form

$$\iint_{Q_T} [g_h(x, t) \bar{u}_h(x, t) + \bar{g}_h(x, t) u_h(x+h, t)]\, dx\, dt = 0.$$

Since when $h_i \to 0$, $g_h(x, t)$ and $\bar{g}_h(x, t)$ are convergent uniformly to $g(x, t)$ and $g_x(x, t)$ respectively in Q_T and $\bar{u}_h(x, t)$ and $u_h(x, t)$ converge weakly to $\bar{u}(x, t)$ and $u(x, t)$ respectively, then passing to limit with $h_i \to 0$, we get the integral equality

$$\iint_{Q_T} [g(x, t) \bar{u}(x, t) + g_x(x, t) u(x, t)]\, dx\, dt = 0$$

for any test function $g(x, t)$. This shows that $\bar{u}(x, t)$ is the generalized derivative of $u(x, t)$ with respect to x, i.e., $\bar{u}(x, t) = u_x(x, t)$.

Similarly we can prove that the above constructed vector functions are appropriate generalized derivatives of $u(x, t)$, i.e., $\tilde{u}(x, t) = u_t(x, t)$, $\bar{\bar{u}}(x, t) = u_{xx}(x, t)$, $\tilde{\bar{u}}(x, t) = u_{xt}(x, t)$ and $\tilde{\tilde{u}}(x, t) = u_{tt}(x, t)$.

From the estimates of Lemma 4, the convergence of $u_h(x, t)$, $\bar{u}_h(x, t)$ and $\tilde{u}_h(x, t)$ to $u(x, t)$, $u_x(x, t)$ and $u_t(x, t)$ is uniform in Q_T respectively.

Let us denote again $G_h(x, t) = \operatorname{grad} F(v_j(t))$, $B_h(x, t) = B(x_j, t, v_j(t))$ in $S_j (j = 0, 1, \cdots, J-1)$ and also $F_h(x, t) = f\left(x_j, t, v_j(t), \frac{\bar{\Delta} v_j(t)}{h}, v_j'(t)\right)$ in $S_j (j = 1, 2, \cdots, J-1)$ and $F_h(x, t) = f\left(h, t, v_1(t), \frac{\bar{\Delta} v_1(t)}{h}, v_1'(t)\right)$ in S_0. When the select sequence $\{h_i\}$ tends to zero, $B_h(x, t) = B(x, t, u_h(x, t))$ uniformly converges to $B(x, t, u(x, t))$ in Q_T; $G_h(x, t) = \operatorname{grad} F(u_h(x, t))$ converges to $\operatorname{grad} F(u(x, t))$ uniformly in Q_T and $F_h(x, t) = f(x, t, u_h(x, t), b\bar{u}_h(x, t) + b'\bar{u}_h(x-h, t), \tilde{u}_h(x, t))$ in Q_T and is uniformly convergent to $f(x, t, u(x, t), u_x(x, t), u_t(x, t))$ in Q_T.

From the vector equation $(1)_h$, we have the equality

$$\int_0^T \sum_{j=1}^{J-1} g_j(t)\Big\{ v_j''(t) - \frac{\Delta_+\Delta_- v_j(t)}{h^2} + \operatorname{grad} F(v_j(t)) - B(x_j, t, v_j(t))v_j'(t)$$

$$-f\Big(x_j, t, v_j(t), b\frac{\Delta_+ v_j(t)}{h} + b'\frac{\Delta_- v_j(t)}{h}, v_j'(t)\Big)\Big\} h\,dt = 0.$$

This can be written in the following form

$$\iint_{Q_T} g_h(x, t)\{\tilde{\tilde{u}}_h(x, t) - \bar{\bar{u}}_h(x, t) + G_h(x, t) - B_h(x, t)\tilde{u}_h(x, t) - F_h(x, t)\}dx\,dt = 0,$$

where $g(x, t)$ is any smooth function in Q_T and $g_h(x, t)$ is the function in slice form defined as before and is convergent to $g(x, t)$ uniformly in Q_T. Passing to limit as $h_i \to 0$, we get the integral relation

$$\iint_{Q_T} g(x, t)\{u_{tt}(x, t) - u_{xx}(x, t) + \operatorname{grad} F(u(x, t)) - B(x, t, u(x, t))u_t(x, t)$$

$$-f(x, t, u(x, t), u_x(x, t), u_t(x, t))\}dx\,dt = 0$$

for any smooth $g(x, t)$. This means that $u(x, t)$ satisfies the system (1) of nonlinear wave equations in generalized sense.

Since the convergence of the finite slice functions is uniform for $u(x, t)$, $u_x(x, t)$ and $u_t(x, t)$, then the nonlinear mutual boundary conditions (2) and the initial conditions (3) are satisfied by the above obtained limiting vector function $u(x, t)$ in classical sense.

Theorem 1. *Under the conditions* (I), (II), (III), (IV) *and* (V), *the problem with the nonlinear mutual boundary conditions* (2) *and initial conditions* (3) *for the system* (1) *of nonlinear wave equations has at least one m-dimensional generalized vector solution* $u(x, t) \in Z \equiv L_\infty(0, T; H^2(0, l)) \cap W_\infty^{(1)}(0, T; H^1(0, l)) \cap W_\infty^{(2)}(0, T; L_2(0, l))$, *which satisfies the system* (1) *in generalized sense and satisfies the conditions* (2) *and* (3) *in classical sense.*

§ 5. Uniqueness of Solution

10. Suppose that there are two m-dimensional vector solutions $u(x, t)$ and $v(x, t) \in W_2^{(2,2)}(Q_T)$ of the nonlinear mutual boundary problem (2) and (3) for the system (1) of nonlinear wave equations. Then $w(x, t) = u(x, t) - v(x, t)$, we have

$$\iint_{Q_T} g\{w_{tt} - w_{xx} - \bar{f}_p^\tau w_x - (B(x, t, u) - \bar{f}_q^\tau)w_t + (\bar{F}_{uu}^\tau - \bar{B}_u^\tau v_t - \bar{f}_u^\tau)w\}dx\,dt = 0, \qquad (37)$$

where $\bar{F}_{uu}^\tau = \int_0^1 \operatorname{grad}^2 F(\tau u + (1-\tau)v)d\tau$, $\bar{B}_u^\tau = \int_0^1 B_u(\tau u + (1-\tau)v)d\tau$, $\bar{f}_u^\tau = \int_0^1 f_u(x, t, \tau u + (1-\tau)v, \tau u_x + (1-\tau)v_x, \tau u_t + (1-\tau)v_t)d\tau$ and similar for $\bar{f}_q^\tau$ and $\bar{f}_p^\tau$ and $g(x, t)$ is any test function in $L_2(Q_T)$. Here the elements of the matrices $\bar{F}_{uu}^\tau$, $\bar{B}_u^\tau v_t$, $\bar{f}_u^\tau$, $\bar{f}_p^\tau$ and $\bar{f}_q^\tau$ are all bounded in Q_T. In fact, since $u(x, t)$ and $v(x, t)$ belong to $W_2^{(2,2)}(Q_T)$ by the interpolation formulas for the functions in Sobolev's spaces, it can be justified that u, u_x, u_t and v, v_x, v_t are Hölder continuous in rectangular domain Q_T and hence are all bounded in Q_T. And satisfies the boundary conditions

$$w_x(0, t) = \tilde{\Phi}_{00}^{\tau} w(0, t) + \tilde{\Phi}_{10}^{\tau} w(l, t),$$
$$-w_x(l, t) = \tilde{\Phi}_{01}^{\tau} w(0, t) + \tilde{\Phi}_{11}^{\tau} w(l, t) \tag{38}$$

and the homogeneous initial conditions

$$w(x, 0) = 0,$$
$$w_t(x, 0) = 0, \tag{39}$$

where $\tilde{\Phi}_{00}^{\tau} = \int_0^1 \operatorname{grad}_0^2 \Phi(\tau u(0, t) + (1-\tau) v(0, t), \tau u(l, t) + (1-\tau) v(l, t), t) d\tau$ and similar for $\tilde{\Phi}_{10}^{\tau}$, $\tilde{\Phi}_{01}^{\tau}$ and $\tilde{\Phi}_{11}^{\tau}$, which are also bounded in $[0, T]$. Hence $w(x, t)$ satisfies a system of linear wave equations in generalized sense and satisfies the linear boundary conditions (38) and the homogeneous initial conditions (39) in classical sense.

Replacing the test function $g(x, t)$ by a test m-dimensional vector function $w_t(x, t)$ and the integral domain Q_T by the rectangular domain Q_t with height $0 < t \leqslant T$, we can verify the following estimates for $w(x, t)$,

$$\sup_{0 \leqslant t \leqslant T} \| w_t(\cdot, t) \|_{L_2(0,l)} + \sup_{0 \leqslant t \leqslant T} \| w_x(\cdot, t) \|_{L_2(0,l)} = 0$$

by the analogous way as used in the previous sections. This shows that $w(x, t) \equiv 0$ in Q_T or $u(x, t) \equiv v(x, t)$ in Q_T.

Theorem 2. *For the nonlinear mutual boundary problem* (2) *and* (3) *of the system* (1) *of nonlinear wave equations, the m-dimensional generalized vector solution $u(x, t) \in W_2^{(2,2)}(Q_T)$ is unique.*

§ 6. Convergence of Finite Slice Solution

11. In the previous section the convergence of the m-dimensional discrete finite slice solutions $v_j(t) (j = 0, 1, \cdots, J)$ of the finite slice scheme $(1)_h$, $(2)_h$ and $(3)_h$ to the m-dimensional generalized vector solution $u(x, t)$ of the nonlinear mutual boundary problem (2) and (3) for the system (1) of nonlinear wave equations takes place only for certain selected sequence $\{h_i\}$ of steplength $h > 0$. Since the solution of the nonlinear boundary problem (2) and (3) for the system (1) of nonlinear wave equations is unique, then the above mentioned convergence will take place for any sequence $\{h_i\}$ of the steplength h, such as $i \to \infty$, $h_i \to 0$. This means that the convergence takes place for $h \to 0$.

Theorem 3. *Under the conditions* (I), (II), (III), (IV) *and* (V), *the m-dimensional finite slice vector solution $v_j(t) (j = 0, 1, \cdots, J)$ of the nonlinear finite slice scheme $(1)_h$, $(2)_h$ and $(3)_h$ converges to the m-dimensional vector function $u(x, t) \in Z$, as $h \to 0$ in the following sense: for any sequence $\{h_i\}$ of the steplength, such that $h_i \to 0$ as $i \to \infty$, $\{v_j(t)\}$, $\left\{\frac{\Delta_+ v_j(t)}{h}\right\}$ and $\{v_j'(t)\}$ are uniformly convergent to $u(x, t)$, $u_x(x, t)$ and $u_t(x, t)$ respectively in Q_T and $\left\{\frac{\Delta_+ \Delta_- v_j(t)}{h}\right\}$, $\left\{\frac{\Delta_+ v_j'(t)}{h}\right\}$ and $\{v_j''(t)\}$ are weakly convergent to $u_{xx}(x, t)$, $u_{xt}(x, t)$ and $u_{tt}(x, t)$ respectively in $L_p(0, T; L_2(0, l))$ for any $2 \leqslant p < \infty$. Furthermore the limiting vector function $u(x, t)$ is the unique generalized global solution of the nonlinear boundary problem* (2) *and* (3) *for the system* (1) *of nonlinear wave equations.*

Hence when h is small, the finite slice solution $v_j(t)$ $(j=0, 1, \cdots, J)$ of the finite slice scheme $(1)_h$, $(2)_h$ and $(3)_h$ may be regarded as the approximate solution of the nonlinear boundary problem (2) and (3) for the system (1) of nonlinear wave equations.

§ 7. Mixed Problem

12. In this section we are going to construct the m-dimensional generalized global vector solution $u(x, t)$ of the boundary problem with the nonlinear mutual mixed boundary conditions

$$u_x(x, t) = \text{grad}\, \Phi(u(0, t), t),$$
$$-u_t(l, t) = \Psi(u_x(l, t), u(0, t), u(l, t), t) \tag{40}$$

and the initial conditions

$$u(0, 0) = \varphi(x),$$
$$u_t(x, 0) = \psi(x) \tag{3}$$

for the system

$$u_{tt} - u_{xx} + \text{grad}\, F(u) = B(x, t, u)u_t + f(x, t, u, u_x, u_t) \tag{1}$$

of nonlinear wave equations in rectangular domain $Q_T = \{0 \leqslant x \leqslant l, 0 \leqslant t \leqslant T\}$, where $u = (u_1, \cdots, u_m)$, $\Psi(p, u_0, u_1, t)$, $f(x, t, u, p, q)$, $\varphi(x)$ and $\psi(x)$ are m-dimensional vector functions, $\Phi(u, t)$ and $F(u)$ are scalar functions, $B(x, t, u)$ is a $m \times m$ matrix, for independent scalar variables $(x, t) \in Q_T$ and vector variables $u_0, u_1, u, p, q \in \mathbb{R}^m$.

Suppose that the conditions (I), (II) and (III) are satisfied. Assume the following conditions for the nonlinear boundary conditions (40) and the initial conditions (3).

(IV$_1$) $\Phi(u, t) \geqslant 0$ is a non-negative scalar function of $t \in [0, T]$ and $u \in \mathbb{R}^m$. $\Phi(u, t)$, $\Phi_t(u, t)$ and $\Phi_{tt}(u, t)$ are three times, twice and once continuously differentiable with respect to the vector variable $u \in \mathbb{R}^m$. And

$$|\Phi_t(u, t)| \leqslant A\{|u|^2 + 1\}, \tag{41}$$

where $A \geqslant 0$ is a constant. The Hessian matrix $H(u, t) = \text{grad}^2\, \Phi(u, t)$ with respect to $u \in \mathbb{R}^m$ is non-negatively definite.

(IV$_2$) $\Psi(p, u_0, u_1, t)$ is a m-dimensional continuously differentiable vector functions of $t \in [0, T]$ and $u_0, u_1, p \in \mathbb{R}^m$. The Jacobi derivative matrix $\Psi_p(p, u_0, u_1, t)$ with respect to $p \in \mathbb{R}^m$ is positively definite, i.e., there is a positive constant $\sigma > 0$, such that

$$(\xi, \Psi_p(p, u_0, u_1, t)\xi) \geqslant \sigma |\xi|^2 \tag{42}$$

for any $t \in [0, T]$ and $u_0, u_1, p, \xi \in \mathbb{R}^m$. Furthermore there are

$$|\Psi(0, u_0, u_1, t)| \leqslant A\{|u_0| + |u_1| + 1\},$$
$$|\Psi_{u_0}(p, u_0, u_1, t)|, \ |\Psi_{u_1}(p, u_0, u_1, t)| \leqslant B(u_0, u_1)\{|p| + 1\}, \tag{43}$$
$$|\Psi_t(p, u_0, u_1, t)| \leqslant B(u_0, u_1)\{|p|^2 + 1\},$$

where $A \geqslant 0$ is a constant, $B(u_0, u_1) \geqslant 0$ is a continuous function of $u_0, u_1 \in \mathbb{R}^m$.

(V′) The m-dimensional initial vector functions $\varphi(x) \in H^2(0, l)$ and $\psi(x) \in H^1(0, l)$ satisfy the boundary conditions (40), i.e.,

$$\begin{aligned} &\varphi'(0)=\operatorname{grad}\Phi(\varphi(0),0),\\ &-\psi(l)=\Psi(\varphi'(l),\varphi(0),\varphi(l),0). \end{aligned} \tag{44}$$

Now we consider the finite slice scheme $(1)_h$ with the finite slice boundary conditions

$$\begin{aligned} &\frac{\Delta_+ v_0(t)}{h}=\operatorname{grad}\Phi(v_0(t),t),\\ &-v_J'(t)=\Psi\left(\frac{\Delta_- v_J(t)}{h},v_0(t),v_J(t),t\right) \end{aligned} \tag{40$_h$}$$

and with the finite slice initial conditions

$$\begin{aligned} &v_j(0)=\varphi_j, \quad j=1,2,\cdots,J\\ &v_j'(0)=\psi_j, \quad j=0,1,\cdots,J-1 \end{aligned} \tag{3$'''$$_h$}$$

and $v_0(0)$ and $v_J'(0)$ are uniquely determined by the systems

$$\begin{aligned} &v_0(0)=\varphi_1-h\operatorname{grad}\Phi(v_0(0),0),\\ &-v_J'(0)=\Psi\left(\frac{\Delta_-\varphi_J}{h},v_0(0),\varphi_J,0\right), \end{aligned} \tag{3^{IV}$_h$}$$

where $\varphi_j=\varphi(x_j)$ and $\psi_j=\psi(x_j)\,(j=0,1,\cdots,J)$.

Since the Hessian matrix of $\Phi(u,t)$ with respect to $u\in\mathbb{R}^m$ is non-negatively definite, then for sufficiently small $h>0$, the first m-dimensional vector equation of $(3^{IV})_h$ has a unique solution $v_0(0)$. And $v_J'(0)$ is uniquely determined by the second m-dimensional vector equation of $(3^{IV})_h$.

The existence and uniqueness of the finite slice solution $v_j(t)$ $(j=0,1,\cdots,J)$ for the finite slice scheme $(1)_h$, $(40)_h$, $(3''')_h$ and $(3^{IV})_h$ can be proved by similar procedure used in the previous sections.

Lemma 5. *Under the conditions* (I), (II), (III), (IV$_1$), (IV$_2$) *and* (V$'$), *for sufficiently small h, the nonlinear finite slice scheme* $(1)_h$, $(40)_h$, $(3''')_h$ *and* $(3^{IV})_h$ *has a unique solution* $v_j(t)\in C^{(2)}([0,T])$, $j=0,1,\cdots,J$.

13. Now we turn to estimate the finite slice solution $v_j(t)\,(j=0,1,\cdots,J)$ of the finite slice scheme $(1)_h$, $(40)_h$, $(3''')_h$ and $(3^{IV})_h$.

In the equality (10) all terms can be estimated by just the same way as used in § 2, no. 4, except the second term of (10). For the terms of (11), we can derive as follows. At first

$$\left(v_0',\frac{\Delta_+v_0}{h}\right)=(v_0',\operatorname{grad}\Phi(v_0,t))=\frac{d}{dt}\Phi(v_0,t)-\Phi_t(v_0,t).$$

From the condition (IV$_1$), we have

$$|\Phi_t(v_0,t)|\leqslant A\{|v_0|^2+1\}.$$

From (12), (13) and (14), we get

$$|\Phi_t(v_0,t)|\leqslant C_{37}\left\{\|\delta v_h\|_2^2+\int_0^t\left(\sum_{j=1}^{J-1}|v_j'|^2h\right)dt+1\right\}.$$

Secondly, we have from the condition (IV$_2$) and from (12), (13) and (14)

$$\begin{aligned} \left(v_J',\frac{\Delta_-v_J}{h}\right)&=-\left(\frac{\Delta_-v_J}{h},\Psi\left(\frac{\Delta_-v_J}{h},v_0,v_J,t\right)\right)\\ &=-\left(\frac{\Delta_-v_J}{h},\tilde{\Psi}_p^\tau\frac{\Delta_-v_J}{h}+\Psi(0,v_0,v_J,t)\right) \end{aligned}$$

$$\leqslant -\sigma\left|\frac{\varDelta_- v_J}{h}\right|^2 - \left(\frac{\varDelta_- v_J}{h},\ \varPsi(0,\ v_0,\ v_J,\ t)\right)$$
$$\leqslant \frac{1}{4\sigma}|\varPsi(0,\ v_0,\ v_J,\ t)|^2$$
$$\leqslant \frac{A^2}{4\sigma}\{|v_0|^2+|v_J|^2+1\}$$
$$\leqslant C_{38}\left\{\|\delta v_h\|_2^2+\int_0^t\left(\sum_{j=1}^{J-1}|v_j'|\,h\right)dt+1\right\},$$

where $\tilde{\varPsi}_p^\tau=\int_0^1\varPsi\left(\frac{\tau\varDelta_- v_J}{h},\ v_0,\ v_J,\ t\right)d\tau$. Then we can proceed the estimation following the way as given in § 2, no. 4 to obtain the analogous results of (17).

Lemma 6. *Under the conditions* (I), (II), (III), (IV$_1$), (IV$_2$) *and* (V'), *for sufficiently small h, the solution $v_j(t)$ ($j=0,\ 1,\ \cdots,\ J$) of the finite slice scheme* $(1)_h$, $(40)_h$, $(3''')_h$ *and* $(3^{IV})_h$ *has the estimates*

$$\sup_{0<t<T}\|v_h(t)\|_\infty+\sup_{0<t\leqslant T}\|v_h'(t)\|_2+\sup_{0<t<T}\|\delta v_h(t)\|_2\leqslant K_8, \tag{45}$$

where K_8 is a constant independent of h and $t\in[0,\ T]$.

For the further estimation, we make the scalar product of $\frac{\varDelta_+\varDelta_- v_j'(t)}{h^2}h$ and scheme $(1)_h$ and sum up the resulting products for $j=1,\ 2,\ \cdots,\ J-1$. We get the equality (18). For the second term of (18), we have (19).

As to the first term of (18), we have

$$\sum_{j=1}^{J-1}\left(\frac{\varDelta_+\varDelta_- v_j'}{h^2},\ v_0''\right)h=-\sum_{j=0}^{J-1}\left(\frac{\varDelta_+ v_j'}{h},\ \frac{\varDelta_+ v_j''}{h}\right)h-\left(\frac{\varDelta_+ v_0'}{h},\ v_0''\right)+\left(\frac{\varDelta_- v_J'}{h},\ v_J''\right). \tag{46}$$

For the term next to the last, we have

$$\left(\frac{\varDelta_+ v_0'}{h},\ v_0''\right)=(\mathrm{grad}^2\,\varPhi v_0',\ v_0'')+(\mathrm{grad}\,\varPhi_t,\ v_0'')$$
$$=\frac{1}{2}\frac{d}{dt}\{(\mathrm{grad}^2\varPhi v_0',\ v_0')+(\mathrm{grad}\,\varPhi_t,\ v_0')\}$$
$$-\{((\mathrm{grad}^2\,\varPhi)_t v_0',\ v_0')+((\mathrm{grad}\,\varPhi_t)_t,\ v_0')\}.$$

For the last term of (46), we have

$$\left(\frac{\varDelta_- v_J'}{h},\ v_J''\right)=-\left(\frac{\varDelta_- v_J'}{h},\ \frac{d}{dt}\varPsi\left(\frac{\varDelta_- v_J}{h},\ v_0,\ v_J,\ t\right)\right)$$
$$=-\left(\frac{\varDelta_- v_J'}{h},\ \varPsi_p\frac{\varDelta_- v_J'}{h}+\varPsi_{u_0}v_0'+\varPsi_{u_1}v_J'+\varPsi_t\right)$$
$$\leqslant-\sigma\left|\frac{\varDelta_- v_J'}{h}\right|^2-\left(\frac{\varDelta_- v_J'}{h},\ \varPsi_{u_0}v_0'+\varPsi_{u_1}v_J'+\varPsi_t\right).$$

Then from the condition (IV$_2$), we obtain

$$\left(\frac{\varDelta_- v_J'}{h},\ v_J''\right)\leqslant-\frac{\sigma}{2}\left|\frac{\varDelta_- v_J'}{h}\right|^2+C_{39}\left\{|v_0'|^4+|v_J'|^4+\left|\frac{\varDelta_- v_J}{h}\right|^4+1\right\}$$
$$\leqslant-\frac{\sigma}{2}\left|\frac{\varDelta_- v_J'}{h}\right|^2+C_{40}\{\|\delta^2 v_h(t)\|_2^2+\|\delta v_h'(t)\|_2^2+1\},$$

where C are constants independent of h. Hence (46) becomes

$$\sum_{j=1}^{J-1}\left(\frac{\Delta_+\Delta_-v'_j}{h^2}, v''_j\right)h \leqslant -\frac{d}{dt}\{\|\delta v'_h(t)\|_2^2+(\text{grad}^2\, \Phi v'_0,\ v'_0)+(\text{grad}\,\Phi_t,\ v'_0)\}$$
$$-\frac{\sigma}{2}\left|\frac{\Delta_-v'_J}{h}\right|^2+C_{40}\{\|\delta^2 v_h(t)\|_2^2+\|\delta v'_h(t)\|_2^2+1\}. \tag{47}$$

In the procedure of estimation of the third term of (18), the contributions of the boundary conditions at the lateral side $x=0$ and $x=l$ of the rectangular domain Q_T are

$$\begin{aligned}&\left|\left(\frac{\Delta_+v'_0}{h},\ \text{grad}\,F(v_0)\right)\right|=\left|\left(\frac{d}{dt}\Phi(v_0,\ t),\ \text{grad}\,F(v_0)\right)\right|\leqslant C_{41}\{\|\delta v'_h(t)\|_2^2+1\},\\&\left|\left(\frac{\Delta_-v'_J}{h},\ \text{grad}\,F(v_J)\right)\right|\leqslant\frac{\sigma}{6}\left|\frac{\Delta_-v'_J}{h}\right|^2+C_{41}\end{aligned} \tag{48}$$

respectively. Also in the procedure of estimation of the fourth term of (18), the contributions of the boundary conditions at the lateral side $x=0$ and $x=l$ are

$$\begin{aligned}&\left|\left(\frac{\Delta_+v'_0}{h},\ B(0,\ t,\ v_0)v'_0\right)\right|=\left|\left(\frac{d}{dt}\Phi(v_0,\ t),\ B(0,\ t,\ v_0)v'_0\right)\right|\\&\qquad\leqslant C_{42}\{\|\delta v'_h(t)\|_2^2+1\},\\&\left|\left(\frac{\Delta_-v'_J}{h},\ B(l,\ t,\ v_J)v'_J\right)\right|\leqslant\frac{\sigma}{6}\left|\frac{\Delta_-v'_J}{h}\right|^2+C_{42}\{\|\delta v'_h(t)\|_2^2+1\}\end{aligned} \tag{49}$$

respectively.

For the last term of (18), the contribution of the boundary condition at $x=0$ is

$$\left|\left(\frac{\Delta_+v'_0}{h}, f_1\right)\right|=\left|\left(\frac{d}{dt}\Phi(v_0,\ t),\ f_1\right)\right|\leqslant C_{43}\{\|\delta v'_h(t)\|_2^2+\|\delta^2 v_h(t)\|_2^2+1\} \tag{50}$$

and the contribution of the boundary condition at $x=l$ is

$$\left|\left(\frac{\Delta_-v'_J}{h}, f_{J-1}\right)\right|\leqslant\frac{\sigma}{6}\left|\frac{\Delta_-v'_J}{h}\right|^2+C_{43}\{\|\delta v'_h(t)\|_2^2+\|\delta^2 v_h(t)\|_2^2+1\}. \tag{51}$$

Using these thus obtained estimations (47)—(51) and following the argument similar as in the proof of Lemma 2 (in § 2, no. 5), we get the further estimates for $v_j(t)$ $(j=0, 1, \cdots, J)$.

Lemma 7. *Under the conditions* (I), (II), (III), (IV$_1$), (IV$_2$) *and* (V'), *for sufficiently samll h, the solution $v_j(t)$ $(j=0,\ 1,\ \cdots,\ J)$ of the finite slice scheme* $(1)_h$, $(40)_h$, $(3''')_h$ *and* $(3^{IV})_h$ *has the estimates*

$$\sup_{0\leqslant t\leqslant T}\|v''_h(t)\|_2+\sup_{0\leqslant t\leqslant T}\|\delta v'_h(t)\|_2+\sup_{0\leqslant t\leqslant T}\|\delta^2 v_h(t)\|_2\leqslant K_9, \tag{52}$$

hence also

$$\sup_{0\leqslant t\leqslant T}\|v'_h(t)\|_\infty+\sup_{0\leqslant t\leqslant T}\|\delta v_h(t)\|_\infty\leqslant K_{10}, \tag{53}$$

where K_9 and K_{10} are constants independent of h.

14. By means of the same arguments used in § 4—6, we can obtain the existence and uniqueness of the generalized global solution $u(x,\ t)\in Z$ of the nonlinear mutual mixed boundary problem (40) and (3) for the system (1) of nonlinear wave equations.

Theorem 4. *Under the conditions* (I), (II), (III), (IV$_1$), (IV$_2$) *and* (V'), *the problem with the nonlinear mutual mixed boundary conditions* (40) *and the initial* *conditions* (3) *for the system* (1) *of nonlinear wave equations has a unique m-dimensional generalized global vector solution* $u(x, t) \in Z \equiv L_\infty(0, T; H^2(0, l)) \cap W_\infty^{(1)}(0, T; H^1(0, l)) \cap W_\infty^{(2)}(0, T; L_2(0, l))$, *which satisfies the system* (1) *in the generalized sense and satisfies the boundary conditions* (40) *and the initial conditions* (3) *in classical sense.*

Similary we have the theorem of convergence of finite slice scheme $(1)_h$, $(40)_h$, $(3''')_h$ and $(3^{IV})_h$.

Theorem 5. *Under the conditions* (I), (II), (III), (IV_1), (IV_2) *and* (V'), *for sufficiently small h, the m-dimensional finite slice vector solution* $v_j(t)$ $(j=0, 1, \cdots, J)$ *of the nonlinear finite slice scheme* $(1)_h$, $(40)_h$, $(3''')_h$ *and* $(3^{IV})_h$ *converges to the m-dimensional vector function* $u(x, t) \in Z$ *as* $h \to 0$ *in the following sense: for any sequence* $h_i \to 0$, $\{v_j(t)\}$, $\left\{\frac{\Delta_+ v_j(t)}{h}\right\}$ *and* $\{v_j'(t)\}$ *are uniformly convergent to* $u(x, t)$, $u_x(x, t)$ *and* $u_t(x, t)$ *respectively in* Q_T *and* $\left\{\frac{\Delta_+\Delta_- v_j(t)}{h^2}\right\}$, $\left\{\frac{\Delta_+ v_j'(t)}{h}\right\}$ *and* $\{v_j''(t)\}$ *are weakly convergent to* $u_{xx}(x, t)$, $u_{xt}(x, t)$ *and* $u_{tt}(x, t)$ *respectively in* $L_p(0, T; L_2(0, l))$ *for any* $2 \leqslant p < \infty$. *Furthermore the limiting vector function* $u(x,t)$ *is the unique generalized global solution of the nonlinear boundary problem* (40) *and* (3) *for the system* (1) *of nonlinear wave equations.*

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