APPLICATION OF THE REGULARIZATION METHOD TO THE NUMERICAL SOLUTION OF ABEL'S INTEGRAL EQUATION*

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§ 1

In the present paper, we shall consider an ill-posed problem, the solution of Abel's integral equation with unbounded kernel

$$Az=\int_0^x \frac{z(s)}{(x-s)^a}\,ds=u(x),\quad (x,\ s)\in[0,\ 1]\times[0,\ 1],\qquad 0<a<1,\ u(0)=0,\qquad (1)$$

where $u(x)$ is a know function in the space $L_2[0,\ 1]$ and $z(s)$ is the unknown function in the space $C[0,\ 1]$. This is an important problem encountered in practice ([1] and [2], Vol. I, 158—160).

It should be pointed out first of all that Abel's integral operator A in equation (1) possesses the properties:

1) The operator A is completely continuous. This is true because

$$\|u\|^2_{L_2}=\int_0^1\Big[\int_0^x \frac{z(s)}{(x-s)^a}\,ds\Big]^2 dx\leqslant\|z\|^2_C\int_0^1\Big[\int_0^x (x-s)^{-a}\,ds\Big]^2 dx=\frac{\|z\|^2_C}{(1-a)^2(3-2a)},$$

and

$$\|u(x+h)-u(x)\|^2_{L_2}=\int_0^1\Big[\int_0^{x+h}\frac{z(s)}{(x+h-s)^a}\,ds-\int_0^x\frac{z(s)}{(x-s)^a}\,ds\Big]^2 dx$$
$$\leqslant\|z\|^2_C\int_0^1\Big[\frac{x^{1-a}-(x+h)^{1-a}+2h^{1-a}}{1-a}\Big]^2 dx\to 0,\ as\ h\to 0.$$

2) The operator A which maps $C[0,\ 1]$ onto $AC[0,\ 1]$ is one-to-one. This follows from the reciprocity formula ([2], Vol. I, 159)

$$z(s)=\frac{\sin\pi a}{\pi}\,\frac{d}{ds}\int_0^s\frac{u(x)}{(s-x)^{1-a}}\,dx.$$

Suppose that the element $z_T(s)\in C_1[0,\ 1]$ is a solution of equation (1) with right-hand member $u(x)=u_T(x)\in AC_1[0,\ 1]$, i.e.,

$$Az_T=u_T,$$

and requires to be found. However, in computation we often know only the approximate right-hand member $u_\delta(x)$ rather than the exact one $u_T(x)$, in such a case, we can speak only of finding an approximate solution $z_\delta(s)$ (i.e., one close to $z_T(s)$). Unfortunately the problem of determining the solution $z(s)$ of equation (1) in the space $C[0,\ 1]$ from the initial data $u(x)$ in the space $L_2[0,\ 1]$ is not well-posed on the pair of spaces $(C,\ L_2)$ in the sense of Hadamard ([3], p. 16). First, it is

* Received February 28, 1984.

obvious that the approximate solution $z_\delta(s)$ cannot be defined as the exact solution of the equation $Az=u_\delta$ with approximate right-hand member

$$u=u_\delta,$$

that is, it cannot be determined by

$$z_\delta=A^{-1}u_\delta,$$

since the approximate element u_δ may fail to belong to the set $AC[0, 1]$. Second, even if such a solution z_δ does exist, it will not possess the property of stability, since the inverse operator A^{-1} is not continuous. To see this, let us suppose that the approximate right-hand member $u_\delta(x)$ has the form

$$u_\delta(x)=u_T(x)+\delta^{\frac{1-a}{3}}\sin\frac{x}{\delta},$$

then

$$\|u_\delta(x)-u_T(x)\|_{L_2}\leqslant\delta^{\frac{1-a}{3}},$$

$$z_\delta(s)=z_T(s)+\frac{\sin\pi a}{\pi}\frac{d}{ds}\int_0^s\frac{\delta^{\frac{1-a}{3}}\sin\frac{x}{\delta}}{(s-x)^{1-a}}\,dx.$$

However, the difference between the solutions

$$\|z_\delta(s)-z_T(s)\|_c\geqslant|z_\delta(\delta^{\frac{1+2a}{3a}})-z_T(\delta^{\frac{1+2a}{3a}})|\geqslant\frac{\sin\pi a}{\pi}\frac{1}{2}\delta^{\frac{-(1-a)}{3}}$$

can be made arbitrarily large for sufficiently small values of δ. Thus, the requirements for a well-posed problem are not satisfied. Consequently, the problem (1) is ill-posed.

§ 2

A method of solving ill-posed problems, widely used in computational work is the regularization method. It consists in constructing a regularizing operator. An operator $R(u, \alpha)$ depending on a parameter α is called a regularizing operator for the equation $Az=u$ in a neighborhood of $u=u_T$ if

1) there exists a positive number δ_1 such that the operator $R(u, \alpha)$ is defined for every $\alpha>0$ and every u in $L_2[0, 1]$ for which

$$\|u-u_T\|_{L_2}\leqslant\delta\leqslant\delta_1.$$

2) there exists a function $\alpha=\alpha(\delta)$ of δ such that, for every $\varepsilon>0$, there exists a number $\delta(\varepsilon)\leqslant\delta_1$ such that the inclusion $u_\delta\in L_2[0, 1]$ and the inequality

$$\|u_\delta-u_T\|_{L_2}\leqslant\delta(\varepsilon)$$

imply

$$\|z_\alpha-z_T\|_c\leqslant\varepsilon,$$

where

$$z_\alpha=R(u_\delta, \alpha(\delta))\quad([3],\ \text{p. } 55).$$

It is obvious that every regularizing operator $R(u_\delta, \alpha(\delta))$ defines a stable method of constructing approximate solutions. Thus, the problem of finding an approximate solution reduces to

1) constructing the operator $R(u, \alpha)$, and

2) selecting the regularization parameter $\alpha=\alpha(\delta)$ from the discrepancy δ.

The regularizing operator for the Fredholm integral equation of the first kind with continuous kernel

$$\int_a^b K(x, s)z(s)ds=u(x) \tag{2}$$

is examined in [4], [5] and [6]. In [4] the operator $R(u, \alpha)$ for equation (2) is constructed by minimizing the so-called smoothing functional

$$M^\alpha[z, u]=\|Az-u\|_{L_2}^2+\alpha\int_a^b [P(s)z(s)^2+K(s)z'(s)^2]ds,$$

and in [5] the parameter α is determined from the discrepancy δ by the condition

$$\|Az_\alpha-u_\delta\|_{L_2}-\delta=0.$$

§ 3

Below, following [3], [4], [5] and [6], we shall use the finite-difference method for Abel's equation to construct a regularizing operator that can easily be realized on a computer. For this we replace equation (1) with its finite-difference approximation

$$A^h z^h=u_T^h$$

on a uniform grid $\omega_x^h\times\omega_s^h$ with step h:

$$\omega_x^h=\{x_i: x_i=ih,\ i=0, 1, \cdots, n\},$$

$$\omega_s^h=\{s_j: s_j=jh,\ j=0, 1, \cdots, n\}, \qquad h=h_n=\frac{1}{n},$$

where

$$A^h=(a_{i,j}) \quad a_{i,j}=\begin{cases}0, & j\geqslant i,\\ \int_{s_j}^{s_{j+1}}(x_i-s)^{-\alpha}ds, & j\leqslant i-1,\ i, j=0, 1, \cdots, n,\end{cases}$$

$$z^h\in W^h=\{z^h:\ z^h=(z_0, z_1, \cdots, z_n)\},$$

$$u^h\in L^h=\{u^h: u^h=(u_0, u_1, \cdots, u_n)\},$$

$$u_T^h=[u_T(x)]^h=(u_T(x_0), u_T(x_1), \cdots, u_T(x_n)),\ u_T(x_0)=0.$$

we shall measure z^h and u^h with norms $\|z^h\|_{W^h}$ and $\|u^h\|_{L^h}$ defined by

$$\|u^h\|_{L^h}^2=(u^h, u^h)_{L^h},\quad \|z^h\|_{W^h}^2=(z^h, z^h)_{W^h},$$

$$(u^h, v^h)_{L^h}=\sum_{i=0}^{n-1}h\frac{u_iv_i+u_{i+1}v_{i+1}}{2},\quad u^h, v^h\in L^h,$$

$$(z^h, y^h)_{W^h}=\sum_{j=0}^{n-1}h\frac{z_jy_j+z_{j+1}y_{j+1}}{2}+\sum_{j=0}^{n-1}h\Delta z_j\Delta y_j,\quad z^h, y^h\in W^h,$$

$$\Delta z_j=\frac{z_{j+1}-z_j}{h},\qquad j=0, 1, \cdots, n+1.$$

We can construct the regularizing operator for equation (1) by minimizing the functional:

$$M_h^\alpha[z^h, u^h]=\|A^hz^h-u^h\|_{L^h}^2+\alpha\|z^h\|_{W^h}^2.$$

Theorem 1. *For every $u^h(u_0=0)$ of L^h and every positive parameter α, there exists a unique element $z_\alpha^h\in W^h$ such that:*

1) *the greatest lower bound of the functional $M_h^\alpha[z^h, u^h]$ is attained with z_α^h, that is*

$$M_h^\alpha[z_\alpha^h, u^h] = \inf_{z^h\in W^h} M_h^\alpha[z^h u^h];$$

2) *the element* z_α^h *must then satisfy the Euler equation*

$$\alpha(z^h, v^h)_{W^h} + (A^h z^h - u^h, A^h v^h)_{L^h} = \alpha(z^h, v^h)_{W^h}$$
$$+((A^h)^* A^h z^h - (A^h)^* u^h, v^h)_{W^h} = 0, \quad \forall v^h \in W^h,$$

or

$$\{\alpha I_{(n+1)\times(n+1)} + (A^h)^* A^h\} z^h = (A^h)^* u^h,$$

where $I_{(n+1)\times(n+1)}$ *is an identity operator (matrix) and* $(A^h)^*$ *is the Hilbert-adjoint operator of* A^h:

$$(A^h)^* A^h = (s_h^1 s_h)^{-1} \widetilde{A}^h A^h = \begin{pmatrix} s_{n\times n}^{(0,0)} & s_{n\times 1}^{(0,1)} \\ s_{1\times n}^{(1,0)} & s_{1\times 1}^{(1,1)} \end{pmatrix} \begin{pmatrix} A_{n\times n}^{(0,0)} & A_{n\times 1}^{(0,1)} \\ A_{1\times n}^{(1,0)} & A_{1\times 1}^{(1,1)} \end{pmatrix},$$

$$\widetilde{A}^h A^h = \begin{pmatrix} A_{n\times n}^{(0,0)} & A_{n\times 1}^{(0,1)} \\ A_{1\times n}^{(1,0)} & A_{1\times 1}^{(1,1)} \end{pmatrix},$$

$$A_{n\times n}^{(0,0)} = \begin{pmatrix} \sqrt{2}\,a_{1,0} & & \cdots & \sqrt{2}\,a_{n-1,0} & a_{n,0} \\ & \sqrt{2}\,a_{2,1} & \cdots & \sqrt{2}\,a_{n-1,1} & a_{n,1} \\ & & \ddots & & \\ & & & \sqrt{2}\,a_{n-1,n-2} & a_{n,n-2} \\ & & & & a_{n,n-1} \end{pmatrix}$$

$$\times \begin{pmatrix} \sqrt{2}\,a_{1,0} & & & \\ \sqrt{2}\,a_{2,0} & \sqrt{2}\,a_{2,1} & & & \\ \sqrt{2}\,a_{n-1,0} & \sqrt{2}\,a_{n-1,1} & \cdots & \sqrt{2}\,a_{n-1,n-2} & \\ a_{n,0} & a_{n,1} & \cdots & a_{n,n-2} & a_{n,n-1} \end{pmatrix},$$

$$A_{n\times 1}^{(0,1)} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad A_{1\times n}^{(1,0)} = (0, 0, \cdots 0), \quad A_{1\times 1}^{(1,1)} = (0),$$

$$(s_h^1 s_h)^{-1} = \begin{pmatrix} s_{n\times n}^{(0,0)} & s_{n\times 1}^{(0,1)} \\ s_{1\times n}^{(1,0)} & s_{1\times 1}^{(1,1)} \end{pmatrix},$$

$$(s_h^1 s_h) = (s_{ij}^1)(s_{ij}) = (c_{ij}) = \begin{pmatrix} 1+\frac{2}{h^2} & \frac{-2}{h^2} & & & \\ \frac{-2}{h^2} & 2+\frac{4}{h^2} & \frac{-2}{h^2} & & \\ & & \ddots & & \\ & & \frac{-2}{h^2} & 2+\frac{4}{h^2} & \frac{-2}{h^2} \\ & & & \frac{-2}{h^2} & 1+\frac{2}{h^2} \end{pmatrix},$$

$$s_{i,j}^1 = s_{j,i}, \quad i, j = 0, 1, \cdots, n,$$

$$s_{0,0}=\sqrt{c_{0,0}},\quad s_{i,i}=\sqrt{c_{i,i}-\left(\frac{c_{i-1,i}}{s_{i-1,i-1}}\right)^2},\qquad i=1,\ 2,\ \cdots,\ n,$$

$$s_{i,j}=\begin{cases}0, & j<i,\\ \dfrac{c_{i,j}}{s_{i,i}}, & j=i+1,\\ 0, & j>i,\ j\neq i+1,\ i=0,\ 1,\ \cdots,\ n;\end{cases}$$

3) *z_α^h is a continuous function of α,*

4) *z_α^h is not equal to zero provided $u^h\neq 0$.*

Proof. Since $M_h^\alpha[z^h,\ u^h]$ is a nonnegative functional, there exists

$$M[\alpha,\ h]=\inf_{z^h\in W^h} M_h^\alpha[z^h,\ u^h].$$

Let $\{z_n^h\}$ denote a minimizing sequence for M_h^α, that is, one such that

$$M[\alpha,\ h]\leqslant M_h^\alpha[z_m^h,\ u^h]\leqslant M[\alpha,\ h]+\frac{1}{m}.$$

We now show that $\{z_m^h\}$ is a Cauchy sequence in the space W^h. Since

$$\begin{aligned}\alpha\left\|\frac{z_m^h-z_{m+p}^h}{2}\right\|_{W^h}^2&=-\alpha\left\|\frac{z_m^h+z_{m+p}^h}{2}\right\|_{W^h}^2+\frac{\alpha}{2}\|z_m^h\|_{W^h}^2+\frac{\alpha}{2}\|z_{m+p}^h\|_{W^h}^2\\&=-M_h^\alpha\left[\frac{z_m^h+z_{m+p}^h}{2},\ u^h\right]+\frac{1}{2}M_h^\alpha[z_m^h,\ u^h]+\frac{1}{2}M_h^\alpha[z_{m+p}^h,\ u^h]\\&\quad+\left\|A^h\frac{z_m^h+z_{m+p}^h}{2}-u^h\right\|_{L^h}^2-\frac{1}{2}\|A^hz_m^h-u^h\|_{L^h}^2-\frac{1}{2}\|A^hz_{m+p}^h-u^h\|_{L^h}^2,\end{aligned}$$

it follows from the convexity property of $\|A^hz^h-u^h\|_{L^h}^2$ that

$$\begin{aligned}\alpha\left\|\frac{z_m^h-z_{m+p}^h}{2}\right\|_{W^h}^2&\leqslant -M[\alpha,\ h]+\frac{1}{2}\left[M[\alpha,\ h]+\frac{1}{m}\right]+\frac{1}{2}\left[M[\alpha,\ h]+\frac{1}{m}\right]\\&=\frac{1}{m}\to 0,\qquad \text{as } m\to\infty.\end{aligned}$$

Consequently, by virtue of the completeness of the space W^h, the sequence $\{z_m^h\}$ converges in it. Let us define

$$z_\alpha^h=\lim_{m\to\infty} z_m^h.$$

The uniqueness of the element z_α^h follows from the fact that $M_h^\alpha[z^h,\ u^h]$ is a nonnegative quadratic functional, and it cannot attain its least value at two distinct elements.

By variational principle it is easily seen that z_α^h must satisfy the Euler equation. The inequality

$$|\alpha I+(A^h)^*A^h|\neq 0$$

follows from the fact that the eigenvalues of $s_{n\times n}^{(0,0)}\times A_{n\times n}^{(0,0)}$ cannot be negative.

Regarding the element z_α^h as a function of α, one can easily see that the function $\dfrac{dz_\alpha^h}{d\alpha}$ also satisfies the equation

$$\{\alpha I+(A^h)^*A^h\}\,\frac{dz_\alpha^h}{d\alpha}=-z_\alpha^h,$$

which differs from Euler's equation only in the right-hand member. Thus, the existence of the derivative $\frac{dz_\alpha^h}{d\alpha}$ implies the continuity of z_α^h.

To see the assertion 4, let us suppose that

$$z_\alpha^h=0.$$

Then
$$(-u^h,\ A^h v^h)_{L^h}=0, \qquad \forall v^h\in W^h.$$

This means that
$$u^h=0.$$

Thus, Theorem 1 is proven.

This theorem shows that we can assign to every element $u^h\in L^h$ an element $z_\alpha^h(s)\in W_2^1[0,\ 1]$:

$$z_\alpha^h(s)=z_{\alpha,j}^h+\frac{z_{\alpha,j+1}^h-z_{\alpha,j}^h}{h}(s-s_j), \qquad s\in[s_j,\ s_{j+1}],\ j=0,\ 1\cdots,\ n-1.$$

The procedure described for obtaining the function $z_\alpha^h(s)$ can be regarded as the result of applying to the element u^h an operator R depending on the parameter α:

$$z_\alpha^h(s)=R(u^h,\ \alpha).$$

Now, we shall find the regularization parameter α as a function $\alpha(\delta)$ for which the operator $R(u_\delta^h,\ \alpha(\delta))$ is a regularizing operator, where

$$\|u_\delta^h-u_T^h\|_{L^h}\leqslant\delta, \qquad u_{\delta,0}^h=0.$$

The regularization parameter α can be determined from the discrepancy δ ([3], p. 108), that is, from

$$\Delta_h(\alpha)\equiv\varphi_h(\alpha)-\left(\delta+\frac{h}{1-a}\|z_\alpha^h\|_{W^h}\right)^2=0,$$

where
$$\varphi_h(\alpha)=\|A^h z_\alpha^h-u_\delta^h\|_{L^h}^2.$$

Theorem 2. *Under the condition*

$$\|u_\delta^h\|_{L^h}^2>\delta^2$$

there exists an $\alpha(s)$ *such that*

$$\Delta_h(\alpha(\delta))\equiv\varphi_h(\alpha(\delta))-\left(\delta+\frac{h}{1-a}\|z_{\alpha(\delta)}^h\|_{W^h}\right)^2=0.$$

Proof. 1) By theorem 1, the function $\Delta_h(\alpha)$ is a continuous function of α;

2) The function $\Delta_h(\alpha)$ is a strictly increasing function. Suppose that $\alpha_1<\alpha_2$. By Theorem 1, we get

$$M_h^{\alpha_2}[z_{\alpha_2}^h,\ u_\delta^h]=\varphi_h(\alpha_2)+\alpha_2\|z_{\alpha_2}^h\|_{W^h}^2>\varphi_h(\alpha_2)+\alpha_1\|z_{\alpha_2}^h\|_{W^h}^2$$
$$>\varphi_h(\alpha_1)+\alpha_1\|z_{\alpha_1}^h\|_{W^h}^2=M_h^{\alpha_1}[z_{\alpha_1}^h,\ u_\delta^h],$$
$$M_h^{\alpha_2}[z_{\alpha_2}^h,\ u_\delta^h]=\varphi_h(\alpha_2)+\alpha_2\|z_{\alpha_2}^h\|_{W^h}^2<\varphi_h(\alpha_1)+\alpha_2\|z_{\alpha_1}^h\|_{W^h}^2.$$

Furthermore, using these inequalities, that is,

$$\varphi_h(\alpha_2)+\alpha_1\|z_{\alpha_2}^h\|_{W^h}^2>\varphi_h(\alpha_1)+\alpha_1\|z_{\alpha_1}^h\|_{W^h}^2$$

and
$$\varphi_h(\alpha_2)+\alpha_2\|z_{\alpha_2}^h\|_{W^h}^2<\varphi_h(\alpha_1)+\alpha_2\|z_{\alpha_1}^h\|_{W^h}^2,$$

we obtain

$$\|z_{\alpha_1}^h\|_{W^h}^2>\|z_{\alpha_2}^h\|_{W^h}^2$$

and
$$\varphi_h(\alpha_2)-\varphi_h(\alpha_1)>\alpha_1[\|z_{\alpha_1}^h\|_{W^h}^2-\|z_{\alpha_2}^h\|_{W^h}^2]>0,$$

from which the strict monotonicity of $\Delta_h(\alpha)$ follows;

3) $$\lim_{\alpha\to\infty} \Delta_h(\alpha) = \|u_\delta^h\|^2_{L^h} - \delta^2.$$

Since the functional $M_h^\alpha[z^h, u_\delta^h]$ attains its minimum when $z^h = z_\alpha^h$, we have

$$\alpha\|z_\alpha^h\|^2_{W^h} \leqslant M_h^\alpha[z_\alpha^h, u_\delta^h] \leqslant \alpha\|O\|^2_{W^h} + \|A^hO - u_\delta^h\|^h_{L^h} = \|u_\delta^h\|^2_{L^h}.$$

Hence $$\lim_{\alpha\to\infty}\|z_\alpha^h\|_{W^h} = 0.$$

This, in turn, implies that

$$\lim_{\alpha\to\infty}\varphi_h(\alpha) = \lim_{\alpha\to\infty}\|A^h z_\alpha^h - u_\delta^h\|^2_{L^h} = \|u_\delta^h\|^2_{L^h}$$

and that $$\lim_{\alpha\to\infty}\Delta_h(\alpha) = \|u_\delta^h\|^2_{L^h} - \delta^2 > 0;$$

4) $$\lim_{\alpha\to 0}\Delta_h(\alpha) \leqslant -\delta^2 < 0.$$

Using the inequality

$$0 \leqslant \varphi_h(\alpha) \leqslant M_h^\alpha[z_\alpha^h, u_\delta^h] \leqslant M_h^\alpha[v^h, u_\delta^h] = \alpha\|v^h\|^2_{W^h},$$

where v^h satisfies $A^h v^h = u_\delta^h$ $(u_{\delta,0}^h = 0)$ (if $A^h v^h = u_\delta^h$ is written out, it is easy to see that such an element v^h exists), and the inequality

$$\Delta_h(\alpha) \leqslant \varphi_h(\alpha) - \delta^2,$$

we immediately get the assertion.

From 1), 2), 3) and 4) it is obvious that the equation

$$\Delta_h(\alpha) = 0$$

has a unique solution $\alpha(\delta)$. This completes the proof of Theorem 2.

Now, we need to show that the operator $R(u_\delta^h, \alpha(\delta))$ is a regularizing operator.

Theorem 3. *Let* $\{\delta_n\}$ *and* $\{u_{\delta_n}^{h_n}\}$ *denote sequences of positive numbers and elements of* L^{h_n}*, respectively, such that*

$$\delta_n \to 0 \quad as \quad n\to\infty,$$

$$\|u_{\delta_n}^{h_n} - u_T^{h_n}\|_{L^{h_n}} \leqslant \delta_n, \qquad u_{\delta_n,0}^{h_n} = 0,$$

and $$\|u_{\delta_n}^{h_n}\|^2_{L^{h_n}} > \delta_n^2.$$

Then

$$\lim_{n\to\infty}\|z_{\alpha(\delta_n)}^{h_n}(s) - z_T(s)\|_C = 0,$$

where $$z_{\alpha(\delta_n)}^{h_n}(s) = R(u_{\delta_n}^{h_n}, \alpha(\delta_n)).$$

Proof. 1) Since the element $z_{\alpha(\delta_n)}^{h_n}$ minimizes the functional $M_{h_n}^{\alpha(\delta_n)}[z^{h_n}, u_{\delta_n}^{h_n}]$, we have

$$M_{h_n}^{\alpha(\delta_n)}[z_{\alpha(\delta_n)}^{h_n}, u_{\delta_n}^{h_n}] = \alpha(\delta_n)\|z_{\alpha(\delta_n)}^{h_n}\|^2_{W^{h_n}} + \|A^{h_n}z_{\alpha(\delta_n)}^{h_n} - u_{\delta_n}^{h_n}\|^2_{L^{h_n}}$$

$$= \alpha(\delta_n)\|z_{\alpha(\delta_n)}^{h_n}\|^2_{W^{h_n}} + \left(\delta_n + \frac{h_n}{1-a}\|z_{\alpha(\delta_n)}^{h_n}\|_{W^{h_n}}\right)^2$$

$$\leqslant \alpha(\delta_n)\|z_T\|^2_{C_1} + \left(\delta_n + \frac{h_n}{1-a}\|z_T\|_{C_1}\right)^2,$$

or $$\{\|z_{\alpha(\delta_n)}^{h_n}\|_{W^{h_n}} - \|z_T\|_{C_1}\}\left\{\frac{2h_n\delta_n}{1-a} + \left[\alpha(\delta_n) + \frac{h_n^2}{(1-a)^2}\right][\|z_{\alpha(\delta_n)}^{h_n}\|_{W^{h_n}} + \|z_T\|]\right\} \leqslant 0.$$

Consequently,

$$\|z_{\alpha(\delta_n)}^{h_n}\|_{W^{h_n}} \leqslant \|z_T\|_{C_1}.$$

Thus, the set $\{z^{h_n}_{\alpha(\delta_n)}(s)\}$ of elements of $W^1_2[0, 1]$, for which

$$\|z^{h_n}_{\alpha(\delta_n)}(s)\|^2_{W^1_2}=\sum_{j=0}^{n-1}\int_{s_j}^{s_{j+1}}[z^{h_n}_{\alpha(\delta_n)}(s)]^2\,ds+\sum_{j=0}^{n-1}\int_{s_j}^{s_{j+1}}\left[\frac{dz^{h_n}_{\alpha(\delta_n)}(s)}{ds}\right]^2 ds\leqslant\|z^{h_n}_{\alpha(\delta_n)}\|^2_{W^{h_n}}$$

is a compact subset of the space $C[0, 1]$.

Thus, $\{z^{h_n}_{\alpha(\delta_n)}(s)\}$ has a subsequence $\{z^{h_{n_k}}_{\alpha(\delta_{n_k})}(s)\}$ that converges (with respect to the metric of $C[0, 1]$) to some element $\bar{z}(s)\in C[0, 1]$:

$$\bar{z}(s)=\lim_{n_k\to\infty} z^{h_{n_k}}_{\alpha(\delta_{n_k})}(s);$$

2) We now show that

$$\bar{z}(s)=z_T(s).$$

Since

$$\|A\bar{z}-Az_T\|_{L_2}\leqslant\|A\bar{z}-Az^{h_{n_k}}_{\alpha(\delta_{n_k})}(s)\|_{L_2}+\|Az^{h_{n_k}}_{\alpha(\delta_{n_k})}(s)-u_T\|_{L_2},$$

using the continuity of operator A and taking the limit as $n_k\to\infty$, we get

$$\|A\bar{z}-Az_T\|_{L_2}=0,$$

or

$$A\bar{z}=Az_T.$$

The uniqueness of the solution of equation (1) implies that

$$z_T(s)=\bar{z}(s)=\lim_{n_k\to\infty} z^{h_{n_k}}_{\alpha(\delta_{n_k})}(s);$$

3) This will be the case for every convergent subsequence of the sequence $\{z^{h_n}_{\alpha(\delta_n)}(s)\}$. It follows that, for every sequence $\{\delta_n\}$ of positive numbers δ_n that converges to zero, the corresponding sequence $\{z^{h_n}_{\alpha(\delta_n)}(s)\}$ converges to the element $z_T(s)$. This completes the proof of the theorem.

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