ESTIMATION OF THE SEPARATION OF TWO MATRICES (II)*

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Abstract

In this paper we give a lower bound of the separation $\operatorname{sep}_F(A, B)$ of two diagonalizable matrices A and B. The key to finding the lower bound of $\operatorname{sep}_F(A, B)$ is to find an upper bound for the condition number $\varkappa(Q)$ of a transformation matrix Q which transforms a diagonalizable matrix A to a diagonal form. The obtained lower bound of $\operatorname{sep}_F(A, B)$ involves the eigenvalues of A and B as well as the departures from normality $\Delta_F(A)$ and $\Delta_F(B)$.

This is a continuation of [6]. In addition to the notation explained in [6] we use $\mathbb{C}^n$ for the n-dimensional column vector space, and $\mathfrak{R}(X)$ for the column space of a matrix X. $\oplus$ denotes the direct sum of subspaces, and $\mathscr{X}^{\perp}$ the orthogonal complement of a subspace $\mathscr{X}$. Besides, X^H stand for conjugate transpose of X.

§ 4. An Upper Bound for the Spectral Condition Number of a Diagonalizable Matrix

Let A and B be diagonalizable matrices with the eigenvalues $\{\lambda_i\}$ and $\{\mu_i\}$ respectively, Q_A and Q_B be transformation matrices which transform A and B to diagonal forms. It is proved that if we set

$$\delta(A, B)=\min_{i,j}|\lambda_i-\mu_j| \tag{4.1}$$

and

$$\varkappa(Q)=\|Q\|_2\|Q^{-1}\|_2, \tag{4.2}$$

then[5, 6]

$$\frac{\delta(A, B)}{\varkappa(Q_A)\varkappa(Q_B)}\leqslant \operatorname{sep}_F(A, B)\leqslant\delta(A, B). \tag{4.3}$$

Therefore, estimation of a lower bound for the separation $\operatorname{sep}_F(A, B)$ is reduced to estimations of upper bounds for the condition numbers $\varkappa(Q_A)$ and $\varkappa(Q_B)$.

In this section we use the characteristic of a diagonalizable matrix A to give an upper bound for the spectral condition number $\inf_Q \varkappa(Q)$ of A, here the inf taking over all Q which similarity transforms A to a diagonal form.

For a nonsingular matrix Q, we set

$$K(Q)=\|Q\|_F\|Q^{-1}\|_F. \tag{4.4}$$

The following lemma delineates the relation between the $K(Q)$ and $\varkappa(Q)$.

Lemma 4.1. *Suppose that $Q\in\mathbb{C}^{m\times m}$ is nonsingular. Then*

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$$1+\frac{K(Q)-m+\sqrt{K^2(Q)-m^2}}{m}\leqslant\varkappa(Q)$$

$$\leqslant 1+\frac{K(Q)-m+\sqrt{[K(Q)-m+2]^2-4}}{2}. \tag{4.5}$$

Proof. Let $K=K(Q)$ and $\varkappa=\varkappa(Q)$. By Theorem 1 of [4],

$$m-2+\varkappa+\varkappa^{-1}\leqslant K\leqslant\frac{1}{2}m(\varkappa+\varkappa^{-1}). \tag{4.6}$$

Combining $\varkappa+\varkappa^{-1}\geqslant 2$ and the first inequality of (4.6), we get $K\geqslant m$. From the second inequality of (4.6),

$$0<\varkappa\leqslant 1-\frac{\sqrt{K^2-m^2}-(K-m)}{m}, \quad \varkappa\geqslant 1+\frac{K-m+\sqrt{K^2-m^2}}{m}; \tag{4.7}$$

and from the first inequality of (4.6),

$$1-\frac{\sqrt{(K-m+2)^2-4}-(K-m)}{2}\leqslant\varkappa\leqslant 1+\frac{K-m+\sqrt{(K-m+2)^2-4}}{2}. \tag{4.8}$$

Observe that

$$\frac{K-m+\sqrt{K^2-m^2}}{m}\leqslant\frac{K-m+\sqrt{(K-m+2)^2-4}}{2},$$

$$0\leqslant\frac{\sqrt{K^2-m^2}-(K-m)}{m}\leqslant\frac{\sqrt{(K-m+2)^2-4}-(K-m)}{2}$$

and
$$\frac{\sqrt{K^2-m^2}-(K-m)}{m}=0 \quad \text{iff } K=m \quad \text{iff } \varkappa=1,$$

hence, from (4.7) and (4.8) we obtain the inequalities (4.5) at once. ∎

Now we cite a theorem proved by Elsner[3], which is a generalization of a result due to Smith[4].

Theorem 4.1. *Suppose that $A\in\mathbb{C}^{m\times m}$ with different eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_r$ of multiplicities $m_1, m_2, \cdots, m_r$ respectively. Let $\mathbb{C}^m=\mathscr{X}_1\oplus\mathscr{X}_2\oplus\cdots\oplus\mathscr{X}_r$, $\mathscr{X}_i$ be the invariant subspace of A corresponding to the λ_i with $\dim(\mathscr{X}_i)=m_i$, $i=1, 2, \cdots, r$. If we set $\mathscr{Y}_i=\bigcap_{j\neq i}\mathscr{X}_j^{\perp}$, $i=1, 2, \cdots, r$, and*

$$\mathscr{Q}=\{Q=(Q_1, Q_2, \cdots, Q_r):\Re(Q_i)=\mathscr{X}_i, \ i=1, \cdots, r\},$$

then

$$\min_{Q\in\mathscr{Q}} K(Q)=\sum_{i=1}^{r}\sum_{j=1}^{m_i}\frac{1}{\sigma_i^{(j)}}, \tag{4.9}$$

where $\{\sigma_i^{(j)}\}_{j=1}^{m_i}$ are the singular values of $P_i^HQ_i$ in which the P_i and Q_i satisfy $\Re(P_i)=\mathscr{Y}_i$, $\Re(Q_i)=\mathscr{X}_i$ and

$$P_i^HP_i=Q_i^HQ_i=I^{(m_i)}, \quad i=1, 2, \cdots, r.$$

The Schur decomposition of any diagonalizable matrix has an important characteristic clarified by the following lemma.

Lemma 4.2. *Let A be an $m\times m$ diagonalizable matrix with Schur decomposition*

$$U^HAU=\Lambda+M\equiv T, \tag{4.10}$$

where U is a unitary matrix, M is a strictly upper triangular matrix (i.e., M is an upper triangular matrix with zeros on its diagonal) and

$$A=\operatorname{diag}(\lambda_1 I^{(m_1)}, \lambda_2 I^{(m_2)}, \cdots, \lambda_r I^{(m_r)}), \quad \lambda_i \neq \lambda_j (i \neq j). \tag{4.11}$$

Then

$$M=\begin{pmatrix} 0 & M_{12} & M_{13} & \cdots & M_{1r} \\ & 0 & M_{23} & \cdots & M_{2r} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & M_{r-1,r} \\ & \mathbf{0} & & & 0 \end{pmatrix}, \quad M_{ij} \in \mathbb{C}^{m_i \times m_j}, \quad 1 \leq i < j \leq r. \tag{4.12}$$

The proof of Lemma 4.2 can be found in [1] ([1], Theorem 2) and [3] ([3], Lemma 2).

Let x, z be right and left eigenvectors of a matrix A corresponding to the eigenvalue λ_i, i.e.,

$$Ax=\lambda_i x, \qquad z^H A=\lambda_i z^H. \tag{4.13}$$

It is well known that if λ_i is a simple eigenvalue then x, z are unique, except for a scalar multiple, and the condition number

$$|S_i|^{-1}=\frac{\|x\|_2 \cdot \|z\|_2}{|z^H x|} \tag{4.14}$$

is uniquely determined ([7], 68—69).

Smith[4] has given the following estimation for $|S_i|^{-1}$.

Theorem 4.2. *Let A be an $m\times m$ matrix with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_m$. If λ_i is a simple eigenvalue of A, then*

$$|S_i|^{-1} \leq \left\{1+\frac{1}{m-1}\left(\frac{\Delta_F(A)}{\min\limits_{j\neq i}|\lambda_i-\lambda_j|}\right)^2\right\}^{\frac{m-1}{2}}, \tag{4.15}$$

where

$$\Delta_F(A)=\sqrt{\|A\|_F^2-\sum_{i=1}^{m}|\lambda_i|^2}.$$

Let A be a diagonalizable matrix. Utilizing Theorems 4.1—4.2 and Lemma 4.2 we shall find an upper bound of $K(Q)$ for a suitable transformation matrix Q of the A, and then utilizing Lemma 4.1 we obtain an upper bound of $\varkappa(Q)$.

Theorem 4.3. *Let A be an $m\times m$ diagonalizable matrix with different eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_r$ of multiplicities $m_1, m_2, \cdots, m_r$ respectively. Then there is a nonsingular matrix Q satisfying*

$$Q^{-1}AQ=\operatorname{diag}(\lambda_1 I^{(m_1)}, \lambda_2 I^{(m_2)}, \cdots, \lambda_r I^{(m_r)}) \tag{4.16}$$

such that

$$K(Q) \leq \sum_{i=1}^{r}\sqrt{m_i}\left\{m_i-1+\left[1+\frac{1}{r-1}\left(\frac{\Delta_F(A)}{\min\limits_{j\neq i}|\lambda_i-\lambda_j|}\right)^2\right]^{r-1}\right\}^{\frac{1}{2}} \equiv K_A \tag{4.17}$$

and

$$\varkappa(Q) \leq 1+\frac{K_A-m+\sqrt{(K_A-m+2)^2-4}}{2} \equiv \varkappa_A, \tag{4.18}$$

where

$$\Delta_F(A)=\sqrt{\|A\|_F^2-\sum_{i=1}^{r} m_i|\lambda_i|^2}=\|M\|_F$$

is the departure from normality of A, and M is the strictly upper triangular matrix of the Schur decomposition of A (see (4.12)).

Proof. 1° We notice that if $X_i, Y_i \in \mathbb{C}^{m\times m_i}$, $X_i^H X_i = Y_i^H Y_i = I^{(m_i)}$ and

$$AX_i = \lambda_i X_i,\ Y_i^H A = \lambda_i Y_i^H, \qquad i=1, 2, \cdots, r, \tag{4.19}$$

then there is a nonsingular $Q\in\mathbb{C}^{m\times m}$ satisfying (4.16) such that

$$K(Q) = \sum_{i=1}^{r} \operatorname{tr}\left[(Y_i^H X_i X_i^H Y_i)^{-\frac{1}{2}}\right]. \tag{4.20}$$

Here $H^{\frac{1}{2}}$ denotes the positive definite square root for a positive definite matrix H.

This conclusion can be proved as follows.

Let $X=(X_1, X_2, \cdots, X_r)$. The matrix X must be nonsingular. In fact, if X is singular, i.e., there exists a nonzero vector $u=(u_1^T, u_2^T, \cdots, u_r^T)^T\in\mathbb{C}^m$ with $u_i\in\mathbb{C}^{m_i}$ $(i=1, 2, \cdots, r)$ such that $Xu=0$, then from (4.19),

$$\begin{pmatrix} I^{(m)} & I^{(m)} & \cdots & I^{(m)} \\ \lambda_1 I^{(m)} & \lambda_2 I^{(m)} & \cdots & \lambda_r I^{(m)} \\ \lambda_1^2 I^{(m)} & \lambda_2^2 I^{(m)} & \cdots & \lambda_r^2 I^{(m)} \\ \vdots & \vdots & & \vdots \\ \lambda_1^{r-1} I^{(m)} & \lambda_2^{r-1} I^{(m)} & \cdots & \lambda_r^{r-1} I^{(m)} \end{pmatrix} \begin{pmatrix} X_1u_1 \\ X_2u_2 \\ \vdots \\ X_ru_r \end{pmatrix} = 0.$$

From this we get $X_iu_i=0$ and then $u_i=0$, $i=1, 2, \cdots, r$. But this contradicties that u is a nonzero vector. Hence X is nonsingular. With the same argument we can prove that the Y is also nonsingular.

Let $\Lambda=\operatorname{diag}(\lambda_1 I^{(m_1)}, \lambda_2 I^{(m_2)}, \cdots, \lambda_r I^{(m_r)})$. From (4.19),

$$AX = X\Lambda, \quad Y^H A = \Lambda Y^H.$$

Thus we have

$$Y^H X\Lambda = Y^H AX = \Lambda Y^H X.$$

Utilizing $\lambda_i \ne \lambda_j$ $(i\ne j)$ we get $Y_j^H X_i = 0$ $(i\ne j)$.

Consequently, if we set $\mathscr{X}_i = \Re(X_i)$ and $\mathscr{Y}_i=\Re(Y_i)$, then $\mathscr{X}_i$ is the invariant subspace of A corresponding to the eigenvalue λ_i with $\dim(\mathscr{X}_i)=m_i$, and

$$\mathscr{Y}_i = \bigcap_{j\ne i} \mathscr{X}_j^{\perp}, \qquad i=1, 2, \cdots, r.$$

Therefore, by Theorem 4.1 there is a nonsingular matrix Q satisfying (4.16) such that the equality (4.20) is valid.

2° Observe that if $X_i, Y_i\in\mathbb{C}^{m\times m_i}$, $X_i^H X_i = Y_i^H Y_i = I^{(m_i)}$ and for the Schur upper triangular form T (see (4.10))

$$TX_i = \lambda_i X_i, \quad Y_i^H T = \lambda_i Y_i^H,$$

then

$$A(UX_i) = \lambda_i (UX_i), \quad (UY_i)^H A = \lambda_i (UY_i)^H$$

and

$$\operatorname{tr}\{[(UY_i)^H (UX_i)(UX_i)^H (UY_i)]^{-\frac{1}{2}}\} = \operatorname{tr}[(Y_i^H X_i X_i^H Y_i)^{-\frac{1}{2}}].$$

Hence, without loss of generality we can assume that $A=T$, i.e.,

$$A=\begin{pmatrix} \lambda_1 I^{(m_1)} & M_{12} & M_{13} & \cdots & M_{1r} \\ & \lambda_2 I^{(m_2)} & M_{23} & \cdots & M_{2r} \\ & & \ddots & \ddots & \vdots \\ & 0 & & \ddots & M_{r-1,r} \\ & & & & \lambda_r I^{(m_r)} \end{pmatrix}. \tag{4.21}$$

3° For $i=1$, we take $X_1=(I^{(m_1)}, 0, \cdots, 0)^T$. Obviously, $AX_1=\lambda_1 X_1$. Then we take

$$Z_1=(I^{(m_1)}, Z_{12}, \cdots, Z_{1r})^H \in \mathbb{C}^{m\times m_1}$$

with $Z_{1j}\in\mathbb{C}^{m_1\times m_j}$ $(2\leqslant j\leqslant r)$ satisfying

$$Z_1^H A=\lambda_1 Z_1^H. \tag{4.22}$$

From (4.22), we have

$$Z_{1j}=\frac{1}{\lambda_1-\lambda_j}(M_{1j}+Z_{12}M_{2j}+\cdots+Z_{1,j-1}M_{j-1,j}), \quad j=2, 3, \cdots, r. \tag{4.23}$$

Let

$$\alpha_j=|\lambda_1-\lambda_j|, \ 2\leqslant j\leqslant r; \quad \mu_{ij}=\|M_{ij}\|_F, \quad 1\leqslant i<j\leqslant r; \tag{4.24}$$

$$\zeta_j=\frac{1}{\alpha_j}(\mu_{1j}+\zeta_2\mu_{2j}+\cdots+\zeta_{j-1}\mu_{j-1,j}), \quad j=2, 3, \cdots, r \tag{4.25}$$

and

$$\hat{A}=\begin{pmatrix} 0 & \mu_{12} & \mu_{13} & \cdots & \mu_{1r} \\ & -\alpha_2 & \mu_{23} & \cdots & \mu_{2r} \\ & & \ddots & \ddots & \vdots \\ & 0 & & \ddots & \mu_{r-1,r} \\ & & & & -\alpha_r \end{pmatrix}. \tag{4.26}$$

Obviously, $\alpha_2, \cdots, \alpha_r>0$, and $x=(1, 0, \cdots, 0)^T\in\mathbb{C}^r$ is a unity right eigenvector of $\hat{A}$ corresponding to the simple eigenvalue zero. It is easy to verify that $z=(1, \zeta_2, \cdots, \zeta_r)^H$ is a left eigenvector of $\hat{A}$ corresponding to the simple eigenvalue zero, here $\zeta_2, \cdots, \zeta_r$ are defined by (4.25). By (4.14),

$$|s_1|^{-1}=\|z\|_2=(1+\zeta_1^2+\cdots\zeta_r^2)^{\frac{1}{2}}. \tag{4.27}$$

But according to Theorem 4.2, we have

$$|s_1|^{-1}\leqslant\left\{1+\frac{1}{r-1}\left(\frac{\Delta_F(\hat{A})}{\min\limits_{2\leqslant j\leqslant r}\alpha_j}\right)^2\right\}^{\frac{r-1}{2}}, \tag{4.28}$$

where $\Delta_F(\hat{A})=\sqrt{\|\hat{A}\|_F^2-\sum_{j=2}^{r}\alpha_j^2}.$

Comparing the equality (4.27) with the inequality (4.28), we get

$$1+\zeta_2^2+\cdots+\zeta_r^2\leqslant\left\{1+\frac{1}{r-1}\left(\frac{\Delta_F(\hat{A})}{\min\limits_{2\leqslant j\leqslant r}\alpha_j}\right)^2\right\}^{r-1}. \tag{4.29}$$

Observe that

$$\|Z_1\|_F^2=m_1+\|Z_{12}\|_F^2+\cdots+\|Z_{1r}\|_F^2$$

and

$$\|Z_{1j}\|_F\leqslant\frac{1}{|\lambda_1-\lambda_j|}(\|M_{1j}\|_F+\|Z_{12}\|_F\|M_{2j}\|_F+\cdots+\|Z_{1,j-1}\|_F\|M_{j-1,j}\|_F)$$
$$\leqslant\frac{1}{\alpha_j}(\mu_{1j}+\zeta_2\mu_{2j}+\cdots+\zeta_{j-1}\mu_{j-1,j})=\zeta_j,\quad j=2,\ 3,\ \cdots,\ r,$$

hence from (4.29) we get

$$\|Z_1\|_F^2\leqslant m_1-1+(1+\zeta_2^2+\cdots+\zeta_r^2)$$
$$\leqslant m_1-1+\left\{1+\frac{1}{r-1}\left(\frac{\Delta_F(\hat{A})}{\min\limits_{j\neq 1}|\lambda_1-\lambda_j|}\right)^2\right\}^{r-1}. \tag{4.30}$$

Substituting the relation

$$\Delta_F^2(\hat{A})=\sum_{1\leqslant i<j\leqslant r}\mu_{ij}^2=\sum_{1\leqslant i<j\leqslant r}\|M_{ij}\|_F^2=\Delta_F^2(A)$$

into (4.30), we obtain

$$\|Z_1\|_F\leqslant\left\{m_1-1+\left[1+\frac{1}{r-1}\left(\frac{\Delta_F(A)}{\min\limits_{j\neq 1}|\lambda_1-\lambda_j|}\right)^2\right]^{r-1}\right\}^{\frac{1}{2}}. \tag{4.31}$$

Let $Y_1=Z_1(Z_1^HZ_1)^{-\frac{1}{2}}$. Evidently, $Y_1^HY_1=I^{(m_1)}$, $Y_1^HA=\lambda_1Y_1^H$ and

$$\operatorname{tr}[(Y_1^HX_1X_1^HY_1)^{-\frac{1}{2}}]=\operatorname{tr}\{[(Z_1^HZ_1)^{-\frac{1}{2}}Z_1^HX_1X_1^HZ_1(Z_1^HZ_1)^{-\frac{1}{2}}]^{-\frac{1}{2}}\}$$
$$=\operatorname{tr}[(Z_1^HZ_1)^{\frac{1}{2}}]\leqslant[m_1\operatorname{tr}(Z_1^HZ_1)]^{\frac{1}{2}}=\sqrt{m_1}\|Z_1\|_F.$$

Substituting (4.31) into the right-hand of the above inequality we get

$$\operatorname{tr}[(Y_1^HX_1X_1^HY_1)^{-\frac{1}{2}}]\leqslant\sqrt{m_1}\left\{m_1-1+\left[1+\frac{1}{r-1}\left(\frac{\Delta_F(A)}{\min\limits_{j\neq 1}|\lambda_1-\lambda_j|}\right)^2\right]^{r-1}\right\}^{\frac{1}{2}}. \tag{4.32}$$

4° Observe that for every natural number i $(2\leqslant i\leqslant r)$ there is a corresponding unitary matrix U_i such that the scalar matrix $\lambda_iI^{(m_i)}$ lies in the left-upper corner of the Schur upper triangular form $T_i=U_i^HAU_i$. Hence from the above proof we reach the following conclusion: For every i $(1\leqslant i\leqslant r)$, there exist X_i and $Y_i\in\mathbb{C}^{m\times m_i}$ satisfying $X_i^HX_i=Y_i^HY_i=I^{(m_i)}$ and the relations (4.19), and we have

$$\operatorname{tr}[(Y_i^HX_iX_i^HY_i)^{-\frac{1}{2}}]\leqslant\sqrt{m_i}\left\{m_1-1+\left[1+\left(\frac{\Delta_F(A)}{\min\limits_{j\neq i}|\lambda_i-\lambda_j|}\right)^2\right]^{r-1}\right\}^{\frac{1}{2}},$$
$$i=1,\ 2,\ \cdots,\ r. \tag{4.33}$$

Substituting the inequalities (4.33) into (4.20) we obtain the estimation (4.17), and then utilizing Lemma 4.1 we get the estimation (4.18). ∎

Remark 4.1. If all eigenvalues of A are simple (i.e., $r=m$ in (4.16)), then by Theorem 4.3 there is a nonsingular matrix Q satisfying

$$Q^{-1}AQ=\operatorname{diag}(\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_m)$$

such that

$$K(Q)\leqslant\sum_{i=1}^{m}\left\{1+\frac{1}{m-1}\left(\frac{\Delta_F(A)}{\min\limits_{j\neq i}|\lambda_i-\lambda_j|}\right)^2\right\}^{\frac{m-1}{2}}. \tag{4.34}$$

This is exactly what Smith has reached conclusion in [4] (Theorem 3 and Theorem 5). Therefore Theorem 4.3 is a generalization of Smith's theorems.

Remark 4.2. In order to compare Theorem 4.3 with the related results of [3] and [6], we consider the case of $r=2$ as follows.

Let $A\in\mathbb{C}^{m\times m}$ with different eigenvalues λ_1 and λ_2 of multiplicities m_1 and m_2, $m_1+m_2=m$. Set $\chi=\dfrac{\Delta_F(A)}{|\lambda_1-\lambda_2|}$. By Theorem 1 of [3], there is a nonsingular matrix Q satisfying

$$Q^{-1}AQ=\operatorname{diag}\,(\lambda_1 I^{(m_1)},\ \lambda_2 I^{(m_2)}) \tag{4.35}$$

such that

$$\varkappa(Q)\leqslant(1+\chi(1+\chi)^{m_1-1}\{1+\chi(1+\chi)^{m_2-1}+\cdots+[\chi(1+\chi)^{m_2-1}]^{m_1-1}\})^2\equiv J; \tag{4.36}$$

by Lemma 3.4 of [6] we have

$$\varkappa(Q)\leqslant(1+\chi)^2\equiv s_1; \tag{4.37}$$

by Lemma 3.5 of [6] we have

$$\varkappa(Q)\leqslant\{1+\chi[1+\sqrt{2}\,\chi+\cdots+(\sqrt{2}\,\chi)^{m_1+m_2-2}]\}^2\equiv s_2; \tag{4.38}$$

but according to Theorem 4.3 of this paper, there is a nonsingular matrix Q satisfying (4.35) such that

$$\varkappa(Q)\leqslant 1+\frac{K_2-m+\sqrt{(K_2-m+2)^2-4}}{2}\equiv s_3, \tag{4.39}$$

where

$$K_2=\sqrt{m_1(m_1+\chi^2)}+\sqrt{m_2(m_2+\chi^2)}. \tag{4.40}$$

It follows from $K_2\leqslant m+\chi^2$ that

$$s_3\leqslant 1+\frac{\chi^2+\chi\sqrt{\chi^2+4}}{2}\equiv\tilde{s}_3. \tag{4.41}$$

Comparing (4.36)—(4.41) we have

$$s_3\leqslant\tilde{s}_3\leqslant s_1\leqslant s_2,\ J.$$

Remark 4.3. According to Theorem 4.3 and the famous Bauer-Fike Theorem ([7], 87), we get the following corollary: Let A be an $m\times m$ diagonalizable matrix with different eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_r$ of multiplicities $m_1, m_2, \cdots, m_r$ respectively. If λ is an eigenvalue of a matrix $A+E$, then there is an eigenvalue λ_i of A such that

$$|\lambda_i-\lambda|\leqslant\varkappa_A\cdot\|E\|_2,$$

here $\varkappa_A$ is denoted by (4.18) and (4.17).

§ 5. Lower Bound of $\mathrm{sep}_F(A, B)$ (III)

By the inequalities (4.3) and Theorem 4.3 we get the following result.

Theorem 5.1. *Let A be an $m\times m$ diagonalizable matrix with different eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_p$ of multiplicities $m_1, m_2, \cdots, m_p$, B be an $n\times n$ diagonalizable matrix with different eigenvalues $\mu_1, \mu_2, \cdots, \mu_q$ of multiplicities $n_1, n_2, \cdots, n_q$, respectively. Let*

$$\delta_i(A)=\min_{k\neq i}|\lambda_i-\lambda_k|\quad(1\leqslant i\leqslant p),$$
$$\delta_j(B)=\min_{l\neq j}|\mu_j-\mu_l|\quad(1\leqslant j\leqslant q),$$

$$\Delta_F(A)=\sqrt{\|A\|_F^2-\sum_{i=1}^{p}m_i|\lambda_i|^2},$$

$$\Delta_F(B)=\sqrt{\|B\|_F^2-\sum_{j=1}^{q}n_j|\mu_j|^2},$$

$$K_A=\sum_{i=1}^{p}\sqrt{m_i}\left\{m_i-1+\left[1+\frac{1}{p-1}\left(\frac{\Delta_F(A)}{\delta_i(A)}\right)^2\right]^{p-1}\right\}^{\frac{1}{2}},$$

$$K_B=\sum_{j=1}^{q}\sqrt{n_j}\left\{n_j-1+\left[1+\frac{1}{q-1}\left(\frac{\Delta_F(B)}{\delta_j(B)}\right)^2\right]^{q-1}\right\}^{\frac{1}{2}},$$

$$\varkappa_A=1+\frac{K_A-m+\sqrt{(K_A-m+2)^2-4}}{2},$$

$$\varkappa_B=1+\frac{K_B-n+\sqrt{(K_B-n+2)^2-4}}{2}$$

and

$$\delta(A,\ B)=\min_{i\neq j}|\lambda_i-\mu_j|.$$

Then

$$\mathrm{sep}_F(A,\ B)\geqslant\frac{\delta(A,\ B)}{\varkappa_A\cdot\varkappa_B}. \tag{5.1}$$

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