# SOLUTION OF THE TWO-DIMENSIONAL STEFAN PROBLEM BY THE SINGULARITY-SEPARATING METHOD*1)

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## Abstract

In this paper the idea of "singularity-separating" presented in [10] is used to solve a two-dimensional phase-change problem. A difference scheme with second-order accuracy everywhere, including the region near the boundary between two phases, is constructed for the above problem. Through the computation it is shown that the singularity-separating method, whose accuracy is high, is efficient for two-dimensional phase-change problem.

## I. Introduction

The Stefan problem, a moving boundary problem for parabolic partial differential equations, is an important subject studied by many scholars for years. It is often met with in engineerings and geophysics. For the multi-dimensional Stefan problem the analytic solution cannot be found except for only a few special cases, and therefore people devote themseft to finding its numerical solution. At present the difference methods and the finite element methods are the main methods for this problem[1-9]. Besides, there is a method in which the original equation is transformed to a new equation by using the internal energy function, and then the difference equations are obtained from the new equation. In the Stefan problem the boundaries among the media with different phases move with time $t$, and so are called moving boundaries. On the moving boundaries the solutions are weakly discontinuous and there exist exothermic processes or endothermic processes. Such a singularity makes it very difficult to find a numerical method with high accuracy for this problem.

We have presented a new numerical method, the singularity-separating method for the Stefan problem——a heat conduction problem with phase change. Its main idea goes as follows: First a curvilinear coordinate transformation is used to turn the moving phase-change boundaries into fixed boundaries of straight lines under the new coordinates. Thus the whole region in the new coordinates is divided into several rectangular subregions by the phase-change boundaries. Then in the subregions a stable difference scheme is constructed for the heat conduction equation

* Received December 17, 1983.
1) Projects supported by the Science Found of the Chinese Academy of Sciences.

under the new coordinates. Since difference equations are constructed in each subregion, there is no difference across discontinuities in the difference equations. Finally a simultaneous system composed of the difference equations in these subregions and the Stefan condition can be solved in order to obtain the solution.

The problem in one dimension has been discussed in [6]. Here we study the case in two dimensions.

## II. Mathematical Formulation of the Problem

The problem with phase change is studied in the region $D=\{(x, y)\,|\,0<x<X, 0<y<H\}$. The solid phase region is denoted as $\Omega_1(t)$ and the liquid phase region as $\Omega_2(t)$. Suppose that there is only one phase-change boundary, which is denoted as $\Gamma(t)$ (see Fig. 1).

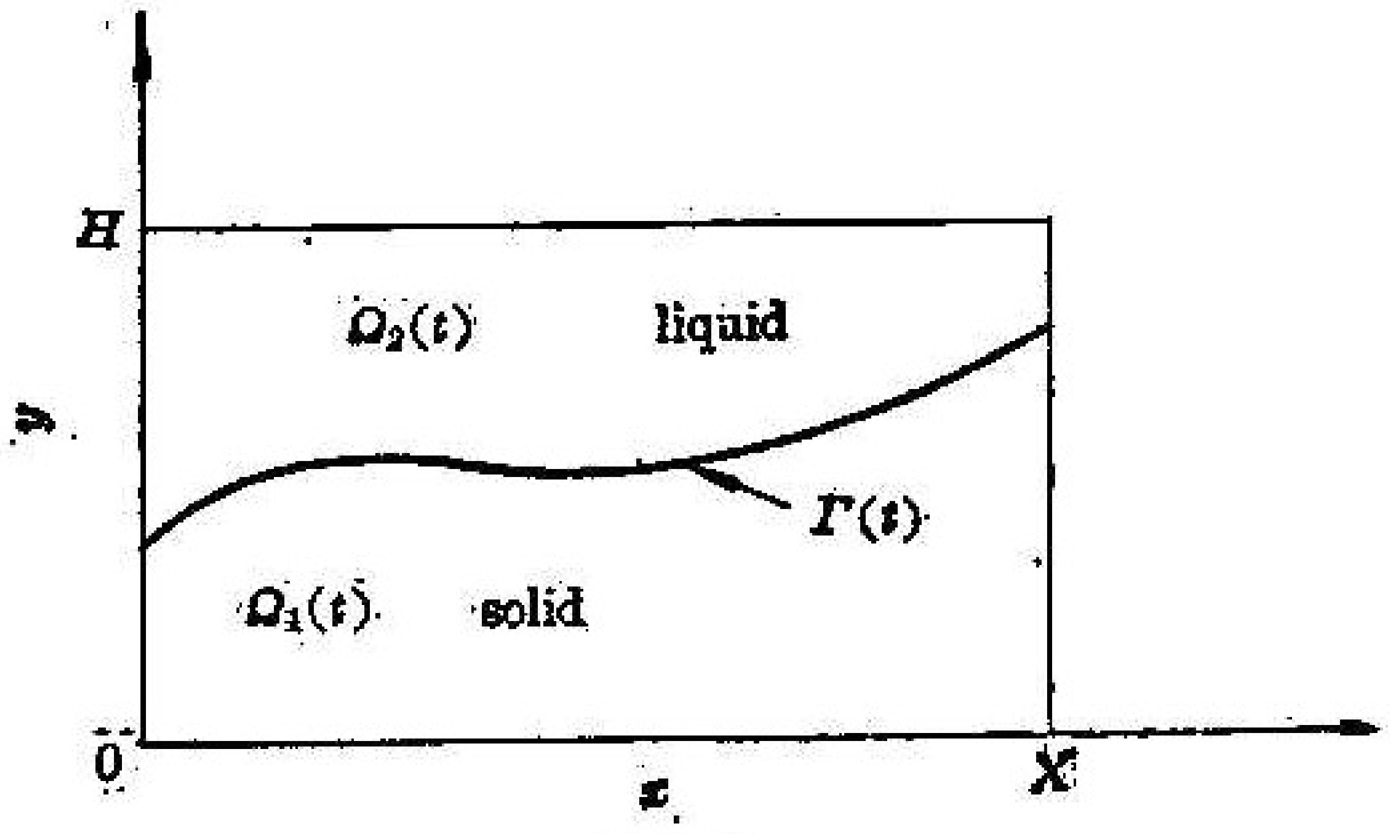


Fig. 1

According to the heat conservation law, the heat conduction problems in two dimensions in the solid region and the liquid region can be respectively described using the following formulae

$$C_1\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left[k_1\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial y}\left[k_1\frac{\partial u}{\partial y}\right],\quad 0<x<X,\ 0<y<f(x, t),\ t>0; \tag{2.1}$$

$$C_2\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left[k_2\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial y}\left[k_2\frac{\partial u}{\partial y}\right],\quad 0<x<X,\ f(x, t)<y<H,\ t>0. \tag{2.2}$$

Here $C_i=C_i(u)$, $i=1, 2$, stand for the specific thermal capacities (that is, the quantity of heat which per volume of substance needs for its temperature to increase by 1°C); $k_i=k_i(u)$, $i=1, 2$, for the coefficients of heat conduction. The subscripts 1 and 2 stand for the solid region and the liquid region respectively.

Suppose that the equation of the phase-change boundary $\Gamma(t)$ is $y=f(x, t)$. On the surface $y=f(x, t)$, the connective condition can be written as

$$u^-(x, f(x, t), t)=u^+(x, f(x, t), t)=u_f; \tag{2.3}$$

and

$$\lambda\frac{d\xi_n}{dt}=\left(k_1\frac{\partial u^-}{\partial n}-k_2\frac{\partial u^+}{\partial n}\right)\Bigg|_{\Gamma(t)}. \tag{2.4}$$

Here $u^-$ and $u^+$ respectively represent the values of $u$ on the lower side and on the upper side of $y=f(x, t)$, $\boldsymbol{n}$ stands for the unit normal, $d\xi_n$ for the variation of distance along $\boldsymbol{n}$ (see Fig. 2), $\lambda$ for the latent heat of phase-change and $u_f$ for the phase-change temperature.

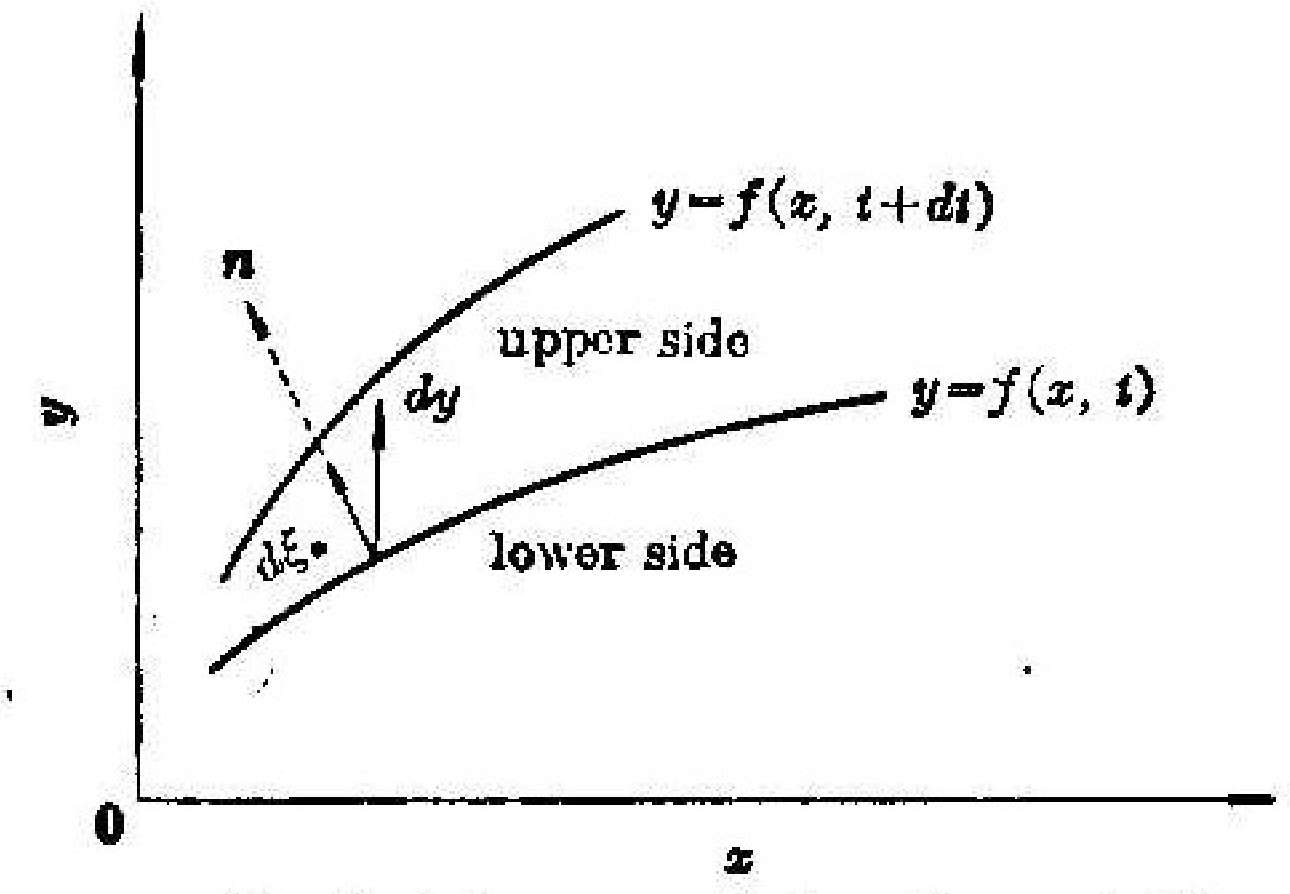


Fig. 2 The normal direction and $d\xi_n$

Formula (2.4) is called the Stefan condition, and can also be written as

$$\lambda\frac{\partial f}{\partial t}=k_1\frac{\partial u^-}{\partial y}-k_2\frac{\partial u^+}{\partial y}-\frac{\partial f}{\partial x}\times\left(k_1\frac{\partial u^-}{\partial x}-k_2\frac{\partial u^+}{\partial x}\right),\quad \text{on } y=f(x, t).$$

Because on $y=f(x, t)$, $\frac{\partial u^\pm}{\partial x}$ and $\frac{\partial u^\pm}{\partial y}$ satisfy the relation

$$\frac{\partial u^\pm}{\partial x}=-\frac{\partial f}{\partial x}\frac{\partial u^\pm}{\partial y}, \tag{2.6}$$

formula (2.5) can also be written as

$$\lambda\frac{\partial f}{\partial t}=\left[1+\left(\frac{\partial f}{\partial x}\right)^2\right]\left(k_1\frac{\partial u^-}{\partial y}-k_2\frac{\partial u^+}{\partial y}\right) \qquad \text{on } y=f(x, t), \tag{2.7}$$

or

$$\lambda\frac{\partial f}{\partial t}=\left[\frac{\partial f}{\partial x}+1\Big/\frac{\partial f}{\partial x}\right]\left(k_2\frac{\partial u^+}{\partial x}-k_1\frac{\partial u^-}{\partial x}\right) \qquad \text{on } y=f(x, t). \tag{2.8}$$

On $x=0$, $y=0$ and $y=H$ the boundary conditions of the first type are given:

$$u(0, y, t)=\phi_0(y, t), \quad 0<y<H,\ t>0; \tag{2.9}$$

$$u(x, 0, t)=\phi_1(x, t), \quad 0<x<X,\ t>0; \tag{2.10}$$

$$u(x, H, t)=\phi_2(x, t), \quad 0<x<X,\ t>0. \tag{2.11}$$

On $x=X$ the boundary condition of the second type is given:

$$\frac{\partial u}{\partial x}=0, \quad 0<y<H,\ t>0. \tag{2.12}$$

And on the region $D$ the initial value

$$u(x, y, 0)=\psi(x, y), \quad 0\leqslant x\leqslant X,\ 0\leqslant y\leqslant H \tag{2.13}$$

is specified. In the following section we shall solve the problem (2.1)—(2.4) and (2.9)—(2.13).

## III. Solution of the Problem

For the convenience of numerical computation the following system of curvilinear coordinates $\{x, z, t\}$ is used to transform the moving boundary into a fixed coordinate surface:

$$\begin{cases} x=x, \\ z=\begin{cases}\dfrac{y}{f(x, t)}, & 0<y<f(x, t), \\ \dfrac{y-f(x, t)}{H-f(x, t)}+1, & f(x, t)<y<H,\end{cases} \\ t=t. \end{cases} \tag{3.1}$$

Let

$$u(x, y, t)=v(x, z, t). \tag{3.2}$$

Clearly, the following relations among $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial z}$, $\cdots$ exist

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}, \tag{3.3}$$

$$\frac{\partial u}{\partial y}=\frac{\partial v}{\partial z}\frac{\partial z}{\partial y}, \tag{3.4}$$

$$\frac{\partial u}{\partial t}=\frac{\partial v}{\partial t}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial t}, \tag{3.5}$$

$$\frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial z}\left[k\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right)\right]\frac{\partial z}{\partial x}+\frac{\partial}{\partial x}\left[k\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right)\right], \tag{3.6}$$

$$\frac{\partial}{\partial y}\left(k\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial z}\left(k\frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial y}. \tag{3.7}$$

Moreover, we have

$$\frac{\partial z}{\partial x}=\begin{cases}-\dfrac{z}{f}\dfrac{\partial f}{\partial x}, & 0<z<1,\\ \dfrac{z-2}{H-f}\dfrac{\partial f}{\partial x}, & 1<z<2,\end{cases} \tag{3.8}$$

$$\frac{\partial z}{\partial y}=\begin{cases}\dfrac{1}{f}, & 0<z<1,\\ \dfrac{1}{H-f}, & 1<z<2,\end{cases} \tag{3.9}$$

$$\frac{\partial z}{\partial t}=\begin{cases}-\dfrac{z}{f}\dfrac{\partial f}{\partial t}, & 0<z<1,\\ \dfrac{z-2}{H-f}\dfrac{\partial f}{\partial t}, & 1<z<2.\end{cases} \tag{3.10}$$

According to these relations, from equations (2.1) and (2.2) we can derive the following heat conduction equations in the new coordinate system:

$$C_1\frac{\partial v}{\partial t}=-C_1\frac{\partial z}{\partial t}\frac{\partial v}{\partial z}+\frac{\partial z}{\partial x}\frac{\partial}{\partial z}\left[k_1\left(\frac{\partial v}{\partial x}+\frac{\partial z}{\partial x}\frac{\partial v}{\partial z}\right)\right]+\frac{\partial}{\partial x}\left[k_1\left(\frac{\partial v}{\partial x}+\frac{\partial z}{\partial x}\frac{\partial v}{\partial z}\right)\right]$$
$$+\frac{\partial z}{\partial y}\frac{\partial}{\partial z}\left(k_1\frac{\partial z}{\partial y}\frac{\partial v}{\partial z}\right), \quad 0<x<X,\ 0<z<1,\ t>0; \tag{3.11}$$

$$C_2\frac{\partial v}{\partial t}=-C_2\frac{\partial z}{\partial t}\frac{\partial v}{\partial z}+\frac{\partial z}{\partial x}\frac{\partial}{\partial z}\left[k_2\left(\frac{\partial v}{\partial x}+\frac{\partial z}{\partial x}\frac{\partial v}{\partial z}\right)\right]+\frac{\partial}{\partial x}\left[k_2\left(\frac{\partial v}{\partial x}+\frac{\partial z}{\partial x}\frac{\partial v}{\partial z}\right)\right]$$
$$+\frac{\partial z}{\partial y}\frac{\partial}{\partial z}\left(k_2\frac{\partial z}{\partial y}\frac{\partial v}{\partial z}\right), \quad 0<x<X,\ 1<z<2,\ t>0. \tag{3.12}$$

And from (2.3) and (2.7) we can derive the following connective conditions in the new coordinate system:

$$v^-(x,1,t)=v^+(x,1,t)=u_f, \quad 0<x<X,\ t>0, \tag{3.13}$$

$$\lambda\frac{\partial f}{\partial t}=\left[1+\left(\frac{\partial f}{\partial x}\right)^2\right]\left(\frac{k_1}{f}\frac{\partial v^-}{\partial z}-\frac{k_2}{H-f}\frac{\partial v^+}{\partial z}\right), \quad 0<x<X,\ t>0. \tag{3.14}$$

When the above coordinate transformation is used, the boundary conditions (2.10) and (2.11) do not change, but (2.9) should be rewritten as

$$v(0,z,t)=\begin{cases}\phi_0(zf(0,t),t), & 0<z<1,\ t>0,\\ \phi_0(f(0,t)+(z-1)(H-f(0,t)),t), & 1<z<2,\ t>0.\end{cases} \tag{3.15}$$

Because the problem in section II can be considered as a two-dimensional heat conduction problem with phase change which has symmetry about $x=X$, (2.12) is equivalent to the following relations:

$$\frac{\partial v(X,z,t)}{\partial x}=0, \quad 0<z<2,\ t>0, \tag{3.16}$$

$$\frac{\partial f(X,t)}{\partial x}=0, \quad t>0. \tag{3.17}$$

The initial value condition (2.13) should be rewritten in the following form

$$v(x,\ z,\ 0)=\begin{cases}\psi(x,\ zf(x,\ 0)), & 0<x<X,\ 0<z<1,\\ \psi(x,\ f(x,\ 0)+(z-1)(H-f(x,\ 0))), & 0<x<X,\ 1<z<2.\end{cases} \tag{3.18}$$

Therefore solving the problem described in section II is equivalent to obtaining $v(x,\ z,\ t)$ and $f(x,\ t)$ which satisfy the relations (3.11)—(3.18), (2.10) and (2.11).

In what follows, we shall give difference schemes for (3.11), (3.12), (3.14), (3.16) and (3.17).

Suppose that the specific thermal capacity and the coefficient of heat conduction are constants in $\Omega_1(t)$ or $\Omega_2(t)$. (If they are not constants, only slight revision on the difference scheme is needed.) To treat the boundary condition of the second type on $x=X$, the following method is used: The lines of mesh are $x=0,\ \Delta x,\ \cdots,\ M\Delta x,\ (M+1)\Delta x;\ z=0,\ \Delta z,\ \cdots,\ L\Delta z,\ (L+1)\Delta z,\ \cdots,\ 2L\Delta z$. Here, $\Delta z=\frac{1}{L}$, $\Delta x=2X/(2M+1)$ $\left(\text{that is } X=\left(M+\frac{1}{2}\right)\Delta x\right)$. $M$ and $L$ are positive integers. For subscripts and superscripts, the following notation is used:

$$F^n_{ij}=F(i\Delta x,\ j\Delta z,\ n\Delta t),\quad G^n_i=G(i\Delta x,\ n\Delta t).$$

Here, $F$ stands for a function of $x,\ z,\ t$ and $G$ for a function of $x,\ t$. The following formula is used to approximate (3.11) and (3.12)

$$\begin{aligned}
C\frac{v^{n+1}_{i,j}-v^n_{i,j}}{\Delta t} &= -C\left(\frac{\partial z}{\partial t}\right)^{n+1/2}_{i,j}\frac{v^{n+1}_{i,j+1}-v^{n+1}_{i,j-1}+v^n_{i,j+1}-v^n_{i,j-1}}{4\Delta z}\\
&+\frac{k}{2\Delta x^2}(v^{n+1}_{i+1,j}-2v^{n+1}_{i,j}+v^{n+1}_{i-1,j}+v^n_{i+1,j}-2v^n_{i,j}+v^n_{i-1,j})\\
&+\frac{k\left(\frac{\partial z}{\partial x}\right)^{n+1/2}_{i+1,j}}{8\Delta x\Delta z}(v^{n+1}_{i+1,j+1}-v^{n+1}_{i+1,j-1}+v^n_{i+1,j+1}-v^n_{i+1,j-1})\\
&-\frac{k\left(\frac{\partial z}{\partial x}\right)^{n+1/2}_{i-1,j}}{8\Delta x\Delta z}(v^{n+1}_{i-1,j+1}-v^{n+1}_{i-1,j+1}+v^n_{i-1,j+1}-v^n_{i-1,j-1})\\
&+\frac{k\left(\frac{\partial z}{\partial x}\right)^{n+1/2}_{i,j}}{8\Delta x\Delta z}(v^{n+1}_{i+1,j+1}-v^{n+1}_{i-1,j+1}-v^{n+1}_{i+1,j-1}+v^{n+1}_{i-1,j-1}\\
&+v^n_{i+1,j+1}-v^n_{i-1,j+1}-v^n_{i+1,j-1}+v^n_{i-1,j-1})\\
&+\frac{k\left(\frac{\partial z}{\partial x}\right)^{n+1/2}_{i,j}}{2\Delta z^2}\left[\left(\frac{\partial z}{\partial x}\right)^{n+1/2}_{i,j+1/2}(v^{n+1}_{i,j+1}-v^{n+1}_{i,j}+v^n_{i,j+1}-v^n_{i,j})\right.\\
&\left.-\left(\frac{\partial z}{\partial x}\right)^{n+1/2}_{i,j-1/2}(v^{n+1}_{i,j}-v^{n+1}_{i,j-1}+v^n_{i,j}-v^n_{i,j-1})\right]\\
&+\frac{k\left(\frac{\partial z}{\partial y}\right)^{n+1/2}_{i,j}}{2\Delta z^2}\left[\left(\frac{\partial z}{\partial y}\right)^{n+1/2}_{i,j+1/2}(v^{n+1}_{i,j+1}-v^{n+1}_{i,j}+v^n_{i,j+1}-v^n_{i,j})\right.\\
&\left.-\left(\frac{\partial z}{\partial y}\right)^{n+1/2}_{i,j-1/2}(v^{n+1}_{i,j}-v^{n+1}_{i,j-1}+v^n_{i,j}-v^n_{i,j-1})\right],
\end{aligned}$$

$$i=1,\ 2,\ \cdots,\ M;\quad j-L^*=1,\ 2,\ \cdots,\ L-1. \tag{3.19}$$

(3.19) stands for the approximate formula to (3.11) if $C=C_1$, $k=k_1$, $L^*=0$ or to

(3.12) if $C=C_2$, $k=k_2$, $L^*=L$.

Equation (3.14) can be approximated by the following formula

$$\left(\frac{\partial f}{\partial t}\right)_i^{n+1}=\frac{1}{\lambda}\left\{\left[\left(\frac{\partial f}{\partial x}\right)_i^{n+1}\right]^2+1\right\}\left(\frac{k_1}{f_i^{n+1}}\frac{v_{i,L-2}^{n+1}-4v_{i,L-1}^{n+1}+3v_{i,L}^{n+1}}{2\Delta z}\right.$$
$$\left.-\frac{k_2}{H-f_i^{n+1}}\frac{-v_{i,L+2}^{n+1}+4v_{i,L+1}^{n+1}-3v_{i,L}^{n+1}}{2\Delta z}\right), \tag{3.20}$$

and (3.16) and (3.17) by the formulae

$$v_{M+1,j}^{n+1}=v_{M,j}^{n+1}, \quad j=0,\ 1,\ \cdots,\ 2L; \tag{3.21}$$

$$f_{M+1}^{n+1}=f_M^{n+1}. \tag{3.22}$$

It is easily known that if $v_{i,j}^n$, $f_i^{n+1/2}$, $f_{x,i}^{n+1/2}$, $f_{t,i}^{n+1/2}$ are given, $v_{i,j}^{n+1}$ can be determined by (2.10), (2.11), (3.13), (3.15), (3.19) and (3.21). Furthermore if $f_i^{n+1}$, $f_{x,i}^{n+1}$ are also known, $f_{t,i}^{n+1}$ can also be determined by (3.20). It is well known that if $f_i^{n+1/2}$, $f_{x,i}^{n+1/2}$, $f_{t,i}^{n+1/2}$, $f_i^{n+1}$ and $f_{x,i}^{n+1}$ possess the accuracy $O(\Delta x^2+\Delta z^2+\Delta t^2)$, the above scheme has a second order accuracy. When $f_i^n$ and $f_{t,i}^{n+1/2}$ are known, $f_i^{n+1}$ can be determined by

$$f_i^{n+1}=f_i^n+\Delta t f_{t,i}^{n+1/2}, \quad i=1,\ 2,\ \cdots,\ M, \tag{3.23}$$

$f_0^{n+1}$ by (2.9) and $f_{M+1}^{n+1}$ by (3.22). And $f_{x,i}^{n+1}$ can be determined by

$$f_{x,i}^{n+1}=\frac{f_{i+1}^{n+1}-f_{i-1}^{n+1}}{2\Delta x}, \quad i=1,\ 2,\ \cdots,\ M. \tag{3.24}$$

(In many cases because $f_{t,0}^{n+1}$ might be obtained from (2.9), $f_{x,0}^{n+1}$ can also be determined by (3.20).) Therefore if $v_{i,j}^n$, $f_i^n$, $f_i^{n+1/2}$, $f_{x,i}^{n+1/2}$ and $f_{t,i}^{n+1/2}$ are given, $v_{i,j}^{n+1}$, $f_i^{n+1}$, $f_{x,i}^{n+1}$ and $f_{t,i}^{n+1}$ with second-order accuracy can also be obtained. But usually only $v_{i,j}^n$, $f_i^n$, $f_{x,i}^n$ and $f_{t,i}^n$ are known. For this reason $f_i^{n+1/2}$, $f_{x,i}^{n+1/2}$, $f_{t,i}^{n+1/2}$ must be determined first from $v_{i,j}^n$, $f_i^n$, $f_{x,i}^n$, $f_{t,i}^n$. Therefore we change the superscripts $n+1$ in all the above formulae into $n+\frac{1}{2}$, change $n+\frac{1}{2}$ into $n$ and keep $n$ unchanged. Then $v_{i,j}^{n+1/2}$, $f_i^{n+1/2}$, $f_{x,i}^{n+1/2}$, $f_{t,i}^{n+1/2}$ can be determined by $v_{i,j}^n$, $f_i^n$, $f_{x,i}^n$, $f_{t,i}^n$, and the errors are $O(\Delta x^2+\Delta z^2+\Delta t^2)$. It is seen that by adding an auxiliary level the results with second-order accuracy can be obtained.

If (3.19) is regarded as a three-level formula in which the step is $\frac{1}{2}\Delta t$, the results with the same accuracy can also be obtained without adding the auxiliary level, but the values on the two levels must be reserved in the process of computation. And at the very beginning of the computation the initial values on the two time levels must be known. However, the amounts of computation of the above two methods are equal.

## IV. Computed Results

A concrete example is computed by using the above method. The problem is as follows:

The computational region is a rectangular region, where some water is freezing gradually. The thermal capacities and the coefficients of heat passage for ice and water are $C_1=500$ Cal./M$^3\cdot$°C, $C_2=730$ Cal./M$^3\cdot$°C; $k_1=3.38$ Cal./M$\cdot$h$\cdot$°C, $k_2=1.85$ Cal./M$\cdot$h$\cdot$°C respectively; the freezing temperature $u_f=0$°C and the latent heat of phase-change is $\lambda=34000$ Cal./M$^3$. The initial value conditions are

$$f(x, 0)=0.011g(1+50000x), \quad 0\leqslant x\leqslant 2, \tag{4.1}$$

$$u(x, y, 0)=\begin{cases}(x^2-4x)(f(x, 0)-y)/f(x, 0), & 0\leqslant y\leqslant f(x, 0),\ 0\leqslant x\leqslant 2, \\ 0, & f(x, 0)\leqslant y\leqslant H,\ 0\leqslant x\leqslant 2.\end{cases} \tag{4.2}$$

Here, $H$ is a positive number which is large enough. The boundary conditions are

$$u(0, y, t)=0, \quad 0\leqslant y\leqslant H,\ t>0, \tag{4.3}$$

$$\frac{\partial u(2, y, t)}{\partial x}=0, \quad 0\leqslant y\leqslant H,\ t>0, \tag{4.4}$$

$$u(x, 0, t)=(x^2-4x)\left[1+\sin\left(\frac{\pi t}{4380}\right)\right], \quad 0<x<2,\ t>0, \tag{4.5}$$

$$u(x, H, t)=0, \quad 0<x<2,\ t>0. \tag{4.6}$$

The problem is to find how the water freezes.

Under the coordinate system $(x, z, t)$, (4.2)—(4.6) can be written as

$$v(x, z, 0)=\begin{cases}(x^2-4x)(1-z), & 0\leqslant z<1,\ 0\leqslant x\leqslant 2, \\ 0, & 1\leqslant z\leqslant 2,\ 0\leqslant x\leqslant 2,\end{cases} \tag{4.7}$$

$$v(0, z, t)=0, \quad 0\leqslant z\leqslant 2,\ t>0, \tag{4.8}$$

$$\left(\frac{\partial v}{\partial x}+\frac{\partial z}{\partial x}\frac{\partial v}{\partial z}\right)\bigg|_{(2,z,t)}=0, \quad 0\leqslant z\leqslant 2,\ t>0, \tag{4.9}$$

$$v(x, 0, t)=(x^2-4x)\left[1+\sin\left(\frac{\pi t}{4380}\right)\right], \quad 0<x<2,\ t>0, \tag{4.10}$$

$$v(x, 2, t)=0, \quad 0<x<2,\ t>0. \tag{4.11}$$

Under the above conditions the temperature in the region $\Omega_2(t)$ is always 0°C; therefore the only thing we must do is to treat the region $\Omega_1(t)$.

This problem is computed on an 013-Computer in the Computing Center, Academia Sinica, by using the following four different groups of steps:

(1) $\Delta x=\frac{4}{21}$ m, $\Delta z=\frac{1}{10}$, $\Delta t=2$h,

(2) $\Delta x=\frac{4}{21}$ m, $\Delta z=\frac{1}{5}$, $\Delta t=2$h,

(3) $\Delta x=\frac{4}{21}$ m, $\Delta z=\frac{1}{10}$, $\Delta t=4$h,

(4) $\Delta x=\frac{4}{21}$ m, $\Delta z=\frac{1}{5}$, $\Delta t=4$h.

It is clear that the result for the first group of steps is the most accurate. The phase-change boundary obtained by using the first group of steps is given in Table 1 and Fig. 3.

The comparison between the result in the first case and those in the other three cases is also given in Tables 2 and 3. It is known from Table 2 that for $t\leqslant 100$ days the difference between the results in the case $\Delta z=\frac{1}{5}$ and in the case $\Delta z=\frac{1}{10}$ is very small. With the increase of $t$, the difference raises, but is still not large. For example, the errors at $t=1100$ days in these cases are only about 0.07, that is, the fractional errors are about 2%. The CPU times on 013-Computer in these cases are given in Table 4. It is known from the table that if the data corresponding to the first 100 days (even 300 days) only are needed, adopting the second group of steps $\left(\Delta x=\frac{4}{21},\ \Delta z=\frac{1}{5},\ \Delta t=2\right)$ is reasonable since we shall obtain quite accurate results and expend a little CPU time.

At $t=100$ days the distributions of the temperature in the $\Omega_1(t)$ are given in Table 5, Figs. 4, 5 and 6. The temperature in $\Omega_2(t)$ remains 0°C. These results are obtained by using the steps $\Delta x=\frac{4}{21}, \Delta z=\frac{1}{21}, \Delta t=2$. Eight isotherms whose temperatures are 0°C, −1°C, −2°C, −3°C, −4°C, −5°C, −6°C and −7°C are drawn in Fig. 4. Fig. 5 gives the variation of temperature $v(x, z, t)$ with variable $x$ at $t=100$ days on the lines $z=0, 0.2, 0.4, 0.6, 0.8$. Fig. 6 shows the variation of temperature $v(x, z, t)$ with variable $z$ at $t=100$ days on $x=j\Delta x$ $(j=2, 4, 6, 8, 10)$.

The numerical result of the above problem shows that the singularity-separating method is very efficient for solution of two dimensional Stefan problems. This method can save the CPU time because the fairly accurate solution can be obtained by using a coarse space mesh and quite a large time step. Therefore the singularity-separating method is a satisfactory numerical method for the Stefan problem.

**Table 1 The variation of the phase-change boundary (unit: m)**

| $t$(days) | $x$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $\Delta x$ | $2\Delta x$ | $3\Delta x$ | $4\Delta x$ | $5\Delta x$ | $6\Delta x$ | $7\Delta x$ | $8\Delta x$ | $9\Delta x$ | $10\Delta x$ |
| | $f$ | | | | | | | | | |
| 0 | 0.0398 | 0.0428 | 0.0446 | 0.0458 | 0.0468 | 0.0476 | 0.0482 | 0.0488 | 0.0493 | 0.0498 |
| 10 | 0.2107 | 0.2758 | 0.3217 | 0.3574 | 0.3848 | 0.4062 | 0.4225 | 0.4342 | 0.4419 | 0.4460 |
| 20 | 0.3238 | 0.4111 | 0.4735 | 0.5227 | 0.5610 | 0.5908 | 0.6135 | 0.6300 | 0.6409 | 0.6467 |
| 30 | 0.4282 | 0.5299 | 0.6030 | 0.6616 | 0.7073 | 0.7431 | 0.7704 | 0.7903 | 0.8035 | 0.8107 |
| 40 | 0.5285 | 0.6406 | 0.7212 | 0.7869 | 0.8381 | 0.8784 | 0.9093 | 0.9318 | 0.9467 | 0.9550 |
| 50 | 0.6253 | 0.7455 | 0.8317 | 0.9030 | 0.9582 | 1.0022 | 1.0357 | 1.0603 | 1.0766 | 1.0857 |
| 100 | 1.0474 | 1.1905 | 1.2906 | 1.3776 | 1.4433 | 1.4972 | 1.5380 | 1.5682 | 1.5884 | 1.5998 |
| 200 | 1.4927 | 1.6506 | 1.7577 | 1.8547 | 1.9249 | 1.9852 | 2.0295 | 2.0631 | 2.0852 | 2.0978 |
| 300 | 1.5442 | 1.7038 | 1.8116 | 1.9098 | 1.9804 | 2.0415 | 2.0862 | 2.1201 | 2.1424 | 2.1551 |
| 400 | 1.7056 | 1.8676 | 1.9759 | 2.0756 | 2.1462 | 2.2085 | 2.2533 | 2.2877 | 2.3102 | 2.3230 |
| 500 | 2.0393 | 2.2071 | 2.3172 | 2.4204 | 2.4912 | 2.5564 | 2.6014 | 2.6373 | 2.6602 | 2.6734 |
| 600 | 2.1921 | 2.3629 | 2.4741 | 2.5791 | 2.6501 | 2.7168 | 2.7620 | 2.7987 | 2.8219 | 2.8352 |
| 700 | 2.2094 | 2.3803 | 2.4915 | 2.5965 | 2.6676 | 2.7343 | 2.7795 | 2.8162 | 2.8394 | 2.8527 |
| 800 | 2.3650 | 2.5370 | 2.6480 | 2.7535 | 2.8239 | 2.8913 | 2.9361 | 2.9732 | 2.9963 | 3.0097 |
| 900 | 2.5650 | 2.7394 | 2.8512 | 2.9578 | 3.0281 | 3.0966 | 3.1411 | 3.1790 | 3.2021 | 3.2157 |
| 1000 | 2.6205 | 2.7961 | 2.9083 | 3.0156 | 3.0860 | 3.1550 | 3.1995 | 3.2378 | 3.2609 | 3.2746 |
| 1100 | 2.6511 | 2.8264 | 2.9384 | 3.0455 | 3.1156 | 3.1846 | 3.2288 | 3.2672 | 3.2902 | 3.3039 |

**Table 2 The difference among the locations of the phase-change boundary obtained by using different steps**

| steps | t (days) | | | | | |
|---|---|---|---|---|---|---|
| | 10 | 50 | 100 | 300 | 500 | 1100 |
| | difference (m) | | | | | |
| $\Delta x=\frac{4}{21}, \Delta z=\frac{1}{10}, \Delta t=2\text{h}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Delta x=\frac{4}{21}, \Delta z=\frac{1}{5}, \Delta t=2\text{h}$ | 0.0003 | 0.0027 | 0.0073 | 0.0162 | 0.0288 | 0.0475 |
| $\Delta x=\frac{4}{21}, \Delta z=\frac{1}{10}, \Delta t=4\text{h}$ | 0.0071 | 0.0137 | 0.0145 | 0.0131 | 0.0118 | 0.0119 |
| $\Delta x=\frac{4}{21}, \Delta z=\frac{1}{5}, \Delta t=4\text{h}$ | 0.0071 | 0.0145 | 0.0177 | 0.0279 | 0.0461 | 0.0644 |

**Table 3 The difference among the temperature fields obtained by using different steps**

| steps | t (days) | | | | | |
|---|---|---|---|---|---|---|
| | 10 | 50 | 100 | 300 | 500 | 1100 |
| | difference (°C) | | | | | |
| $\Delta x=\frac{4}{21}, \Delta z=\frac{1}{10}, \Delta t=2\text{h}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\Delta x=\frac{4}{21}, \Delta z=\frac{1}{5}, \Delta t=2\text{h}$ | 0.00003 | 0.00135 | 0.00362 | 0.00084 | 0.01448 | 0.02116 |
| $\Delta x=\frac{4}{21}, \Delta z=\frac{1}{10}, \Delta t=4\text{h}$ | 0.00190 | 0.01097 | 0.01469 | 0.00068 | 0.00650 | 0.00406 |
| $\Delta x=\frac{4}{21}, \Delta z=\frac{1}{5}, \Delta t=4\text{h}$ | 0.00185 | 0.02032 | 0.01048 | 0.00045 | 0.00922 | 0.01710 |

**Table 4 The CPU times in different cases**

| time / steps / t (days) | $\Delta x=\frac{4}{21}$ m | | | |
|---|---|---|---|---|
| | $\Delta z=\frac{1}{10}$ $\Delta t=2\text{h}$ | $\Delta z=\frac{1}{5}$ $\Delta t=2\text{h}$ | $\Delta z=\frac{1}{10}$ $\Delta t=4\text{h}$ | $\Delta z=\frac{1}{5}$ $\Delta t=4\text{h}$ |
| 10 | 1m. 38s | 17s | 39s | 7s |
| 1100 | 2h. 59m | 31m. 25s | 1h. 11m. 15s | 12m. 20s |

**Table 5 The distribution of temperature in $\Omega_1(t)$ at $t=100$ days**

| $z$ | $x$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | $\Delta x$ | $2\Delta x$ | $3\Delta x$ | $4\Delta x$ | $5\Delta x$ | $6\Delta x$ | $7\Delta x$ | $8\Delta x$ | $9\Delta x$ | $10\Delta x$ |
| | $v$ | | | | | | | | | | |
| 0 | 0 | −1.44 | −2.74 | −3.90 | −4.91 | −5.77 | −6.49 | −7.07 | −7.50 | −7.79 | −7.94 |
| $\Delta z$ | 0 | −1.25 | −2.38 | −3.37 | −4.24 | −4.98 | −5.60 | −6.09 | −6.46 | −6.71 | −6.83 |
| $2\Delta z$ | 0 | −1.09 | −2.06 | −2.90 | −3.63 | −4.26 | −4.78 | −5.19 | −5.51 | −5.72 | −5.82 |
| $3\Delta z$ | 0 | −0.95 | −1.77 | −2.47 | −3.08 | −3.60 | −4.03 | −4.38 | −4.63 | −4.81 | −4.89 |
| $4\Delta z$ | 0 | −0.82 | −1.50 | −2.08 | −2.58 | −2.99 | −3.34 | −3.62 | −3.82 | −3.97 | −4.04 |
| $5\Delta z$ | 0 | −0.70 | −1.25 | −1.71 | −2.10 | −2.43 | −2.70 | −2.92 | −3.08 | −3.19 | −3.25 |
| $6\Delta z$ | 0 | −0.58 | −1.01 | −1.36 | −1.65 | −1.90 | −2.10 | −2.26 | −2.39 | −2.47 | −2.51 |
| $7\Delta z$ | 0 | −0.46 | −0.77 | −1.01 | −1.22 | −1.39 | −1.54 | −1.65 | −1.74 | −1.79 | −1.83 |
| $8\Delta z$ | 0 | −0.33 | −0.52 | −0.68 | −0.80 | −0.91 | −1.00 | −1.07 | −1.13 | −1.16 | −1.18 |
| $9\Delta z$ | 0 | −0.18 | −0.27 | −0.34 | −0.40 | −0.45 | −0.49 | −0.52 | −0.55 | −0.57 | −0.58 |
| $10\Delta z$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

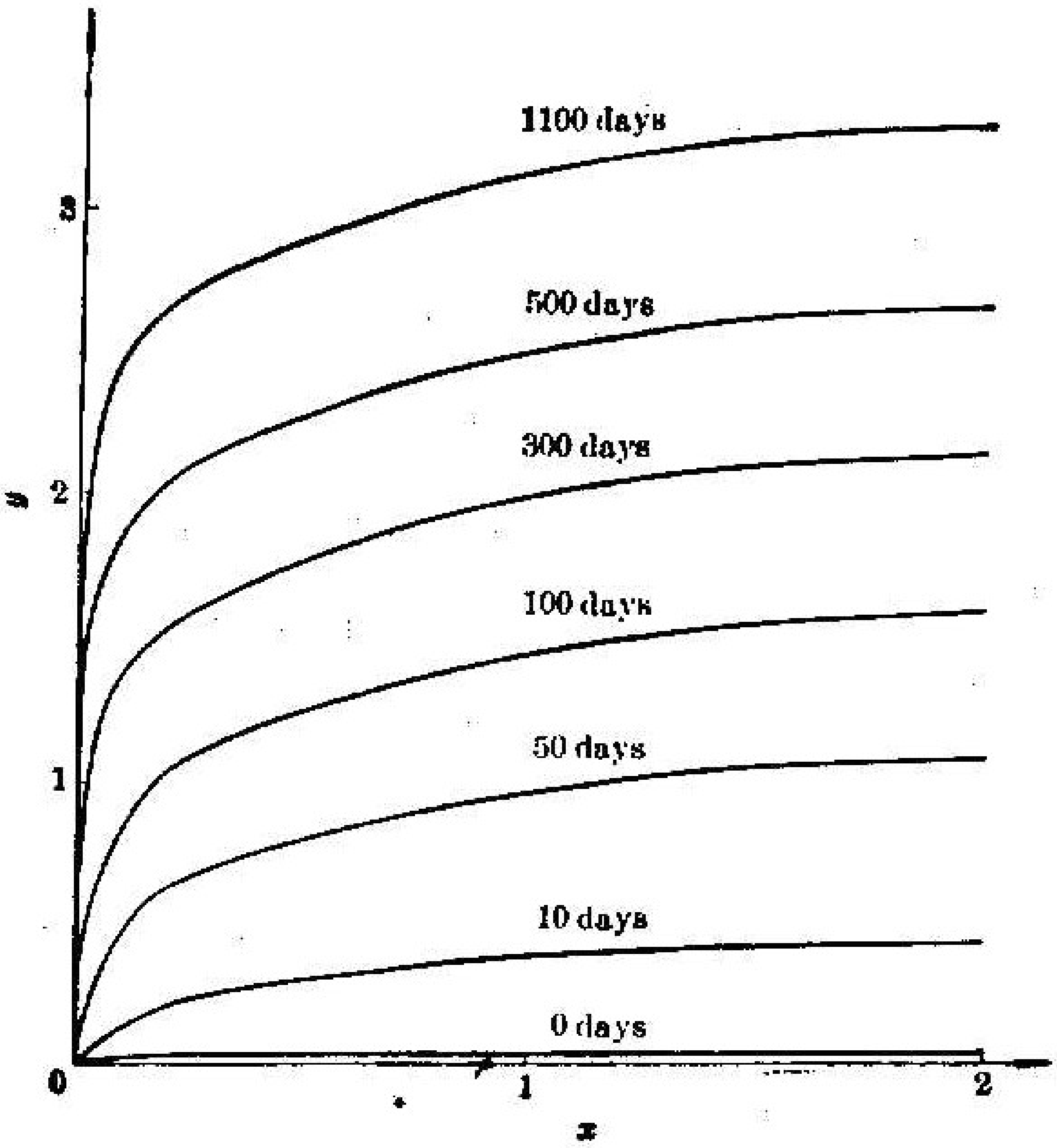


Fig. 3 The variation of phase-change boundary

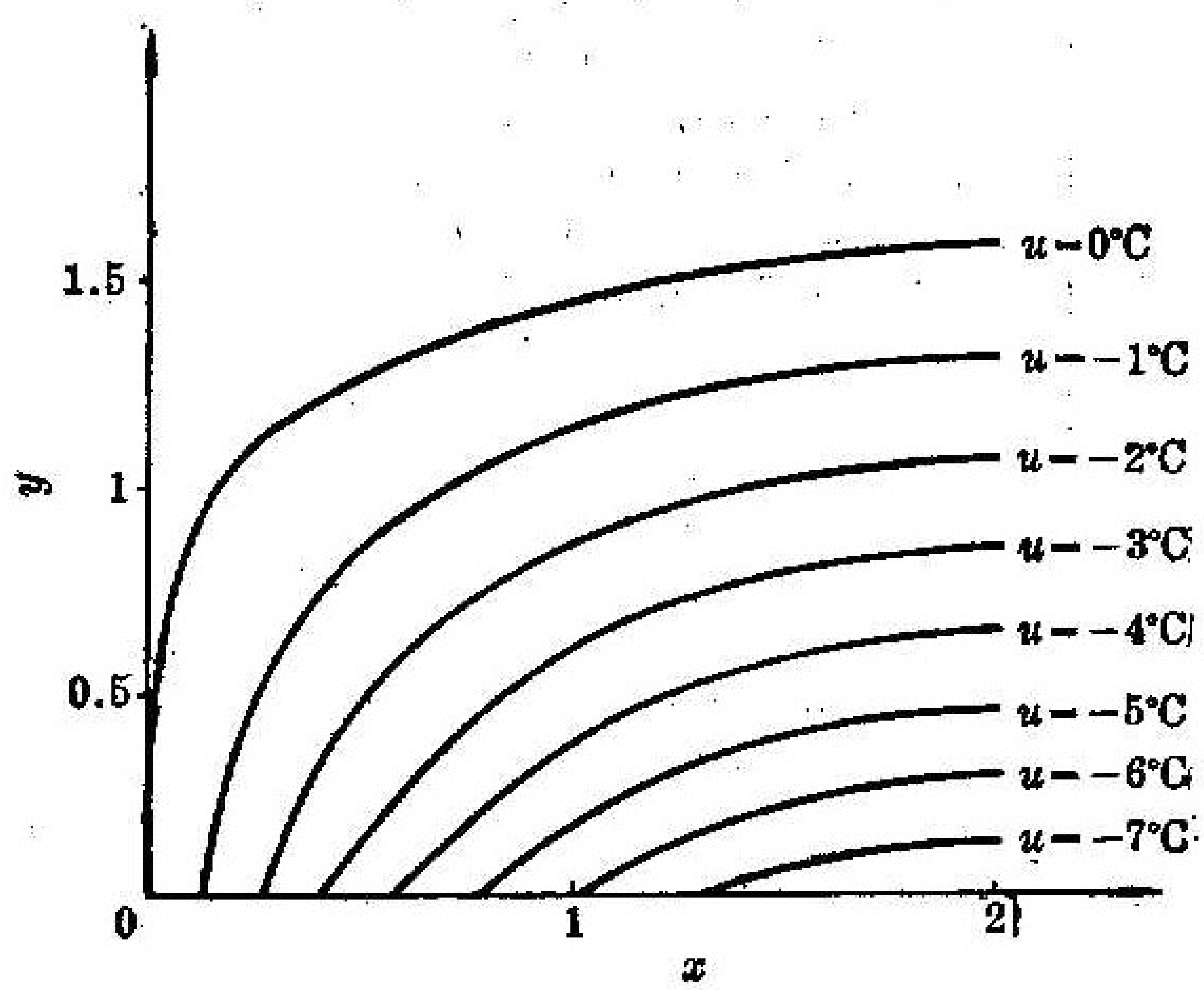


Fig. 4 The isotherms in $\Omega_1(t)$ at $t=100$ days

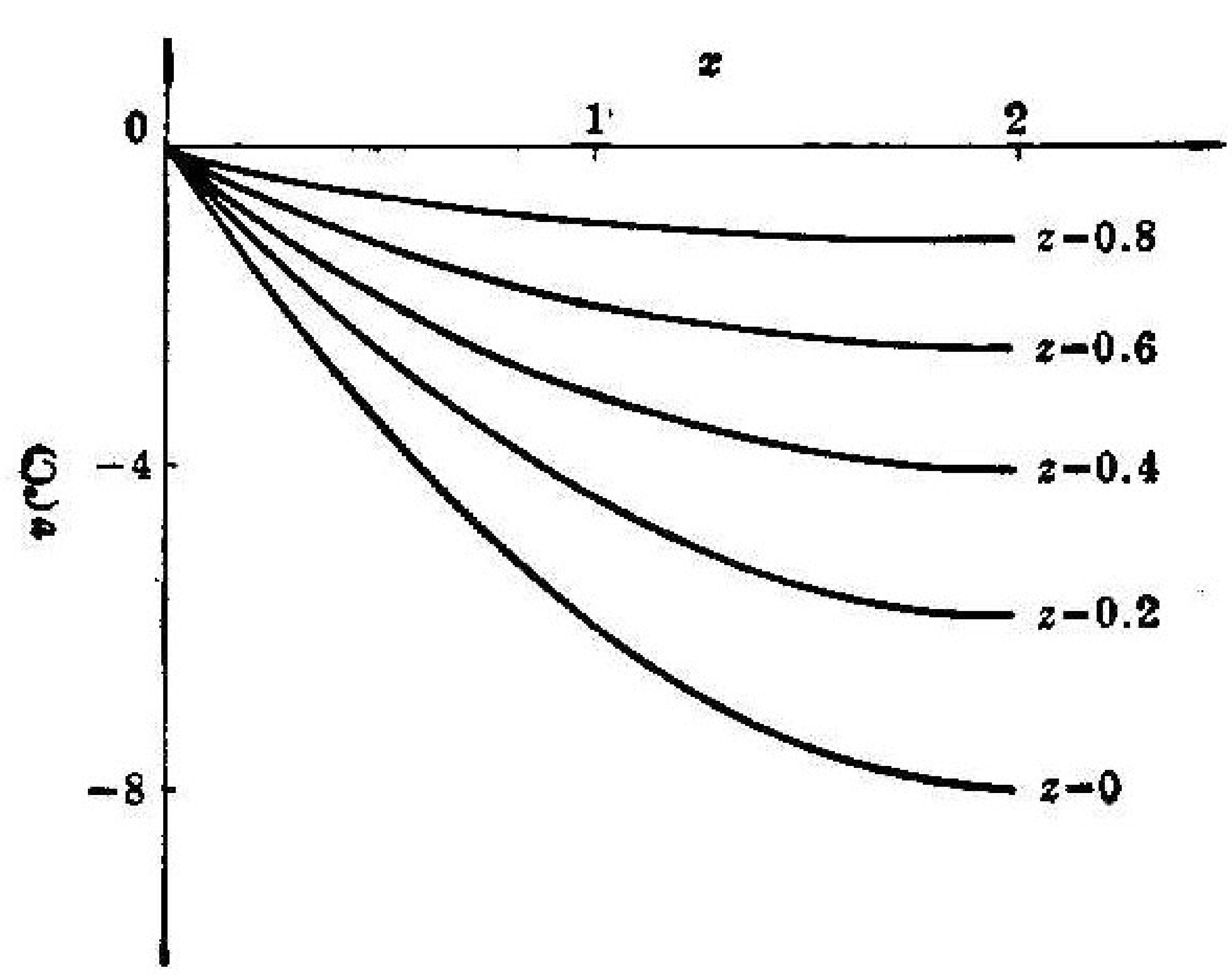


Fig. 5 The variation of the temperature with the variable $x$ at $t=100$ days

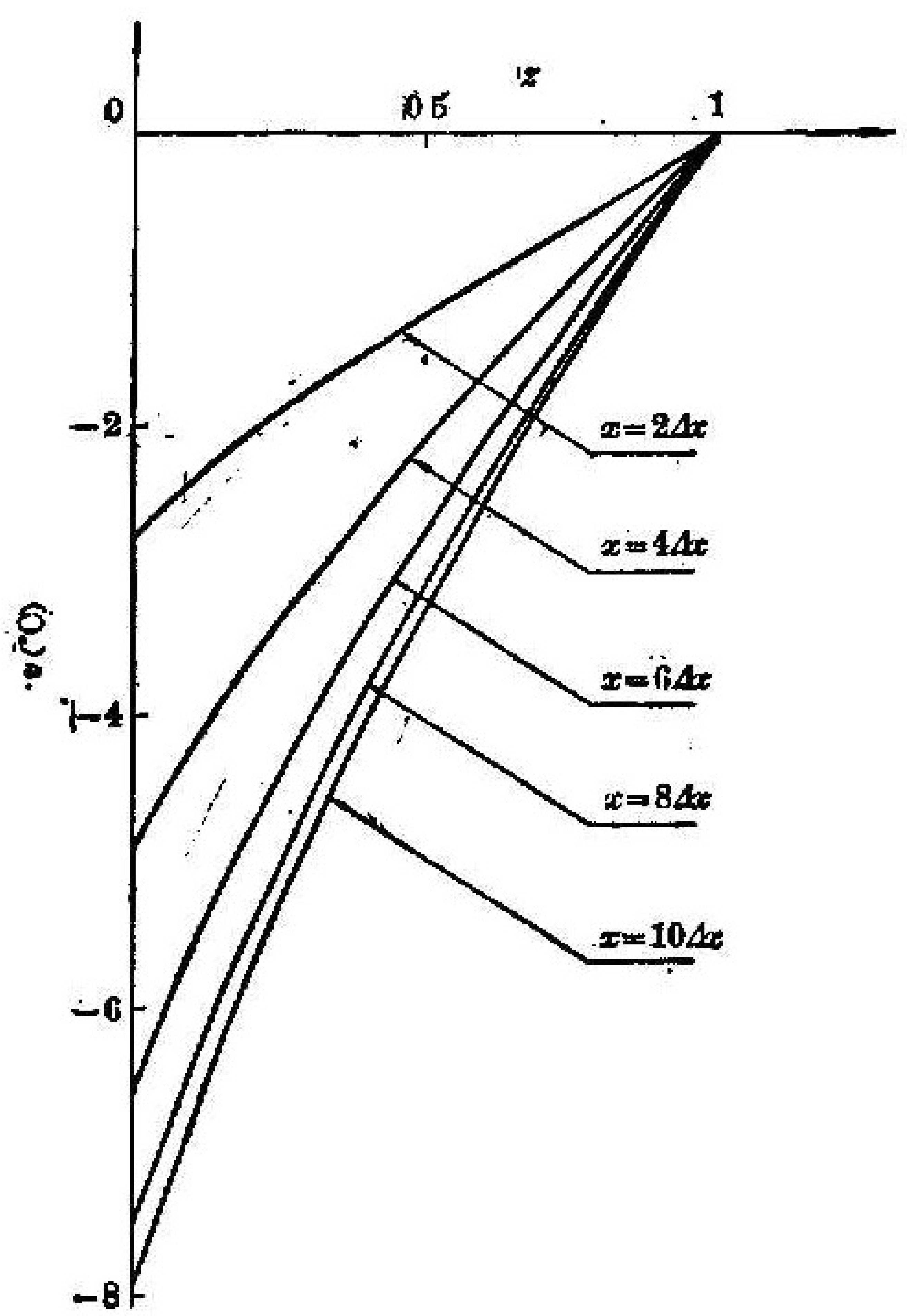


Fig. 6 The variation of the temperature with the variable $z$ at $t=100$ days

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