$W^{1,\infty}$-INTERIOR ESTIMATES FOR FINITE ELEMENT METHOD ON REGULAR MESH*

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Abstract

For a large class of piecewise polynomial subspaces S^h defined on the regular mesh, $W^{1,\infty}$-interior estimate $\|u_h\|_{1,\infty,\Omega_0}\leqslant c\|u_h\|_{-s,\Omega_1}$, $u_h\in S^h(\Omega_1)$ satisfying the interior Ritz equation $B(u_h,\varphi)=0$, $\forall\varphi\in\mathring{S}^h(\Omega_1)$, $\Omega_0\subset\subset\Omega_1\subset\subset\Omega$, is proved. For the finite element approximation u_h (of degree $r-1$) to u, we have $W^{1,\infty}$-interior error estimate $\|u-u_h\|_{1,\infty,\Omega_0}\leqslant ch^{r-1}(\|u\|_{r,\infty,\Omega_1}+\|u\|_{1,\Omega})$. If the triangulation is strongly regular in Ω_1 and $r=2$ we obtain $W^{1,\infty}$-interior superconvergence

$$\max_{x\in X}|D(u-\bar{u}_h)(x)|\leqslant ch^2(|\ln h|\,\|u\|_{3,\infty,\Omega_1}+\|u\|_{2,\Omega}).$$

§ 1. Introduction

Let Ω be an n-dimensional bounded domain with the boundary $\partial\Omega$. Denote the norm and semi-norm of the Sobolev space $W^{k,p}(\Omega)$, $1\leqslant p\leqslant\infty$, respectively, by

$$\|u\|_{k,p,\Omega}=\sum_{|\alpha|\leqslant k}\|D^\alpha u\|_{L^p(\Omega)},\quad |u|_{k,p,\Omega}=\sum_{|\alpha|=k}\|D^\alpha u\|_{L^p(\Omega)}.$$

We simply write $W^{k,2}=H^k$, $\|u\|_{k,2,\Omega}=\|u\|_{k,\Omega}$ if $p=2$.

We consider the elliptic boundary value problem

$$\begin{cases} Lu=-D_j(a_{ij}D_iu+a_{0j}u)+a_{i0}D_iu+a_{00}u=f, & \text{in } \Omega\\ u=0, & \text{on } \partial\Omega\end{cases}\tag{1.1}$$

and a bilinear form

$$B(u,v)=\int_\Omega\sum_{i,j=0}^n a_{ij}D_iuD_jv\,dx,\quad D_0u=u,$$

where the coefficients a_{ij} are suitably smooth in $\bar{\Omega}$. Suppose that

$$B(v,v)\geqslant c\|v\|^2_{1,\Omega},\quad c>0,\ \forall v\in\mathring{H}^1(\Omega).\tag{1.2}$$

On a regular (i. e. quasi-uniform) mesh-domain Ω_h of Ω we give a finite dimensional subspace $S^h\subset C(\bar{\Omega})$, consisting of piecewise polynomials of degree $r-1$, and

$$\mathring{S}^h(\Omega_1)=\{\varphi\in S^h(\Omega)\,|\,\text{supp}\,\varphi\subseteq\bar{\Omega}_1\},\quad \Omega_1\subset\subset\Omega.$$

An approximate solution $u_h\in S^h(\Omega)$ to u satisfies the interior Ritz equation

$$B(u-u_h,\varphi)=0,\quad \forall\varphi\in\mathring{S}^h(\Omega_1).\tag{1.3}$$

An important special case occurs when $Lu=0$. Then $u_h\in S^h(\Omega)$ satisfies[1]

$$B(u_h,\varphi)=0,\quad \forall\varphi\in\mathring{S}^h(\Omega_1).\tag{1.4}$$

Such u_h will play a central role in deriving the interior error estimates. For the regular mesh in Ω_1, J. Nitsche and A. Schatz[1] first proved L^2-interior estimate

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$$\|u_h\|_{1,\Omega_0}\leqslant c\|u_h\|_{-s,\Omega_1},\quad \Omega_0\subset\subset\Omega_1, \tag{1.5}$$

where $s\geqslant 0$ is an integer, arbitrary but fixed, and $\|u_h\|_{-s,\Omega}$ negative norm. For the uniform mesh, J. Bramble, J. Nitsche and A. Schatz[2] later proved L^∞-interior estimate

$$\|u_h\|_{0,\infty,\Omega_0}\leqslant c\|u_h\|_{-s,\Omega_1}. \tag{1.6}$$

For the regular mesh, A. Schatz and L. Wahlbin[8] also proved it by the technique of estimating derivatives on annuluses. The present paper extends these results and proves the following

Fundamental Lemma. *Suppose that the triangulation is regular in $\Omega_1\subset\subset\Omega$, and $u_h\in S^h$ satisfies* (1.3). *Then*

$$\|u_h\|_{1,\infty,\Omega_0}\leqslant c\|u_h\|_{-s,\Omega_1}. \tag{1.7}$$

Using the lemma, we may derive $W^{1,\infty}$-interior error estimate (Theorem 1) and $W^{1,\infty}$-interior superconvergence (Theorem 2) for the general problem (1.1).

§ 2. Some Assumptions

[7] and [8] discussed a priori estimate and the solvability of solution $u\in W^{2,p}(\Omega)$, $1<p<\infty$, for the problem (1.1). We obtained

Lemma 1[9]. *Let $\Omega\in C^{1,1}$, $a_{ij}\in W^{1,\infty}(\Omega)$, $i+j\neq 0$, $a_{00}\in L^\infty(\Omega)$, $f\in L^p(\Omega)$, $1<p<\infty$, and $u\in W^{2,p}(\Omega)$ is a unique solution of* (1.1). *Then*

$$\|u\|_{2,p,\Omega}\leqslant c\tilde{p}^\lambda\|f\|_{0,p,\Omega}, \tag{2.1}$$

where $\tilde{p}=\max(p, p')$, $p'=p/(p-1)$, and the constants λ and c are independent of p and f.

Let $\Omega=G$ be a sphere with radius R suitably small. Suppose that the Green function $g(x, y)$ for (1.1) exists such that

$$|D^\alpha g(x, y)|\leqslant\begin{cases} c(|\ln|x-y||+1), & n=2 \text{ and } \alpha=0,\\ c|x-y|^{2-n-|\alpha|}, & n>2 \text{ or } |\alpha|=1.\end{cases} \tag{2.2}$$

By the Green function $g(x, y)$, the solution u of (1.1) can be expressed by

$$u(x)=\int_G g(x, y)f(y)dy. \tag{2.3}$$

If $1\leqslant q<n/(n-1)$, we have

$$|u|_{1,q,G}\leqslant c\left(\int_G\int_G|x-y|^{(1-n)q}|f(y)|dx\,dy\right)^{1/q}\left(\int_G|f(y)|dy\right)^{1/q'}\leqslant c\|f\|_{0,1,G}. \tag{2.4}$$

We now turn to the finite dimensional subspace $S^h(\Omega)$[1] and make the following assumptions (for $1\leqslant p\leqslant\infty$):

A1. For each $u\in W^{1,p}(G)$, $1\leqslant t\leqslant r$, there exists a $\varphi\in S^h(\Omega_1)$ such that

$$\|u-\varphi\|_{s,p,G}\leqslant ch^{t-s}\|u\|_{t,p,G},\quad s=0,\ 1. \tag{2.5}$$

A2. Let $\omega\in C_0^\infty(G_0)$ and $u_h\in S^h(G)$, $G_0\subset\subset G\subset\subset\Omega$. Then there exists $\varphi\in \mathring{S}^h(G)$ such that

$$\|\omega u_h-\varphi\|_{1,p,G}\leqslant ch\|u_h\|_{1,p,G}. \tag{2.6}$$

A3. For each $h\in(0, 1]$, there exists a mesh-domain G_1, $G_0\subset\subset G_1\subset\subset G$, such that, for all $\varphi\in S^h(\Omega_1)$,

$$\|\varphi\|_{s,p,G_1}\leqslant ch^{t-s}\|\varphi\|_{t,p,G_1},\quad 0\leqslant t\leqslant s\leqslant r, \tag{2.7}$$

$$\|\varphi\|_{0,p,G_1}\leqslant ch^{n\left(\frac{1}{p}-\frac{1}{q}\right)}\|\varphi\|_{0,q,G_1},\quad 1\leqslant q\leqslant p\leqslant\infty. \tag{2.8}$$

In particular, taking $p=\infty$, $q=|\ln h|$ (then $h^{-n/q}=e^n$) we have

$$\|\varphi\|_{0,\infty,G_1}\leqslant c\|\varphi\|_{0,q,G_1}. \tag{2.9}$$

Much has been done relating to L^∞-convergence of the Ritz projection $u_h\in \mathring{S}^h(\Omega)$. For the Poisson equation, for example, one has (cf. [5])

$$\|u-u_h\|_{s,\infty,\Omega}\leqslant ch^{t-s}|\ln h|^{\bar r}\|u\|_{t,\infty,\Omega},\quad s=0,\ 1,\quad 1\leqslant t\leqslant r \tag{2.10}$$

and the refined estimate[4]

$$\|u-u_h\|_{0,\infty,\Omega}\leqslant c|\ln h|^{\bar r}\inf_{\varphi}\|u-\varphi\|_{0,\infty,\Omega}, \tag{2.11}$$

where $$\bar r=\begin{cases}1, & r=2,\\ 0, & r>2.\end{cases}$$

Recently, R. Rannacher and R. Scott[6] discussed the difficult case $r=2$ on the convex polygonal domain Ω in the plane and successfully proved that the Ritz projection $Pu=u_h\in\mathring{S}^h(\Omega)$ is stable in $\mathring{W}^{1,p}(\Omega)$, $2\leqslant p\leqslant\infty$, namely,

$$\|Pu\|_{1,p,\Omega}\leqslant c\|u\|_{1,p,\Omega}; \tag{2.12}$$

then

$$\|u-Pu\|_{1,p,\Omega}\leqslant ch\|u\|_{2,p,\Omega},\quad 2\leqslant p\leqslant\infty, \tag{2.13}$$

$$\|u-Pu\|_{0,p,\Omega}\leqslant ch^2\|u\|_{2,p,\Omega},\quad 2\leqslant p<\infty. \tag{2.14}$$

They also pointed out that the results can be extended to the general elliptic operator L and the smooth domain $\Omega\in C^{1,1}$.

Using a duality argument, we can derive (2.12)—(2.14) for $1<p\leqslant 2$ if Ω is smooth. In particular, from (2.1) with $p'=|\ln h|$, we have

$$\|Pu\|_{1,p,\Omega}\leqslant c|\ln h|^{\lambda}\|u\|_{1,p,\Omega},$$
$$\|u-Pu\|_{0,p,\Omega}\leqslant ch|\ln h|^{\lambda}\|u\|_{1,p,\Omega}. \tag{2.15}$$

§ 3. Proof of Fundamental Lemma

In view of (1.5) we only prove the following

$$\|u_h\|_{1,\infty,G_0}\leqslant c\|u_h\|_{1,G}. \tag{3.1}$$

Let $G_0\subset\subset G_1\subset\subset G\subset\subset G'$ be concentric spheres and $\omega\in C_0^\infty(G_1)$ with $\omega=1$ on G_0, $\tilde u=\omega u_h$. Note that since $\operatorname{supp}\tilde u\subset G_1$ we can suitably change the mesh in $G\backslash G_1$ such that the boundary nodes of the mesh-domain G_h belong to ∂G. The change does not affect $\tilde u$, u_h and the proofs that follow. Therefore we can define the Ritz projection operator P in $\mathring{S}^h(G)$ (and conjugate P^*) such that

$$\|P\tilde u\|_{1,p,G}\leqslant c\|\tilde u\|_{1,p,G}.$$

We have

$$\begin{aligned}\|u_h\|_{1,p,G_0}&\leqslant\|\tilde u\|_{1,p,G_1}\leqslant c|\tilde u|_{1,p,G_1}\leqslant c|\tilde u-P\tilde u|_{1,p}+c|P\tilde u|_{1,p}\\ &\leqslant c\inf_{\varphi}\|\tilde u-\varphi\|_{1,p,G}+c|P\tilde u|_{1,p,G_1}\\ &\leqslant ch\|u_h\|_{1,p,G}+c|P\tilde u|_{1,p,G_1},\quad 2\leqslant p\leqslant\infty.\end{aligned} \tag{3.2}$$

To estimate $P\tilde u$ we construct a conjugate problem

$$L^*v=f, \text{ in } G, \quad v=0, \text{ on } \partial G,$$

and for $z=P\tilde{u}$, $\psi=\xi Dv$ and $\xi\in C_0^\infty(G)$ with $\xi=1$ on G_1 it is easy to calculate

$$\begin{aligned}-(D(\xi z),\ f)&=B(D(\xi z),\ v)=B(\xi z,\ Dv)+B'(\xi z,\ v)\\&=B(P\tilde{u},\ \psi)+(F(z,\ \xi),\ Dv)+B'(\xi z,\ v),\end{aligned} \tag{3.3}$$

and

$$B(P\tilde{u},\ \psi)=B(\tilde{u},\ P^*\psi)=B(u_h,\ \omega P^*\psi-\varphi)+(F(u_h,\ \omega),\ P^*\psi-\psi+\psi), \tag{3.4}$$

where
$$B'(z,\ v)=\int_\Omega\Big(\sum_{i+j\neq 0}Da_{ij}D_izD_jv-a_{00}D(zv)\Big)dx,$$

$$F(u,\ \xi)=D_j(a_{ij}uD_i\xi)+a_{ij}D_iuD_j\xi+(a_{0i}-a_{i0})uD_i\xi.$$

From (3.3), (3.4), (2.6), (2.12) and (2.13), for $1<p<\infty$ and $1<t<\infty$ arbitrary but fixed, we have

$$\begin{aligned}|(D(\xi z),\ f)|&\leqslant c\|u_h\|_{1,p}(\|\omega P^*\psi-\varphi\|_{1,p'}+\|P^*\psi-\psi\|_{0,p'})\\&\quad+c(\|u_h\|_{1,t}\|\psi\|_{0,t'}+\|P\tilde{u}\|_{1,t}\|v\|_{1,t'})\end{aligned} \tag{3.5}$$

$$\begin{aligned}&\leqslant ch\|u_h\|_{1,p}(\|P^*\psi\|_{1,p'}+\|\psi\|_{1,p'})+c\|u_h\|_{1,t}\|v\|_{1,t'}\\&\leqslant ch\|u_h\|_{1,p}\|v\|_{2,p'}+c\|u_h\|_{1,t}\|v\|_{1,t'}.\end{aligned} \tag{3.5a}$$

Taking $1<t<n$, $\frac{1}{p}=\frac{1}{t}-\frac{1}{n}$, and using

$$\|v\|_{1,t',G}\leqslant c\|v\|_{2,p',G}\leqslant c\|f\|_{0,p',G},$$

(3.5a), (3.2) and the inverse estimate, we have

$$\|P\tilde{u}\|_{1,p,G_1}\leqslant\|D(\xi z)\|_{0,p,G}\leqslant ch\|u_h\|_{1,p}+c\|u_h\|_{1,t}$$

and

$$\|u_h\|_{1,p,G_0}\leqslant ch\|u_h\|_{1,p}+c\|u_h\|_{1,t}\leqslant c\|u_h\|_{1,t,G}. \tag{3.6}$$

If $2k<n\leqslant 2k+2$, taking $t=2$, $\frac{1}{p_1}=\frac{1}{2}-\frac{1}{n}$ in (3.6) and iterating k times by (3.6) (with different G_0 and G) we can derive

$$\|u_h\|_{1,p_k,G_0}\leqslant c\|u_h\|_{1,G},\quad \frac{1}{p_k}=\frac{1}{2}-\frac{k}{n},\quad n\geqslant 2. \tag{3.7}$$

Therefore we consider two cases:

1) $n=2k+1$. Taking $t=p_k=2n$, $t'<n/(n-1)$ and $p=|\ln h|$, from (2.4), (2.1), (2.14), (2.15) and (3.5) we have

$$\|v\|_{1,t',G}\leqslant c\|f\|_{0,1,G}\leqslant c\|f\|_{0,p',G},$$

$$\|v\|_{2,p',G}\leqslant c|\ln h|^\lambda\|f\|_{0,p',G},$$

and

$$h|P^*\psi|_{1,p'}+\|P^*\psi-\psi\|_{0,p',G}\leqslant ch|\ln h|^\lambda\|\psi\|_{1,p'}\leqslant ch|\ln h|^{2\lambda}\|f\|_{0,p',G}.$$

Then

$$|P\tilde{u}|_{1,\infty,G_1}\leqslant c|P\tilde{u}|_{1,p,G_1}\leqslant c\|D(\xi z)\|_{0,p,G}\leqslant c(h|\ln h|^{2\lambda}\|u_h\|_{1,\infty,G}+\|u_h\|_{1,G}).$$

By (3.2), (3.7) and the inverse estimate, (3.1) for $n=2k+1$ is proved.

2) $n=2k+2$. Taking $t=p_k=n$, and using the imbedding theorem and a priori estimate, for $q>n$ arbitrary but fixed we have

$$\|v\|_{1,t',G}\leqslant c\|v\|_{2,q',G}\leqslant c\|f\|_{0,q',G}.$$

Taking $p=q$ in (3.5) and using (3.7) and the inverse estimate, then we have

$$\|P\tilde{u}\|_{1,q,G_0}\leqslant ch\|u_h\|_{1,q,G}+c\|u_h\|_{1,G},$$

and

$$\|u_h\|_{1,q,G_0}\leqslant c\|u_h\|_{1,G},\quad q>n. \tag{3.8}$$

Now, taking $p=|\ln h|$, $t=q>n$ in (3.5) and noticing

$$\|v\|_{1,q',G}\leqslant c\|f\|_{0,p'},\quad \|P^*\psi\|_{1,p'}\leqslant c|\ln h|^{2\lambda}\|f\|_{0,p',G},$$

we obtain

$$|P\tilde{u}|_{1,\infty,G_1}\leqslant ch|\ln h|^{2\lambda}\|u_h\|_{1,\infty,G}+c\|u_h\|_{1,G}.$$

Using (3.2), (3.8) and the inverse estimate, we obtain

$$\|u_h\|_{1,\infty,G_0}\leqslant ch|\ln h|^{2\lambda}\|u_h\|_{1,\infty,G}+c\|u_h\|_{1,G}\leqslant c\|u_h\|_{1,q,G}+c\|u_h\|_{1,G}\leqslant c\|u_h\|_{1,G'}.$$

Finally, by combining the above two cases, the fundamental lemma is proved.

§ 4. Some Applications

We now go back to the primary problem (1.1). Let Ω be suitably smooth, and there are negative norm estimates of $e=u-u_h$,

$$\|e\|_{-s,\Omega}\leqslant ch^{s+t}\|u\|_{t,\Omega},\quad 0\leqslant s\leqslant r-2,\ 1\leqslant t\leqslant r. \tag{4.1}$$

Let $G_0\subset\subset G_1\subset\subset G_2\subset\subset G$ be concentric spheres, $\omega\in C_0^\infty(G_2)$ with $\omega=1$ on G_1. Denote $\tilde{u}=\omega u$ (cf. § 3). The local Ritz projection $P\tilde{u}\in\mathring{S}^h(G)$ satisfies ($\tilde{e}=\tilde{u}-P\tilde{u}$)

$$B(\tilde{e},\varphi)=0,\quad \forall\varphi\in\mathring{S}^h(G) \tag{4.2}$$

and has a negative norm estimate

$$\|\tilde{e}\|_{-s,G}\leqslant ch^{s+t}\|\tilde{u}\|_{t,G},\quad 0\leqslant s\leqslant r-2,\ 1\leqslant t\leqslant r. \tag{4.3}$$

Noticing $\tilde{u}=u$ on G_1,

$$B(u_h-P\tilde{u},\varphi)=B(\tilde{e},\varphi)-B(e,\varphi)=0,\quad \forall\varphi\in\mathring{S}^h(G_1)$$

and using the fundamental lemma lead us to

$$\begin{aligned}\|u_h-P\tilde{u}\|_{1,\infty,G_0}&\leqslant c\|u_h-P\tilde{u}\|_{-s,G_1}\leqslant c(\|\tilde{e}\|_{-s,G}+\|e\|_{-s,G})\\&\leqslant c(h^{t+s}\|u\|_{t,G}+\|e\|_{-s,\Omega}).\end{aligned} \tag{4.4}$$

Below we derive two useful results.

1) $W^{1,\infty}$-interior estimate.

Theorem 1. *Let $\Omega_0\subset\subset\Omega_1\subset\subset\Omega$; then*

$$\|u-u_h\|_{1,\infty,\Omega_0}\leqslant ch^{r-1}(\|u\|_{r,\infty,\Omega_1}+\|u\|_{1,\Omega}). \tag{4.5}$$

Proof. From (4.2) and assumption (2.12) we have

$$\|u-P\tilde{u}\|_{1,\infty,G_0}\leqslant\|\tilde{u}-P\tilde{u}\|_{1,\infty,G}\leqslant ch^{r-1}\|\tilde{u}\|_{r,\infty,G}\leqslant ch^{r-1}\|u\|_{r,\infty,G}.$$

Taking $t=1$ in (4.4), then yields

$$\|u-u_h\|_{1,\infty,G_0}\leqslant\|\tilde{u}-P\tilde{u}\|_{1,\infty,G_0}+\|u_h-P\tilde{u}\|_{1,\infty,G_0}\leqslant ch^{r-1}(\|u\|_{r,\infty,G}+\|u\|_{1,\Omega}).$$

The subdomain Ω_0 can be covered by a finite number of G_0. The theorem is thus proved.

From (2.11) and the fundamental lemma one has

$$\|u-u_h\|_{0,\infty,\Omega_0}\leqslant ch^r(|\ln h|^{\bar{r}}\|u\|_{r,\infty,\Omega_1}+\|u\|_{2,\Omega}), \tag{4.6}$$

which was derived by A. Schatz and L. Wahlbin in 1977[3].

2) $W^{1,\infty}$-interior superconvergence.

For the sake of simplicity, we only consider the two-dimensional bounded domain Ω and the linear triangular element. Let the triangulation be strongly regular in $\Omega_1\subset\subset\Omega$[10], i.e. each quadrilateral consisting of two adjacent triangles τ_1 and τ_2 has a deviation $O(h^2)$ from some parallelogram. The middle points x on the common side of such τ_1 and τ_2 form a set M. Denote the mean value of the gradient of u_h at x by

$$D\bar{u}_h(x)=((Du_h)_{\tau_1}+(Du_h)_{\tau_2})/2.$$

If the mesh is uniform in the subdomain $\Omega_1\subset\subset\Omega$, from a well-known estimate[2] on the difference ratio $\partial_h^\alpha e$ one can derive an interior superconvergence

$$\max_{x\in X}|D(u-\bar{u}_h)(x)|\leqslant ch^2(\|u\|_{5,\Omega_1}+\|u\|_{2,\Omega}). \tag{4.7}$$

We now extend the result to the strongly regular mesh.

Theorem 2. *Let the triangulation be strongly regular in* Ω_1, $\Omega_0\subset\subset\Omega_1\subset\subset\Omega$, $X=M\cap\Omega_0$, $u_h\in\mathring{S}^h(\Omega)$ *be the linear finite element approximation to* u; *then*

$$\max_{x\in X}|D(u-\bar{u}_h)(x)|\leqslant ch^2(|\ln h|\,\|u\|_{3,\infty,\Omega_1}+\|u\|_{2,\Omega}). \tag{4.8}$$

Proof. Let w_I be a linear interpolation of $w=\tilde{u}$. We know that[10]

$$\max_{x\in X}|D(w-\bar{w}_I)(x)|\leqslant ch^2\|u\|_{3,\infty,G_1}, \tag{4.9}$$

and $(\zeta=Pw-w_I)$

$$B(\zeta,\varphi)=B(w-w_I,\varphi)\leqslant ch^2\|w\|_{3,\infty,G_1}\|\varphi\|_{1,1,G}. \tag{4.10}$$

Following [6] we construct

$$L^*g=D\delta_z,\quad g\in\mathring{H}^1(G),$$

where δ_z is a smooth δ-function and $(\varphi, D\delta_z)=D\varphi(z)$, $\forall\varphi\in S^h(G)$. Then

$$D\zeta(z)=(\zeta, D\delta_z)=B(\zeta,g)=B(\zeta,P^*g)\leqslant ch^2\|w\|_{3,\infty,G_1}\|P^*g\|_{1,1,G}. \tag{4.11}$$

We now prove

$$\|P^*g\|_{1,1,G}\leqslant c|\ln h|,\quad P^*g\equiv g_h. \tag{4.12}$$

In fact, using a weighted norm method[6] we have

$$\begin{aligned}\left(\int_G|Dg_h|\,dx\right)^2&=\int_G\sigma^{-2}\,dx\cdot\int_G\sigma^2|Dg_h|^2\,dx\\&\leqslant c|\ln h|\left(\int\sigma^2|Dg|^2\,dx+\int\sigma^2|D(g-g_h)|^2\,dx\right)\\&\leqslant c|\ln h|(|\ln h|+1)\leqslant c|\ln h|^2.\end{aligned}$$

The details are omitted here.

From (4.9), (4.11), (4.12) and (4.4) we obtain

$$\begin{aligned}\max_{x\in X\cap G_0}|D(u-\bar{u}_h)(x)|&\leqslant\max_{x\in X}|D(w-\bar{w}_I)(x)|+\|\tilde{u}_I-P\tilde{u}\|_{1,\infty,G}+\|P\tilde{u}-u_h\|_{1,\infty,G}\\&\leqslant ch^2(|\ln h|\,\|u\|_{3,\infty,G}+\|u\|_{2,\Omega}).\end{aligned}$$

The theorem is proved.

Theorem 2 can be extended to some other finite elements[11, 13-17]. Using these results we can study the superconvergence of finite element approximations to nonlinear elliptic problems[12].

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