SOME ADVANCES IN THE STUDY OF ERROR EXPANSION FOR FINITE ELEMENTS*

I. Eigenvalue Error Expansion

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Abstract

For the eigenvalue problem on a smooth domain we prove that the Ric hardson extrapolation increases the accuracy from second to third order for linear finite elements, and from fourth to fifth order for quadratic finite elements, without modification of the scheme near the boundary.

§ 1. Introduction

As an introductory model problem the simple eigenvalue problem

$$-\Delta u=\lambda u \text{ in } \Omega,\ u=0 \text{ on } \partial\Omega,\quad \int_\Omega u^2\,dx=1 \tag{1}$$

on a smooth domain $\Omega\subset R^2$ will be investigated in the first part of this paper.

Let $T_h=\{K\}$ be a regular triangulation of Ω of width h with all its boundary vertices on $\partial\Omega$. Corresponding to T_h, we define the following finite element space of degree 1 or 2,

$$S_h=\{V_h\in C(\Omega^h)\,|\,V_h \text{ linear/quadratic on each } K\in T_h,\ V_h\equiv 0 \text{ on } \partial\Omega^h\},$$
$$\Omega^h=\cup\{K\in T_h\}.$$

The finite element eigenvalue problem associated with (1) is determined by

$$(\nabla u_h, \nabla\varphi_h)=\lambda_h(u_h, \varphi_h),\quad \forall\varphi_h\in S_h, \tag{2}$$
$$u_h\in S_h,\quad (u_h, u_h)=1.$$

We recall the eigenvalue errors:

$$\lambda_h-\lambda=\begin{cases} O(h^2) & \text{for linear elements,}\\ O(h^4) & \text{for quadratic elements}^{1)}.\end{cases}$$

Our purpose is to prove further error expansions

$$\lambda_h-\lambda=\begin{cases} h^2e+O(h^3) & \text{for linear elements,} & (3)\\ h^4e+O(h^5) & \text{for quadratic elements}^{1)}. & (4)\end{cases}$$

The second part of this paper develops to the solution problems

$$-\Delta u=f \text{ in } \Omega,\ u=0 \text{ on } \partial\Omega$$

and

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1) For isoparametric elements

$$-\Delta u+u=f \text{ in } \Omega, \quad \frac{\partial u}{\partial n}=0 \text{ on } \partial\Omega$$

on a smooth or convex polygonal domain Ω. Let

T_h——triangulation with locally uniform meshes/piecewise uniform meshes,

z——nodal points of T_h having positive distance from $\partial\Omega$/the corner points of $\partial\Omega$/the interior vextex p of the macro triangulation (see [1]),

u^h——linear finite element solution of u,

$i_h u$——piecewise linear interpolant of u,

R_h and R'_h——remainders in the following error expansions:

$$(u^h-u)(z)=h^2 e(z)+R_h(z),$$

$$\partial_s(u^h-i_h u)(z)=h^2\,\partial_s e(z)+R'_h(z).$$

We will see that R_h and R'_h are of higher order, though the order will depend on the smoothness of u and the uniformity of T_h. We assemble some results as follows. (see [1]).

T_h	R_h	R'_h
uniform	$O(h^4)\|u\|_{4+s,\infty}$	$O(h^4)\|u\|_{5+s,\infty}$
piecewise uniform	$O\left(h^4\ln\frac{1}{h}\right)\|u\|_{4+s,\infty}$	$O(h^3)\|u\|_{4+s,\infty}$
	$O\left(h^3\ln\frac{1}{h}\right)\|u\|_{3,\infty}$	
	$o(h^2)$ if $u\in W^{3,2+s}(\Omega_0)\cap H^2(\Omega)$	$o(h^2)$ if $u\in W^{4,2+s}(\Omega_0)\cap H^2(\Omega)$
locally uniform	$O\left(h^3\ln\frac{1}{h}\right)\|u\|_{3,\infty}$	$O(h^3)\|u\|_{4+s,\infty}$

where $s>0$, $\Omega_0\subset\Omega$. Notice that even at the point p where the triangulation is only regular in the usual sense, the known error behavior

$$(u^h-u)(p)=O\left(h^2\ln\frac{1}{h}\right)$$

may also be improved to

$$\frac{1}{3}(4u^{h/2}-u^h)(p)=u(p)+O(h^3).$$

So the Richardson extrapolation increases the accuracy in almost all cases.

We will also see in the second part that an extended expansion of the form

$$(u^h-u)(z)=h^2e^{(1)}(z)+h^4e^{(2)}(z)+O(h^4)$$

even holds for interior nodal points z.

Our conclusion is that the finite element method fits Richardson extrapolation well.

§ 2. Expansion of Bramble Functional and the Intergral Error Version

In section 3 the eigenvalue error will be reduced to some integral errors, which can be expanded by means of the following

Expansion of Bramble lemma:

Let P_k be the set of all polynomials of degree k. Then

(i) *Suppose that F is a bounded linear functional on $W^{k+2,p}(\Omega)$* and satisfies

$$F(P)=0, \quad \forall P\in P_k.$$

Then one can expand

$$F(u)=\sum_{|\alpha|=k+1}\frac{F(x^\alpha)}{\alpha!\operatorname{mes}\Omega}\int_\Omega D^\alpha u(x)dx+R, \quad |R|\leqslant c|u|_{k+2,p,\Omega}. \tag{5}$$

(ii) *Suppose that F is a bounded linear functional on $W^{k+3,p}(\Omega)$ and satisfies*

$$F(P)=0,\ \forall P\in P_k,\ F(x^\alpha)=0,\ \forall|\alpha|=k+2.$$

Suppose also that Ω is symmetric with respect to the origin. Then one can expand

$$F(u)=\sum_{|\alpha|=k+1}\frac{F(x^\alpha)}{\alpha!\operatorname{mes}\Omega}\int_\Omega D^\alpha u(x)dx+R, \quad |R|\leqslant c|u|_{k+3,p,\Omega}. \tag{6}$$

Proof. Fou $u\in W^{k+2,p}(\Omega)$ we define the linear functional

$$\bar F(u)=F(u)-\sum_{|\alpha|=k+1}\frac{F(x^\alpha)}{\alpha!\operatorname{mes}\Omega}\int_\Omega D^\alpha u(x)dx.$$

In order to prove that $\bar F$ satisfies the Bramble condition

$$\bar F(P)=0, \quad \forall P\in P_{k+1} \tag{7}$$

we first assume $P\in P_k$. This leads to

$$D^\alpha P=0 \text{ with } |\alpha|=k+1,$$

and hence

$$\bar F(P)=F(P)=0, \quad \forall P\in P_k.$$

Now we assume $P=x^\alpha$ with $|\alpha|=k+1$. Then

$$D^\beta(x^\alpha)=0, \quad \forall|\beta|=k+1, \quad \beta\neq\alpha,$$

$$D^\alpha(x^\alpha)=\alpha!,$$

and

$$\bar F(x^\alpha)=F(x^\alpha)-\frac{F(x^\alpha)}{\alpha!\operatorname{mes}\Omega}\int_\Omega \alpha!dx=0,$$

and hence (7) is satisfied. Thus, Bramble lemma gives (5).

Let us further assume

$$P=x^\alpha \text{ with } |\alpha|=k+2.$$

Then $D^\beta(x^\alpha)$ ($|\beta|=k+1$) is a homogeneous polynomial of degree 1 and hence an odd function. Thus, by the symmetry of Ω,

$$\int_\Omega D^\beta(x^\alpha)dx=0, \quad \forall|\beta|=k+1,$$

and hence

$$\bar F(x^\alpha)=F(x^\alpha)=0, \quad \bar F(P)=0, \quad \forall P\in P_{k+2}.$$

Thus, Bramble lemma gives (6).

Before applying the Bramble functional expansion to the integral error we recall some properties of interpolant on the reference element.

Lemma 1. *Let $u(\xi)$ be a function on $[-1, 1]$, $i^{(1)}u$ the linear interpolant with nodes $\{-1, 1\}$, and $i^{(2)}u$ the quadratic interpolant with nodes $\{-1, 0, 1\}$. Then*

(i) *the oddness of u implies the oddness of* $i^{(k)}u$, $k=1, 2$;

(ii) $i^{(1)}(\xi^2)=1$, $i^{(2)}(\xi^3)=\xi$, $i^{(2)}(\xi^4)=\xi^2$;

(iii) $\int_{-1}^{1}(I-i^{(1)})(\xi^2)d\xi=-\frac{4}{3}$, $\int_{-1}^{1}(I-i^{(1)})(\xi^2)\xi\,d\xi=0$,

$$\int_{-1}^{1}(I-i^{(2)})(\xi^3)d\xi=\int_{-1}^{1}(I-i^{(2)})(\xi^5)d\xi=\int_{-1}^{1}(I-i^{(2)})(\xi^4)\xi\,d\xi$$
$$=\int_{-1}^{1}(I-i^{(2)})(\xi^3)\xi^2\,d\xi=0,$$
$$\int_{-1}^{1}(I-i^{(2)})(\xi^4)d\xi=\int_{-1}^{1}(I-i^{(2)})(\xi^3)\xi\,d\xi=-\frac{4}{15}.$$

Lemma 2. *Let* $u(\xi, \eta)$ *be a function on the reference triangle*

$$\overline{K}_0=\{(\xi, \eta)\,|\,0\leqslant\xi\leqslant1,\ 0\leqslant\eta\leqslant1,\ \xi+\eta\leqslant1\},$$

$i^{(1)}u$ *the linear interpolant with nodes* $\{(0, 1), (0, 0), (1, 0)\}$. *Then*

(i) $i^{(1)}(\xi^2)=\xi$, $i^{(1)}(\eta^2)=\eta$, $i^{(1)}(\xi\eta)=0$;

(ii) $\int_{K_0}(I-i^{(1)})(\xi^2)d\xi\,d\eta=\int_{K_0}(I-i^{(1)})(\eta^2)d\xi\,d\eta=-\frac{1}{12}$,

$$\int_{K_0}(I-i^{(1)})(\xi\eta)d\xi\,d\eta=\frac{1}{24}.$$

Proof. (i) follows from the fact that the interpolant nodes lie on the line equations

$$\xi(\xi-1)=0,\quad \eta(\eta-1)=0,\quad \xi\eta=0.$$

Then (ii) follows by a direct calculation.

Lemma 3. *Let* $u(\xi, \eta)$ *be a function on the two symmetric triangles*

$$\overline{K}=\{(\xi, \eta)\,|\,\xi+\eta\geqslant0,\ \xi\leqslant1, \eta\leqslant1\},\quad \overline{K}'=\{(\xi, \eta)\,|\,\xi+\eta\leqslant0,\ \xi\geqslant-1,\ \eta\geqslant-1\},$$

$i^{(2)}u$ *the piecewise quadratic interpolant on* $\overline{T}=\overline{K}\cup\overline{K}'$. *Then*

(i) *the oddness of u implies the oddness of* $i^{(2)}u$;

(ii) $i^{(2)}(\xi^3)=\xi$, $i^{(2)}(\eta^3)=\eta$, $i^{(2)}(\xi^4)=\xi^2$, $i^{(2)}(\eta^4)=\eta^2$, $i^{(2)}(\xi^3\eta)=i^{(2)}(\xi\eta^3)=\xi\eta$,

$$i^{(2)}(\xi^2\eta)=\xi(|\xi+\eta|-1),\quad i^{(2)}(\xi\eta^2)=\eta(|\xi+\eta|-1),$$
$$i^{(2)}(\xi^2\eta^2)=\xi^2+\eta^2+\xi\eta-|\xi+\eta|;$$

(iii) $\int_{\overline{T}}(I-i^{(2)})(y^\alpha)dy=0$ *for* $|\alpha|=3$ *with* $y=(\xi, \eta)$,

$$\int_{\overline{T}}(I-i^{(2)})(\xi^2\eta^2)d\xi\,d\eta=\frac{4}{9},$$
$$\int_{\overline{T}}(I-i^{(2)})(\xi^2)d\xi\,d\eta=\int_{\overline{T}}(I-i^{(2)})(\eta^2)d\xi\,d\eta=-\frac{8}{15},$$
$$\int_{\overline{T}}(I-i^{(2)})(\xi^3)\xi\,d\xi\,d\eta=-\frac{4}{15},$$
$$\int_{\overline{T}}(I-i^{(2)})(\xi^3\eta)d\xi\,d\eta=\int_{\overline{T}}(I-i^{(2)})(\xi\eta^3)d\xi\,d\eta=0,$$
$$\int_{\overline{T}}(I-i^{(2)})(\eta^3)\xi\,d\xi\,d\eta=0,$$
$$\int_{\overline{T}}(I-i^{(2)})(\xi^2\eta)\xi\,d\xi\,d\eta=\frac{4}{15},\quad \int_{\overline{T}}(I-i^{(2)})(\xi\eta^2)d\xi\,d\eta=-\frac{4}{45}.$$

Proof. Let y be the interpolant nodes of $\overline{T}$. Then $-y$ are also the interpolant nodes of $\overline{T}$. Hence

$$(i^{(2)}u)(-y)=u(-y)=-u(y)=-(i^{(2)}u)(y),$$

and (i) follows. (ii) follows from the fact that y lie on the line equations

$$\xi(\xi-1)(\xi+1)=0, \quad \eta(\eta-1)(\eta+1)=0$$

and that the interpolant nodes of $\overline{K}$ lie on the line equations

$$(\xi-1)(\eta-1)\xi=0, \quad (\xi-1)(\eta-1)\eta=0.$$

Then (iii) follows from (i) and (ii).

We now give the integral error versions for the Bramble functional expansion.

Lemma 4. *Let $u\in C^3[-1, 1]$, $v\in C^2[-1, 1]$. Then*

(i) $\int_{-1}^{1} (u-i^{(1)}u)d\xi=-\frac{1}{3}\int_{-1}^{1} u''d\xi+O(1)\ |u|_{3,\infty}$;

(ii) $\int_{-1}^{1} (u-i^{(1)}u)v\,d\xi=-\frac{1}{3}\int_{-1}^{1} u''v\,d\xi+O(1)\ (|v|_{0,\infty}|u|_{3,\infty}+|v|_{1,\infty}|u|_{3,\infty}$
$+|v|_{2,\infty}|u|_{2,\infty})$.

Proof. Let

$$F(u)=\int_{-1}^{1} (u-i^{(1)}u)d\xi$$

be the Bramble functional. Then

$$F(u)=0, \quad \forall u\in P_1, \quad F(\xi^2)=-\frac{4}{3},$$

and hence (i) is a particular version of (5) with $k=1$.

Let

$$F_1(u)=\int_{-1}^{1} (u-i^{(1)}u)\xi\,d\xi, \quad F_2(u)=\int_{-1}^{1} |u-i^{(1)}u|d\xi, \quad F_3(u)=\int_{-1}^{1}\xi u\,d\xi.$$

Then

$$F_1(u)=F_2(u)=0,\ \forall u\in P_1,\ F_1(\xi^2)=0,\ F_3(u)=0,\ \forall u\in P_0.$$

Thus, Bramble lemma gives

$$|F_1(u)|\leqslant c|u|_{3,\infty}, \quad |F_2(u)|\leqslant c|u|_{2,\infty}, \quad |F_3(u)|\leqslant c|u|_{1,\infty}.$$

We now expand

$$v(\xi)=v(0)+v'(0)\xi+O(1)\ |v|_{2,\infty}.$$

Then

$$\int_{-1}^{1} (u-i^{(1)}u)v\,d\xi=v(0)F(u)+v'(0)F_1(u)+O(1)\ |v|_{2,\infty}F_2(u)$$

$$=-\frac{1}{3}v(0)\int_{-1}^{1} u''d\xi+O(1)\ (|v|_{0,\infty}|u|_{3,\infty}+|v|_{1,\infty}|u|_{3,\infty}+|v|_{2,\infty}|u|_{2,\infty}),$$

$$v(0)\int_{-1}^{1} u''\,d\xi=\int_{-1}^{1} vu''\,d\xi-v'(0)F_3(u'')+O(1)\ |v|_{2,\infty}|u|_{2,\infty},$$

and (ii) follows.

Lemma 5. *Let $u\in C^6[-1, 1]$, $v\in C^3[-1, 1]$. Then*

(i) $\int_{-1}^{1} (u-i^{(2)}u)d\xi=-\frac{1}{180}\int_{-1}^{1} u^{(4)}d\xi+O(1)\ |u|_{6,\infty}$;

(ii) $\int_{-1}^{1}(u-i^{(2)}u)v\,d\xi=-\frac{1}{180}\int_{-1}^{1}u^{(4)}v\,d\xi-\frac{1}{45}\int_{-1}^{1}u^{(3)}v'\,d\xi+O(1)\sum_{k=0}^{3}|v|_{k,\infty}|u|_{6-k,\infty}$.

Proof. Let

$$F(u)=\int_{-1}^{1}(u-i^{(2)}u)\,d\xi.$$

Then

$$F(u)=0,\quad \forall u\in P_2,\quad F(\xi^3)=F(\xi^5)=0,\quad F(\xi^4)=-\frac{4}{15},$$

and hence (i) is a particular version of (6) with $k=3$.

Let

$$F_1(u)=\int_{-1}^{1}(u-i^{(2)}u)\xi\,d\xi,\quad F_2(u)=\int_{-1}^{1}(u-i^{(2)}u)\xi^2\,d\xi,$$

$$F_3(u)=\int_{-1}^{1}\xi u\,d\xi,\qquad F_4(u)=\int_{-1}^{1}|u-i^{(2)}u|\,d\xi.$$

Then

$$F_1(u)=F_2(u)=F_4(u)=0,\quad \forall u\in P_2,\quad F_1(\xi^4)=F_2(\xi^3)=0,$$

$$F_1(\xi^3)=-\frac{4}{15},\quad F_3(u)=0,\quad \forall u\in P_0.$$

Thus, (6) gives

$$F_1(u)=-\frac{4}{15}\,\frac{1}{3!\times 2}\int_{-1}^{1}u^{(3)}d\xi+O(1)\,|u|_{5,\infty},$$

$$|F_2(u)|\le c|u|_{4,\infty},\qquad |F_4(u)|\le c|u|_{3,\infty},\qquad |F_3(u)|\le c|u|_{1,\infty}.$$

We now expand

$$v(\xi)=v(0)+v'(0)\xi+\frac{1}{2}v''(0)\xi^2+O(1)\,|v|_{3,\infty}.$$

Then

$$\int_{-1}^{1}(u-i^{(2)}u)v\,d\xi=v(0)F(u)+v'(0)F_1(u)+\frac{1}{2}v''(0)F_2(u)+O(1)\,|v|_{3,\infty}F_3(u)$$

$$=-\frac{1}{180}v(0)\int_{-1}^{1}u^{(4)}d\xi-\frac{1}{45}v'(0)\int_{-1}^{1}u^{(3)}d\xi$$

$$+O(1)\sum_{k=0}^{3}|v|_{k,\infty}|u|_{6-k,\infty},$$

$$v(0)\int_{-1}^{1}u^{(4)}d\xi=\int_{-1}^{1}vu^{(4)}d\xi-v'(0)F_3(u)^{(4)})+O(1)\,|v|_{2,\infty}|u|_{4,\infty},$$

$$v'(0)\int_{-1}^{1}u^{(3)}d\xi=\int_{-1}^{1}v'u^{(3)}d\xi-v''(0)F_3(u^{(3)})+O(1)\,|v|_{3,\infty}|u|_{3,\infty},$$

and (ii) follows.

Lemma 6. *Let $u\in C^3(\overline{K}_0)$, $v\in C^1(\overline{K}_0)$. Then*

(i) $\int_{K_0}(u-i^{(1)}u)\,d\xi\,d\eta=-\frac{1}{12}\int_{K_0}(\partial_\xi^2+\partial_\eta^2-\partial_\xi\partial_\eta)u\,d\xi\,d\eta+O(1)\,|u|_{3,\infty}$;

(ii) $\int_{K_0}(u-i^{(1)}u)v\,d\xi\,d\eta=-\frac{1}{12}\int_{K_0}(\partial_\xi^2+\partial_\eta^2-\partial_\xi\partial_\eta)u\cdot v\,d\xi\,d\eta$
$+O(1)\,(|v|_{0,\infty}|u|_{3,\infty}+|v|_{1,\infty}|u|_{2,\infty})$.

Proof. Let

$$F(u)=\int_{K_0}(u-i^{(1)}u)\,dy.$$

Then

$$F(u)=0,\quad \forall u\in P_1,\quad F(\xi^2)=F(\eta^2)=-\frac{1}{12},\quad F(\xi\eta)=\frac{1}{24},$$

and hence (i) is a particular version of (5) with $k=1$.

(ii) follows from

$$\int_{K_0}(u-i^{(1)}u)\,v\,dy=v(0)F(u)+\int_{K_0}(v(y)-v(0))\,(u-i^{(1)}u)\,dy$$

$$=-\frac{1}{12}\int_{K_0}(\partial_\xi^2+\partial_\eta^2-\partial_\xi\partial_\eta)u\cdot v\,dy$$

$$+O(1)\,(|v|_{0,\infty}|u|_{3,\infty}+|v|_{1,\infty}|u|_{2,\infty}).$$

Lemma 7. *Let $u\in C^5(\overline{T})$, $v\in C^2(\overline{T})$. Then*

(i) $\displaystyle\int_{\overline{T}}(u-i^{(2)}u)\,d\xi\,d\eta=-\frac{1}{180}\int_{\overline{T}}(\partial_\xi^4+\partial_\eta^4-5\partial_\xi^2\partial_\eta^2)u\,d\xi\,d\eta+O(1)\,|u|_{5,\infty}$;

(ii) $\displaystyle\int_{\overline{T}}(u-i^{(2)}u)\,v\,d\xi\,d\eta=-\frac{1}{180}\int_{\overline{T}}(\partial_\xi^4+\partial_\eta^4-5\partial_\xi^2\partial_\eta^2)u\cdot v\,d\xi\,d\eta$

$$-\frac{1}{90}\int_{\overline{T}}(\partial_\xi^3-3\partial_\xi^2\partial_\eta+\partial_\xi\partial_\eta^2)u\cdot\partial_\xi v\,d\xi\,d\eta$$

$$-\frac{1}{90}\int_{\overline{T}}(\partial_\eta^3-3\partial_\xi\partial_\eta^2+\partial_\xi^2\partial_\eta)u\cdot\partial_\eta v\,d\xi\,d\eta+O(1)\sum_{k=0}^{2}|v|_{k,\infty}|u|_{5-k,\infty}.$$

Proof. Let

$$F(u)=\int_{\overline{T}}(u-i^{(2)}u)\,dy.$$

Then

$$F(u)=0,\quad \forall u\in P_2,\quad F(y^\alpha)=0,\quad \forall|\alpha|=3,$$

$$F(\xi^4)=F(\eta^4)=-\frac{8}{15},\quad F(\xi^3\eta)=F(\xi\eta^3)=0,\quad F(\xi^2\eta^2)=\frac{4}{9},$$

and hence (i) is a particular version of (5) with $k=3$.

Let

$$F_1(u)=\int_{\overline{T}}(u-i^{(2)}u)\,\xi\,d\xi\,d\eta.$$

Then

$$F_1(u)=0,\quad \forall u\in P_2,\quad F_1(\xi^3)=-\frac{4}{15},\quad F_1(\eta^3)=0,\quad F_1(\xi^2\eta)=\frac{4}{15},$$

$$F_1(\xi\eta^2)=-\frac{4}{45}.$$

Thus, (5) gives

$$F_1(u)=-\frac{1}{90}\int_{\overline{T}}(\partial_\xi^3-3\partial_\xi^2\partial_\eta+\partial_\xi\partial_\eta^2)u\,d\xi\,d\eta+O(1)\,|u|_{4,\infty}.$$

Similarly,

$$F_2(u)=\int_{\overline{T}}(u-i^{(2)}u)\,\eta\,d\xi\,d\eta=-\frac{1}{90}\int_{\overline{T}}(\partial_\eta^3-3\partial_\xi\partial_\eta+\partial_\xi^2\partial_\eta)u\,d\xi\,d\eta+O(1)\,|u|_{4,\infty}.$$

We now expand

$$v(\xi,\eta)=v(0,0)+\partial_\xi v(0,0)\xi+\partial_\eta v(0,0)\eta+O(1)\,|v|_{2,\infty}.$$

Then

$$\int_{\bar{T}}(u-i^{(2)}u)v\,d\xi\,d\eta=v(0,0)F(u)+\partial_\xi v(0,0)F_1(u)+\partial_\eta v(0,0)F_2(u)$$
$$+O(1)\,|v|_{2,\infty}\int_{\bar{T}}|u-i^{(2)}u|\,d\xi\,d\eta,$$

and (ii) follows.

Lemmas 4—7 relate to the case of the reference element. It is easy, however, to give the versions for the arbitrary element by mapping the later into the reference element.

Let $s\subset\Omega$ be an arbitrary fixed line segment with endpoints a_1 and a_2, h the length of s, t the unit vector along s. We use the notation $\partial=\frac{\partial}{\partial t}$. Define

$$x=\frac{1-\xi}{2}a_1+\frac{1+\xi}{2}a_2$$

which maps s into $[-1, 1]$ and satisfies

$$\frac{dx}{d\xi}=\frac{h}{2}t,\quad ds=\frac{h}{2}d\xi.$$

Let

$$\bar{u}(\xi)=u\Big(\frac{1-\xi}{2}a_1+\frac{1+\xi}{2}a_2\Big)$$

for the given function u on s. Then

$$\bar{u}'(\xi)=\nabla u\cdot\frac{dx}{d\xi}=\frac{h}{2}\nabla u\cdot t=\frac{h}{2}\partial u,\quad \bar{u}^{(k)}(\xi)=\Big(\frac{h}{2}\Big)^k\partial^k u,$$

and $\overline{iu}=i\bar{u}$. Therefore Lemmas 4 and 5 lead to

Proposition 1. Let $u\in C^3(s)$, $v\in C^2(s)$. Then

$$\int_s(u-i_h^{(1)}u)v\,ds=-\frac{h^2}{12}\int_s\partial^2u\cdot v\,ds+O(h^4)\|v\|_{0,\infty}\|u\|_{3,\infty}+O(h^5)\|v\|_{2,\infty}\|u\|_{3,\infty}.$$

Proposition 2. Let $u\in C^6(s)$, $v\in C^3(s)$. Then

$$\int_s(u-i_h^{(2)}u)v\,ds=-\frac{h^4}{2880}\int_s\partial^4u\cdot v\,ds-\frac{h^4}{720}\int_s\partial^3u\,\partial v\,ds+O(h^7)\|v\|_{3,\infty}\|u\|_{6,\infty}.$$

We turn to the more complex case where $K\subset\Omega_h$ is an arbitrary fixed triangle with three vertices a_i $(i=1, 2, 3)$ in counter-clockwise ordering. Accordingly, let s_i be the side of K opposite to a_i, h_i the length of s_i, H_i the height on s_i, n_i the outer normal unit vector along s_i, and t_i the tangent unit vector along s_i. We use notation

$$\partial_i=\frac{\partial}{\partial t_i}=t_i\cdot\nabla,\quad A=\text{mes}\,K.$$

Define the linear mapping

$$x=\eta a_1+\xi a_2+(1-\xi-\eta)a_3$$

which maps K into $\bar{K}_0$ (the reference triangle in Lemma 2). Then

$$\frac{\partial x}{\partial\xi}=a_2-a_3=-h_1t_1,\quad \frac{\partial x}{\partial\eta}=a_1-a_3=h_2t_2,\quad \left|\frac{D(x_1,x_2)}{D(\xi,\eta)}\right|=\frac{\text{mes}\,K}{\text{mes}\,\bar{K}_0}=2A.$$

Let

$$\bar{u}(\xi,\eta)=u(\eta a_1+\xi a_2+(1-\xi-\eta)a_3)$$

for the given function u on K. Then

$$\partial_\xi \bar{u} = \nabla u \frac{\partial x}{\partial \xi} = \nabla u(-h_1 t_1) = -h_1 \partial_1 u, \quad \partial_\eta \bar{u} = h_2 \partial_2 u,$$

$$|\bar{u}|_{k,\infty} \leq C h^k |u|_{k,\infty}, \quad \int_K u\, dx = 2A \int_{K_0} \bar{u}\, d\xi\, d\eta,$$

and $\overline{iu} = i\bar{u}$. Therefore, Lemma 6 leads to

Proposition 3. Let $u \in C^3(K)$, $v \in C^1(K)$. Then

$$\int_K (u - i_h^{(1)} u) v\, dx = -\frac{1}{12} \int_K (h_1^2 \partial_1^2 + h_2^2 \partial_2^2 + h_1 h_2 \partial_1 \partial_2) u \cdot v\, dx + O(h^5) \|v\|_{1,\infty} \|u\|_{3,\infty}.$$

Let K and K' be the adjacent elements with vertices a_1, a_2, a_3 and $a_1' = a_2$, $a_2' = a_1$, a_3' respectively. In order to map

$$T = K \cup K'$$

into the reference element $\bar{T} = \bar{K} \cup \bar{K}'$, we suppose that K and K' form a parallelogram. Then, the linear mapping

$$x = \frac{1-\xi}{2} a_1 + \frac{1-\eta}{2} a_2 + \frac{\xi + \eta}{2} a_3$$

maps K into $\bar{K}$, and, by the relation

$$a_3' = a_3 + (a_1 - a_3) + (a_2 - a_3),$$

maps simultaneously K' into $\bar{K}'$. We have

$$\frac{\partial x}{\partial \xi} = \frac{a_3 - a_1}{2} = -\frac{h_2}{2} t_2, \quad \frac{\partial x}{\partial \eta} = \frac{h_1}{2} t_1, \quad \left| \frac{D(x_1, x_2)}{D(\xi, \eta)} \right| = \frac{\operatorname{mes} K}{\operatorname{mes} \bar{K}} = \frac{A}{2}.$$

Let

$$\bar{u}(\xi, \eta) = u\left(\frac{1-\xi}{2} a_1 + \frac{1-\eta}{2} a_2 + \frac{\xi+\eta}{2} a_3 \right)$$

for a given function u on T. Then

$$\partial_\xi \bar{u} = -\frac{h_2}{2} \partial_2 u, \quad \partial_\eta \bar{u} = \frac{h_1}{2} \partial_1 u, \quad |\bar{u}|_{k,\infty} \leq C h^k |u|_{k,\infty},$$

$$\int_T u\, dx = \frac{A}{2} \int_{\bar{T}} \bar{u}\, d\xi\, d\eta.$$

Therefore, Lemma 7 leads to

Proposition 4. Let $u \in C^5(T)$, $v \in C^2(T)$, then

$$\begin{aligned} \int_T (u - i_h^{(2)} u) v\, dx = & -\frac{1}{2880} \int_T (h_1^4 \partial_1^4 + h_2^4 \partial_2^4 - 5 h_1^2 h_2^2 \partial_1^2 \partial_2^2) u \cdot v\, dx \\ & - \frac{1}{1440} \int_T h_2 \partial_2 v (h_2^3 \partial_2^3 + 3 h_2^2 h_1 \partial_2^2 \partial_1 + h_1^2 h_2 \partial_1^2 \partial_2) u\, dx \\ & - \frac{1}{1440} \int_T h_1 \partial_1 v (h_1^3 \partial_1^3 + 3 h_2 h_1^2 \partial_2 \partial_1^2 + h_1 h_2^2 \partial_1 \partial_2^2) u\, dx \\ & + O(h^7) \|v\|_{2,\infty} \|u\|_{5,\infty}. \end{aligned}$$

Some other lemmas have yet to be prepared.

Lemma 8. *On the triangle K, there holds*

$$n_i = \frac{1}{2A} h_i h_{i+1} (n_i \cdot n_{i+1} t_i - t_{i+1}),$$

where the index $i+1$ is used mod (3).

Proof. Since t_i and t_{i+1} are independent, there exist $\alpha_1^{(i)}$ and $\alpha_2^{(i)}$ such that

$$n_i = \alpha_1^{(i)} t_i + \alpha_2^{(i)} t_{i+1}.$$

Note that $n_j \cdot t_j = 0$. We obtain

$$\alpha_1^{(i)} = \frac{n_i \cdot n_{i+1}}{t_i \cdot n_{i+1}}, \qquad \alpha_2^{(i)} = \frac{1}{\eta_i \cdot t_{i+1}}.$$

Then the lemma follws from

$$t_i \cdot n_{i+1} = \frac{H_{i+1}}{h_i} = \frac{2A}{h_i h_{i+1}}, \quad t_{i+1} \cdot n_i = -\frac{2A}{h_i h_{i+1}}.$$

Lemma 9. *There exist constants β_i and γ_i ($i=1, 2, 3$) such that, for $v \in C^2(K)$,*

$$\partial_1 \partial_2 v = \sum_{i=1}^{3} \beta_i \partial_i^2 v, \quad \Delta v = \sum_{i=1}^{3} \gamma_i \partial_i^2 v.$$

Proof. Let

$$\partial_1 \partial_2 v = D^2 v(t_1, t_2), \qquad \partial_i^2 v = D^2 v(t_i, t_i).$$

Since

$$\sum h_i t_i = 0$$

we have

$$h_3^2 \partial_3^2 v = h_1^2 D^2 v(t_1, t_1) + h_2^2 D^2 v(t_2, t_2) + 2h_1 h_2 D^2 v(t_1, t_2),$$

and hence

$$\partial_1 \partial_2 v = D^2 v(t_1, t_2) = \frac{1}{2h_1 h_2} (h_3^2 \partial_3^2 v - h_1^2 \partial_1^2 v - h_2^2 \partial_2^2 v).$$

Noticing

$$\Delta v = \frac{\partial^2 v}{\partial t_1^2} + \frac{\partial^2 v}{\partial n_1^2}$$

we obtain

$$\frac{\partial^2 v}{\partial n_1^2} = D^2 v(n_1, n_1) = D^2 v(\alpha_1^{(1)} t_1 + \alpha_2^{(1)} t_2, \alpha_1^{(1)} t_1 + \alpha_2^{(1)} t_2)$$
$$= (\alpha_1^{(1)})^2 \partial_1^2 v + (\alpha_2^{(1)})^2 \partial_2^2 v + 2\alpha_1^{(1)} \alpha_2^{(1)} \partial_1 \partial_2 v = \sum \gamma_i \partial_i^2 v.$$

Lemma 10. *There holds, for $v \in C^1(K)$,*

$$h_2 \int_{s_1} v\, ds - h_1 \int_{s_2} v\, ds = \frac{h_1 h_2 h_3}{2A} \int_K \partial_3 v\, dx.$$

Proof. Following [1, 4] we have

$$\int_K \partial_3 v\, dx = \int_{\partial K} v \cdot (t_3 \cdot n) ds = \int_{s_1} v \cdot (t_3 \cdot n_1) ds + \int_{s_2} v \cdot (t_3 \cdot n_2) ds,$$

and Lemma 10 follows from Lemma 8.

§ 3. Eigenvalue Error Expansion for Linear Elements

Let $S_h^{(1)}$ be the linear finite element space, and $u^h \in S_h^{(1)}$ the H_0^1-projection of u:

$$(\nabla u^h, \nabla \varphi_h) = (\nabla u, \nabla \varphi_h), \quad \forall \varphi_h \in S_h^{(1)}.$$

There holds, by definition (2),

$$\lambda(u, u_h) = (\nabla u, \nabla u_h) = (\nabla u^h, \nabla u_h) = \lambda_h(u^h, u_h),$$

and, consequently,

$$\lambda=\lambda_h(u^h, \bar{u}_h) \text{ with } \bar{u}_h=\frac{u_h}{(u, u_h)}.$$

Hence, using the known errors for eigenvalue and eigenfunction, one obtains

$$\begin{aligned}\lambda_h-\lambda&=\lambda_h(u, \bar{v}_h)-\lambda_h(u^h, \bar{u}_h)=\lambda_h(u-u^h, \bar{u}_h)\\&=(u-u^h, \lambda u)+(u-u^h, \lambda_h\bar{u}_h-\lambda u)\\&=(\nabla u, \nabla(u-u^h))+O(h^4)\\&=(\nabla(u-i_hu), \nabla(u-u^h))+O(h^4).\end{aligned}$$

We now assume a subdomian $\Omega_h\subset\Omega^h=\bigcup\{K\in T_h\}$, which satisfies

$$\text{mes }(\Omega\backslash\Omega_h)=O(h).$$

Then

$$\lambda_h-\lambda=\int_{\Omega_h}\nabla(u-i_hu)\nabla(u-u^h)\,dx+O(h^3)\,\|u\|^2_{2,\infty} \tag{8}$$

holds independently of whatever division we have in $\Omega\backslash\Omega_h$. Thus, as an example, Ω_h may be assumed to be an interior polygonal domain and $T_{h/2}$ is constructed by subdividing each triangle $K\in\Omega_h$ into four congruent triangles, but, on $\Omega^h\backslash\Omega_h$, the triangulation is only assumed to be regular in the usual sense.

Integrating by parts over Ω_h, (8) becomes

$$\lambda_h-\lambda=\lambda\sum_{K\subset\Omega_h}\int_K(u-i_hu)u\,dx+\sum_{K\subset\Omega_h}\int_{\partial K}(u-i_hu)n\cdot\nabla(u-u^h)\,ds+O(h^3)\,\|u\|^2_{2,\infty}.$$

We apply Proposition 3 to expand the area integral

$$\int_K(u-i_hu)u\,dx=-\frac{1}{12}\int_K(h_1^2\partial_1^2+h_2^2\partial_2^2+h_1h_2\partial_1\partial_2)u\cdot u\,dx+O(h^5)\,\|u\|_{1,\infty}\|u\|_{3,\infty}.$$

It remains to expand the line integral

$$\int_{\partial K}(u-i_hu)n\cdot\nabla(u-u^h)\,ds=\sum_{i=1}^{3}\int_{s_i}(u-i_hu)n_i\cdot\nabla(u-u^h)\,ds.$$

By means of Lemma 8,

$$\int_{s_i}(u-i_hu)n_i\cdot\nabla(u-u^h)\,ds=\alpha_1^{(i)}\int_{s_i}(u-i_hu)\,\partial_i(u-u^h)\,ds+\alpha_2^{(i)}\int_{s_i}(u-i_hu)\,\partial_{i+1}(u-u^h)\,ds.$$

We expand, by Proposition 1,

$$\begin{aligned}\int_{s_1}(u-i_hu)\,\partial_1(u-u^h)\,ds&=-\frac{h_1^2}{12}\int_{s_1}\partial_1^2u\partial_1(u-u^h)\,ds+O(h^4)\,\|u-u^h\|_{1,\infty}\|u\|_{3,\infty}\\&\quad+O(h^5)\,\|u-u^h\|_{3,\infty,K}\|u\|_{3,\infty}\\&=-\frac{h_1^2}{12}\int_{s_1}\partial_1^2u\partial_1(u-u^h)\,ds+O(h^5)\,\|u\|^2_{3,\infty},\end{aligned}$$

and, similarly,

$$\int_{s_1}(u-i_hu)\,\partial_2(u-u^h)\,ds=-\frac{h_1^2}{12}\int_{s_1}\partial_1^2u\partial_2(u-u^h)\,ds+O(h^5)\,\|u\|^2_{3,\infty}.$$

We apply Lemma 10 to change the line integral on s_1 into a line integral on s_2:

$$\begin{aligned}\int_{s_1}\partial_1^2u\partial_2(u-u^h)\,ds&=\frac{h_1}{h_2}\int_{s_1}\partial_1u\partial_2(u-u^h)\,ds+\frac{h_1h_3}{2A}\int_K\partial_3(\partial_1u\partial_2(u-u^h))\,dx\\&=\frac{h_1}{h_2}\int_{s_2}\partial_1^2u\partial_2(u-u^h)\,ds+\frac{h_1h_3}{2A}\int_K\partial_1^2u\partial_3\partial_2u\,dx\end{aligned}$$

$$+O(h^3)\|u\|_{3,\infty}\|u\|_{2,\infty}.$$

Therefore,

$$\lambda_h-\lambda=-\frac{\lambda}{12}\sum_{K\subset\Omega_h}\int_K(h_1^2\partial_1^2+h_2^2\partial_2^2+h_1h_2\partial_1\partial_2)u\cdot u\,dx$$
$$+\sum_{K\subset\Omega_h}\sum_{i=1}^{3}\frac{h_i^3}{24A}\Big(\frac{h_ih_{i+1}h_{i+2}}{2A}\int_K\partial_i^2u\partial_{i+1}\partial_{i+2}u\,dx-n_in_{i+1}h_{i+1}\int_{s_i}\partial_i^2u\partial_i(u-u^h)\,ds$$
$$+h_i\int_{s_{i+1}}\partial_i^2u\partial_{i+1}(u-u^h)\,ds\Big)+O(h^3)\|u\|_{3,\infty}^2. \tag{9}$$

This representation contains the principal information we will need about the behavior of eigenvalue error. The key problem here is how to reduce the line integrals to a higher order in h.

If the average gradient error is of higher order:

$$\int_{s_j}\partial_j(u-u^h)\,ds=O(h^3), \tag{10}$$

then the representation (9) reduces to the sum of the area integrals:

$$\lambda_h-\lambda=-\frac{\lambda}{12}\sum_{K\subset\Omega_h}\int_K(h_1^2\partial_1^2+h_2^2\partial_2^2+h_1h_2\partial_1\partial_2)u\cdot u\,dx$$
$$+\sum_{K\subset\Omega_h}\sum_{i=1}^{3}\frac{h_i^4h_{i+1}h_{i+2}}{48A^2}\int_K\partial_i^2u\partial_{i+1}\partial_{i+2}u\,dx+O(h^3)\|u\|_{3,\infty}^2,$$

and the Richardson extrapolation applies.

We now construct a special triangulation T_h without hypothesis (10). That is, T_h is uniform when restricted in Ω_h but keeps flexible on $\Omega^h\backslash\Omega_h$. Then the following simplifications occur. All line integrals in (9) are cancelled on edges interior to Ω_h since $\partial_i=\partial_i'$. The remaining boundary integrals on $\partial\Omega_h$ are of higher order $O(h^3)$. Hence one obtains an eigenvalue error expansion without line integral:

$$\lambda_h-\lambda=h^2\Big(-\frac{\lambda}{12}\int_\Omega\tau_1^2\partial_1^2+\tau_2^2\partial_2^2+\tau_1\tau_2\partial_1\partial_2\Big)u\cdot u\,dx$$
$$+\frac{1}{48\sigma^2}\sum_{i=1}^{3}\tau_i^4\tau_{i+1}\tau_{i+2}\int_\Omega\partial_i^2u\partial_{i+1}\partial_{i+2}u\,dx)+O(h^3)\|u\|_{3,\infty}^2,$$

where we set $h_i=\tau_ih$ and $A=\sigma h^2$.

We should mention that a similar error expansion has been observed in [3] (see also [2]).

§ 4. Eigenvalue Error Expansion for Quadratic Elements

Let $S_h^{(2)}$ be the quadratic finite element space, and $u^h\in S_h^{(2)}$ the H_0^1-projection of u. Then by the same argument as in § 3,

$$\lambda_h-\lambda=(\nabla(u-i_hu),\nabla(u-u^h))+O(h^6)$$
$$=\int_{\Omega_h}\nabla(u-i_hu)\nabla(u-u^h)\,dx+O(h^5)\|u\|_{3,\infty}^2$$
$$=-\sum_{K\subset\Omega_h}\int_K(u-i_hu)\Delta(u-u^h)\,dx$$
$$+\sum_{K\subset\Omega_h}\int_{\partial K}(u-i_hu)n\cdot\nabla(u-u^h)\,ds+O(h^5)\|u\|_{3,\infty}^2.$$

Lemmas 9 and 8 yield

$$\lambda_h-\lambda=-\sum_{i=1}^{3}\gamma_i\sum_{K\subset\Omega_h}\int_K(u-i_hu)\partial_i^2(u-u^h)\,dx+\sum_{i=1}^{3}\sum_{K\subset\Omega_h}\Big(\alpha_1^{(i)}\int_{s_i}(u-i_hu)\partial_i(u-u^h)\,ds$$
$$+\alpha_2^{(i)}\int_{s_i}(u-i_hu)\partial_{i+1}(u-u^h)\,ds\Big)+O(h^5)\,\|u\|_{3,\infty}^2. \tag{11}$$

We first consider the area integrals and estimate its contribution on the boundary elements with $s_i\subset\partial\Omega_h$:

$$\Big|\sum_{s_i\subset\partial\Omega_h}\int_K(u-i_hu)\partial_i^2(u-u^h)\,dx\Big|\leqslant Ch^{-1}h^2\,|u-i_hu|_{0,\infty}|u-u^h|_{2,\infty}\leqslant Ch^5\|u\|_{3,\infty}^2 \tag{12}$$

(since the number of boundary elements is of order $O(h^{-1})$). We use notations

$$L_1(u)=-\frac{1}{2880}(\tau_3^4\partial_3^4+\tau_2^4\partial_2^4-5\tau_3^2\tau_2^2\partial_3^2\partial_2^2)u,$$

$$L_2^{(1)}(u)=-\frac{\tau_2}{1440}(\tau_2^3\partial_2^3+3\tau_2^2\tau_3\partial_2^2\partial_3+\tau_3^2\tau_2\partial_3^2\partial_2)u,$$

$$L_3^{(1)}(u)=-\frac{\tau_3}{1440}(\tau_3^3\partial_3^3+3\tau_2\tau_3^2\partial_2\partial_3^2+\tau_3\tau_2^2\partial_3\partial_2^2)u,$$

and denote $s_i\not\subset\partial\Omega_h$ by $s_i\subset\Omega_h^0=\Omega_h\backslash\partial\Omega_h$. Then, by Proposition 4 (with $s_1=K\cap K'$), we expand the area integrals

$$\sum_{s_1\subset\Omega_h^0}\int_K(u-i_hu)\partial_1^2u\,dx=h^4\Big(\int_{\Omega_h}L_1(u)\partial_1^2u\,dx+\int_{\Omega_h}L_2^{(1)}(u)\partial_2\partial_1^2u\,dx$$
$$+\int_{\Omega_h}L_3^{(1)}(u)\partial_3\partial_1^2u\,dx\Big)+O(h^5)\,\|u\|_{4,\infty}\|u\|_{5,\infty},$$

and noting that the constant $\partial^2u^h=\partial_1'^2u^h$ on s_1,

$$\sum_{s_1\subset\Omega_h^0}\int_K(u-i_hu)\partial_1^2u^h\,dx=h^4\sum_{K\subset\Omega_h}\int_K L_1(u)\partial_1^2u^h dx+O(h^5)\,\|u\|_{3,\infty}\|u\|_{5,\infty},$$

and hence, noting that $\partial_1^2(u-u^h)=O(h)$,

$$\sum_{s_1\subset\Omega_h^0}\int_K(u-i_hu)\partial_1^2(u-u^h)\,ds$$
$$=h^4\int_{\Omega_h}(L_2^{(1)}(u)\partial_2\partial_1^2u+L_3^{(1)}(u)\partial_3\partial_1^2u)\,dx+O(h^5)\,\|u\|_{4,\infty}\|u\|_{5,\infty}. \tag{13}$$

It remains to expand the line integrals. By Proposition 2,

$$\int_{s_1}(u-i_hu)\partial_1(u-u^h)\,ds$$
$$=c_1h_1^4\int_{s_1}\partial_1^4u\partial_1(u-u^h)\,ds+c_2h_1^4\int_{s_1}\partial_1^3u\partial_1^2(u-u^h)\,ds+O(h^7)\,\|u\|_{4,\infty}\|u\|_{6,\infty}$$
$$=c_2h_1^4\int_{s_1}\partial_1^2(u-u^h)\partial_1^3u\,ds+O(h^7)\,\|u\|_{4,\infty}\|u\|_{6,\infty},$$

and, similarly,

$$\int_{s_1}(u-i_hu)\partial_2(u-u^h)\,ds=c_2h_1^4\int_{s_1}\partial_1\partial_2(u-u^h)\partial_1^3u\,ds+O(h^7)\,\|u\|_{4,\infty}\|u\|_{6,\infty}.$$

By Lemmas 9 and 10,

$$\int_{s_1}\partial_1\partial_2(u-u^h)\partial_1^3u\,ds=\sum_{i=1}^{3}\beta_i^{(1)}\int_{s_1}\partial_i^2(u-u^h)\partial_1^3u\,ds,$$

$$\int_{s_1}\partial_2^2(u-u^h)\partial_1^3u\,ds=\frac{h_1}{h_2}\int_{s_1}\partial_2^2(u-u^h)\partial_1^3u\,ds+\frac{h_1h_3}{2A}\int_K\partial_3(\partial_2^2(u-u^h)\partial_1^3u)\,dx,$$

$$\int_K\partial_3(\partial_2^2(u-u^h)\partial_1^3u)\,dx=\int_K\partial_3\partial_2^2u\partial_1^3u\,dx+O(h^3)\|u\|_{3,\infty}\|u\|_{4,\infty}.$$

Notices

$$\sum_{K\subset\Omega_h}\int_{s_i}\partial_i^2(u-u^h)\partial_1^3u\,ds=\sum_{s_i\subset\partial\Omega_h}\int_{s_i}\partial_i^2(u-u^h)\partial_1^3u\,ds=O(h)\|u\|_{3,\infty}^2$$

(since the line integrals are cancelled on edges interior to Ω_h), one obtains

$$\sum_{K\subset\Omega_h}\int_{s_1}\partial_2^2(u-u^h)\partial_1^3u\,ds=\frac{h_1h_3}{2A}\int_{\Omega_h}\partial_3\partial_2^2u\partial_1^3u\,dx+O(h)\|u\|_{3,\infty}\|u\|_{4,\infty},$$

and, similarly,

$$\sum_{K\subset\Omega_h}\int_{s_1}\partial_3^2(u-u^h)\partial_1^3u\,ds=-\frac{h_1h_2}{2A}\int_{\Omega_h}\partial_2\partial_3^2u\partial_1^3u\,dx+O(h)\|u\|_{3,\infty}\|u\|_{4,\infty}.$$

Then

$$\sum_{K\subset\Omega_h}\int_{s_1}(u-i_hu)\partial_1(u-h^h)\,ds=O(h^5)\|u\|_{4,\infty}\|u\|_{6,\infty},$$

$$\begin{aligned}\sum_{K\subset\Omega_h}\int_{s_1}&(u-i_hu)\partial_2(u-u^h)\,ds\\&=c_2h_1^4\Big(\beta_2^{(1)}\frac{h_1h_3}{2A}\int_{\Omega_h}\partial_3\partial_2^2u\partial_1^3u\,dx-\beta_3^{(1)}\frac{h_1h_2}{2A}\int_{\Omega_h}\partial_2\partial_3^2u\partial_1^3u\,dx\Big)+O(h^5)\|u\|_{4,\infty}\|u\|_{6,\infty}\\&=\frac{c_2h_1^5}{2A}\int_{\Omega_h}(\beta_2^{(1)}h_3\partial_3\partial_2^2u-\beta_3^{(1)}h_2\partial_2\partial_3^2u)\partial_1^3u\,dx+O(h^5)\|u\|_{4,\infty}\|u\|_{6,\infty}.\end{aligned}$$

Combining with (11), (12) and (13) we obtain

$$\begin{aligned}\lambda_h-\lambda=&-h^4\sum_{i=1}^3\gamma_i\int_\Omega(L_2^{(i)}(u)\partial_{i+1}\partial_i^2u+L_3^{(i)}(u)\partial_{i+2}\partial_i^2u)\,dx\\&-\frac{h^4}{5760\sigma}\sum_{i=1}^3\frac{\tau_i^5}{n_it_{i+1}}\int_\Omega(\beta_2^{(i)}\tau_{i+2}\partial_{i+2}\partial_{i+1}^2u-\beta_3^{(i)}\tau_{i+1}\partial_{i+1}\partial_{i+2}^2u)\partial_i^3u\,dx\\&+O(h^5)\|u\|_{4,\infty}\|u\|_{6,\infty}\end{aligned}$$

which leads to

$$\frac{1}{15}(16\lambda_{h/2}-\lambda_h)=\lambda+O(h^5)\|u\|_{5,\infty}\|u\|_{6,\infty}.$$

§ 5. The Less Regular Case

In case the eigenfunction u is less regular:

$$u\in H^2(\Omega)\cap H_0^1(\Omega),$$

we still have, for linear elements,

$$\lambda_h-\lambda=h^2e(u)+o(h^2). \tag{14}$$

This can be proved as follows. Let

$$F_h(u)=\frac{1}{h^2}(\nabla(u-i_hu),\nabla(u-u^h)).$$

Then

$$\lambda_h-\lambda=h^2F_h(u)+o(h^4)\|u\|_{2,2}^2.$$

Notice that u can be approximated in $H^2(\Omega)$ by

$$\tilde{u}\in C^3(\Omega)\cap H_0^1(\Omega),$$

where $\Delta u=f\in L^2(\Omega)$, $\Delta\tilde{u}=\tilde{f}\in C_0^1(\Omega)$, $\|\tilde{f}-f\|_{0,2}\to 0$. By § 3,

$$F_h(\tilde{u})=e(\tilde{u})+o(1)\|\tilde{u}\|_{3,\infty}^2,$$

then $F_h(\tilde{u})-F_{h'}(\tilde{u})=o(1)$, $\forall h,\ h'\ll 1$,

$$\begin{aligned}|F_h(u)-F_h(\tilde{u})|&=\frac{1}{h^2}|(\nabla(u-\tilde{u}-i_h(u-\tilde{u})),\nabla(u-u^h))\\&\quad+(\nabla(\tilde{u}-i_h\tilde{u}),\nabla(u-\tilde{u}-(u-\tilde{u})^h))|\\&\leqslant c(\|u\|_{2,2}+\|\tilde{u}\|_{2,2})\|u-\tilde{u}\|_{2,2},\end{aligned}$$

$$F_h(u)-F_{h'}(u)=F_h(u)-F_h(\tilde{u})+F_h(\tilde{u})-F_{h'}(\tilde{u})+F_{h'}(\tilde{u})-F_{h'}(u),$$

and hence

$$F_h(u)=e(u)+o(1).$$

References

[1] H. Blum, Q. Lin, R. Rannacher, Preprint A 85/10, Univ. des Saarlandes, Saarbrüchen, 1985 (accepted by Numer. Math.).
[2] Q. Lin, T. Lu, *Bonn Math. Schriften*, Nr. 158, 1984.
[3] Q. Lin, T. Lu. J. P. Wang, Mannuscript 1985.
[4] Q. Lin, J. ch. xu, *J. Comp. Math.*, **3: 2** (1985), 115—133.