EXTRAPOLATION COMBINED WITH MULTIGRID METHOD FOR SOLVING FINITE ELEMENT EQUATIONS*1)

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Abstract

An algorithm combining the MG method with two types of extrapolation is given for solving finite element equations with any initial triangulation. A high order approximation to the solution of PDEs can be obtained at the cost of order $O(N)$ of computational work.

§ 1. Introduction

Two types of extrapolation are suggested in [1] for solving boundary value problems by successively refining meshes:

Type 1 for gaining a higher order approximation to the solution of PDEs;

Type 2 for gaining a good initial approximation in iteration.

These extrapolations are based theoretically upon the asymptotic expansion

$$u^h = u^I + c_1 h^{\alpha_1} + c_2 h^{\alpha_2} + \cdots, \quad 0<\alpha_1<\alpha_2<\cdots, \tag{1}$$

where u^h, u^I represent the discrete solution and the interpolation function of the solution of PDEs for linear finite element. It has been known that[2,3]

$$u^h(z) = u^I(z) + w(z)h^2 + O(h^2 \ln h) \tag{2}$$

holds if the solution of PDEs is smooth enough. The numerical experiments and some theoretical analysis in [4] show that asymptotic expansions also hold for less regular problems. In order to make the extrapolation of type 1 effective, the discrete solution must be accurate enough and this should cost an order of $O(N \ln N)$ of computational work for ordinary MG methods (N the number of nodes). Now an algorithm combining the MG method with type 2 extrapolation is given and its order of computational work is reduced to $O(N)$.

When we finished the paper, we learned that some authors[5,6] also worked on the same topic. But their results are limited to special regular domains and special initial partition.

§ 2. Algorithm and Analysis

Let Ω be a plane polygon. A series of nested triangulations of Ω are produced

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as follows: An initial partition Δ_0 divides Ω into a few large triangles. Then, successive midpoint refinements produce a series of partitions $\Delta_0, \Delta_1, \cdots, \Delta_k, \cdots$ with corresponding mesh sizes $h_0, h_1, \cdots, h_k, \cdots$ and $h_{k-1}=2h_k$. A series of linear finite element equations,

$$A_k u_k = f_k \tag{3}$$

corresponding to the partition Δ_k, are solved one by one. Now, an algorithm is given as follows.

1. For $k=0, 1$, solve $\tilde{u}_0 = A_0^{-1} f_0$, $\tilde{u}_1 = A_1^{-1} f_1$ directly.
2. For $k \geqslant 2$, take the initial approximation

$$u_k^0 = \Pi(\tilde{u}_{k-2}, \tilde{u}_{k-1}) \tag{4}$$

and then perform MG iteration r times to obtain $\tilde{u}_k$.

3. If $\tilde{u}_k$ is accurate enough according to some stopping criteria such as given in [1], stop and go to do the type 1 extrapolation; otherwise go to step 2.

The MG algorithm is referred to [7]. This paper mainly deals with the initial choice of (4).

Theorem. *Let constants c_1 and c_2 satisfy, for $k=2, 3, \cdots$*

$$\rho_k \leqslant \rho < 1, \tag{5}$$

$$\|u_k - \Pi(u_{k-2}, u_{k-1})\|_{L_2(\Omega)} \leqslant c_1 h^\alpha, \quad \alpha > 0, \tag{6}$$

$$\|\Pi(u_{k-2}, u_{k-1}) - \Pi(\tilde{u}_{k-2}, \tilde{u}_{k-1})\|_{L_2(\Omega)} \leqslant c_2(\|u_{k-2} - \tilde{u}_{k-2}\|_{L_2(\Omega)} + \|u_{k-1} - \tilde{u}_{k-1}\|_{L_2(\Omega)}). \tag{7}$$

Constant ρ_k stands for the convergence factor of the MG iteration on Δ_k in the sense of L_2-norm. Then, when r makes $2c_2\rho^r < 1$,

$$\|u_k - \tilde{u}_k\|_{L_2(\Omega)} \leqslant c(\rho) h^\alpha, \quad k = 0, 1, 2, \tag{8}$$

holds with $c(\rho) = c_1\rho^r/(1 - c_2\rho^r)$.

Proof. By induction. For $j=0, 1$, $u_j = \tilde{u}_j$ and (8) is trivial. Suppose now (8) is true for $j \leqslant k-1$; then, for $j=k$,

$$\begin{aligned}
\|u_k^0 - u_k\|_{L_2(\Omega)} &= \|\Pi(\tilde{u}_{k-2}, \tilde{u}_{k-1}) - u_k\|_{L_2(\Omega)} \\
&\leqslant \|\Pi(u_{k-2}, u_{k-1}) - u_k\|_{L_2(\Omega)} + \|\Pi(u_{k-2}, u_{k-1}) - \Pi(\tilde{u}_{k-2}, \tilde{u}_{k-1})\|_{L_2(\Omega)} \\
&\leqslant c_1 h^\alpha + c_2(\|u_{k-2} - \tilde{u}_{k-2}\|_{L_2(\Omega)} + \|u_{k-1} - \tilde{u}_{k-1}\|_{L_2(\Omega)}) \\
&\leqslant (c_1 + 2c_2 c(\rho)) h^\alpha,
\end{aligned}$$

$$\|u_k - \tilde{u}_k\|_{L_2(\Omega)} \leqslant \rho^r \|u_k^0 - u_k\|_{L_2(\Omega)} \leqslant \rho^r (c_1 + 2c_2 c(\rho)) h^\alpha = c(\rho) h^\alpha.$$

The proof is thus completed.

The norm in the above theorem can be replaced by other norms as long as the corresponding (5), (6) and (7) hold.

§ 3. The Choice of Initial Approximations

Suppose that

$$u^h(z) = u^1(z) + w(z) h^2 + O(h^\tau), \quad z \in \Omega$$

with $\tau > 2$. We show how to define $\Pi(u_{k-2}, u_{k-1})$ such that (6) and (7) hold for $\alpha > 2$.

Mesh sizes of Δ_{k-2}, Δ_{k-1} and Δ_k are $4h$, $2h$ and h; correspondingly, interpolation functions of u are denoted by u^{4I}, u^{2I} and u^I. Thus,

$$u^{4h}(z)=u^{4I}(z)+16w(z)h^2+O(h^\tau),$$
$$u^{2h}(z)=u^{2I}(z)+4w(z)h^2+O(h^\tau).$$

From now on Δ_k also represents the set of nodes of the k-th partition.

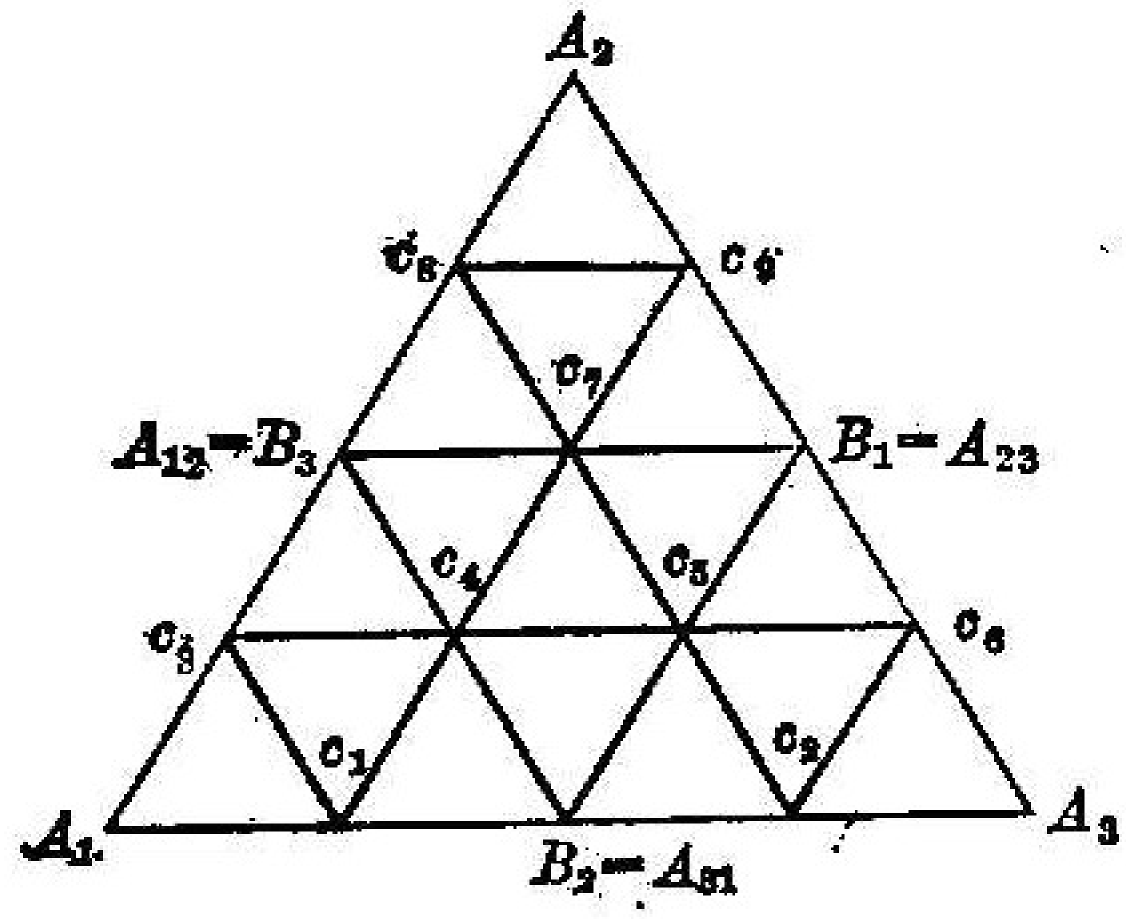

Fig. 1 Typical element T of Δ_k

1. For $z\in\Delta_{k-2}$, $u^{4I}(z)=u^{2I}(z)=u^I(z)=u(z)$, it is easy to see that

$$u_k(z)=\frac{5}{4}\,u_{k-1}(z)-\frac{1}{4}\,u_{k-2}(z)+O(h^\tau).$$

So, for $z=A_1$, A_2 and A_3 in Fig. 1, define

$$\Pi(u_{k-2},\,u_{k-1})(z)=\frac{5}{4}\,u_{k-1}(z)-\frac{1}{4}\,u_{k-2}(z).$$

2. For $z\in\Delta_{k-1}\backslash\Delta_{k-2}$, say $z=B_1$,

$$u^{4h}(B_1)=\frac{1}{2}(u(A_2)+u(A_3))+16w(B_1)h^2+O(h^\tau),$$
$$u^{2h}(B_1)=u(B_1)+4w(B_1)h^2+O(h^\tau),$$
$$u^h(B_1)=u(B_1)+w(B_1)h^2+O(h^\tau),$$

combined with

$$u(A_i)=\frac{4}{3}\,u^{2h}(A_i)-\frac{1}{3}\,u^{4h}(A_i)+O(h^\tau) \tag{9}$$

lead to

$$u^h(B_1)=u^{2h}(B_1)+\frac{1}{8}[u^{2h}(A_2)-u^{4h}(A_2)+u^{2h}(A_3)-u^{4h}(A_3)]+O(h^\tau).$$

So, define

$$\Pi(u_{k-2},\,u_{k-1})(B_1)=u_{k-1}(B_1)+\frac{1}{8}[u_{k-1}(A_2)-u_{k-2}(A_2)+u_{k-1}(A_3)-u_{k-2}(A_3)]$$

and similarly for $z=B_2$, B_3.

3. For $z\in\Delta_k\backslash\Delta_{k-1}$, i.e. $z=c_1$, c_2, $\cdots$, c_9, take $\hat{u}$ as the quadratic interpolation function of u with nodes A_1, A_2, A_3, B_1, B_2 and B_3. Then the values of $\hat{u}$ at c_1, c_2, $\cdots$, c_9 can be expressed as:

$$
\begin{pmatrix} \hat{u}(c_1) \\ \hat{u}(c_2) \\ \hat{u}(c_3) \\ \hat{u}(c_4) \\ \hat{u}(c_5) \\ \hat{u}(c_6) \\ \hat{u}(c_7) \\ \hat{u}(c_8) \\ \hat{u}(c_9) \end{pmatrix} = \begin{pmatrix} \frac{3}{8} & 0 & -\frac{1}{8} & 0 & \frac{3}{4} & 0 \\ -\frac{1}{8} & 0 & \frac{3}{8} & 0 & \frac{3}{4} & 0 \\ \frac{3}{8} & -\frac{1}{8} & 0 & 0 & 0 & \frac{3}{4} \\ 0 & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{8} & -\frac{1}{8} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\ 0 & -\frac{1}{8} & \frac{3}{8} & \frac{3}{4} & 0 & 0 \\ -\frac{1}{8} & 0 & -\frac{1}{8} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{8} & \frac{3}{8} & 0 & 0 & 0 & \frac{3}{4} \\ 0 & \frac{3}{8} & -\frac{1}{8} & \frac{3}{4} & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} u(A_1) \\ u(A_2) \\ u(A_3) \\ u(B_1) \\ u(B_2) \\ u(B_3) \end{pmatrix}.
$$

On T,

$$\|u-\hat{u}\|_{0,\infty}\leqslant ch^3$$

and for $z=c_1, c_2, \cdots, c_9$

$$u^h(z)=\hat{u}(z)+w(z)h^2+O(h^\alpha),$$
$$u^{2h}(z)=\hat{u}(z)+4w(z)h^2+O(h^\alpha), \quad \alpha=\min(\tau, 3)$$

hold. Therefore

$$u^h(z)=\frac{3}{4}\hat{u}(z)+\frac{1}{4}u^{2h}(z)+O(h^\alpha).$$

In the expression of $\hat{u}(z)$, combining (9) and

$$u(A_{ij})=u^{2h}(A_{ij})+\frac{1}{6}[u^{2h}(A_i)-u^{4h}(A_i)+u^{2h}(A_j)-u^{4h}(A_j)]+O(h^\tau),$$
$$i, j=1, 2, 3, B_1=A_{23}, B_2=A_{31}, B_3=A_{12}, \tag{10}$$

one can get the approximation to $u^h(z)$ with order $O(h^\alpha)$.

In detail, for $c_1, c_2, c_3, c_6, c_8, c_9$, say c_1,

$$u^h(c_1)=\frac{19}{32}u^{2h}(A_1)-\frac{3}{16}u^{4h}(A_1)+\frac{11}{16}u^{2h}(B_2)-\frac{1}{32}u^{2h}(A_3)$$
$$-\frac{1}{16}u^{4h}(A_3)+O(h^\alpha).$$

So, define

$$\Pi(u_{k-2}, u_{k-1})(c_1)=\frac{19}{32}u_{k-1}(A_1)-\frac{3}{16}u_{k-2}(A_1)+\frac{11}{16}u_{k-1}(B_2)$$
$$-\frac{1}{32}u_{k-1}(A_3)-\frac{1}{16}u_{k-2}(A_3).$$

Similarly, for c_4, c_5, c_7, say c_4,

$$u^h(c_4)=\frac{3}{8}u^{2h}(A_1)-\frac{3}{16}u^{4h}(A_1)-\frac{1}{16}(u^{2h}(A_2)+u^{2h}(A_3))$$
$$-\frac{1}{32}(u^{4h}(A_2)+u^{4h}(A_3))+\frac{1}{2}(u^{2h}(B_2)+u^{2h}(B_3))+O(h^\alpha)$$

and thus define

$$\Pi(u_{k-2}, u_{k-1})(c_4)=\frac{3}{8}u_{k-1}(A_1)-\frac{3}{16}u_{k-2}(A_1)-\frac{1}{16}(u_{k-1}(A_2)+u_{k-1}(A_3))$$
$$-\frac{1}{32}(u_{k-2}(A_2)+u_{k-2}(A_3))+\frac{1}{2}(u_{k-1}(B_2)+u_{k-1}(B_3)).$$

Defining $\Pi(u_{k-2}, u_{k-1})$ as above, we get

$$\|u_k-\Pi(u_{k-2}, u_{k-1})\|_{L_2(\Omega)}\leqslant Ch^\alpha, \quad \alpha=\min(\tau, 3).$$

Now we come to show that the interpolation defined above also satisfies (7). Let f be a linear function on T with values f_1, f_2 and f_3 at three vertices, then

$$\|f\|^2_{L_2(T)}=\frac{\mathrm{meas}(T)}{6}(f_1^2+f_2^2+f_3^2+f_1f_2+f_2f_3+f_3f_1)$$

holds and when f is a piecewise linear function on Δ_k,

$$\frac{N_k}{12}\sigma_k\sum_{z\in\Delta_k}f^2(z)\leqslant\|f\|^2_{L_2(\Omega)}\leqslant\frac{N_k^*}{3}\sigma_k^*\sum_{z\in\Delta_k}f^2(z)$$

hold where σ_k^*, σ_k are the maximal and minimal area among all elements and N_k^*, N_k are the maximal and minimal value among numbers of elements around each node of Δ_k. For midpoint refinement, $\sigma_k^*/\sigma_k=\sigma_0^*/\sigma_0$, $N_k^*=\max(6, N_0^*)$, $N_k=\max(6, N_0)$.

Because of the linearity of Π, denoting $\delta_1=u_{k-1}-\tilde{u}_{k-1}$ and $\delta_2=u_{k-2}-\tilde{u}_{k-2}$, we have

$$\|\Pi(u_{k-2}, u_{k-1})-\Pi(\tilde{u}_{k-2}, \tilde{u}_{k-1})\|^2_{L_2(\Omega)}$$
$$=\|\Pi(\delta_2, \delta_1)\|^2_{L_2(\Omega)}\leqslant\frac{N_k^*}{3}\sigma_k^*\sum_{z\in\Delta_k}[\Pi(\delta_2, \delta_1)(z)]^2$$
$$\leqslant CN_k^*\sigma_k^*[\sum_{z\in\Delta_{k-1}}\delta_1^2(z)+\sum_{z\in\Delta_{k-2}}\delta_2^2(z)]$$
$$\leqslant C\frac{N_k^*}{N_k}\sigma_k^*\left[\frac{1}{\sigma_{k-1}}\|\delta_1\|^2_{L_2(\Omega)}+\frac{1}{\sigma_{k-2}}\|\delta_2\|^2_{L_2(\Omega)}\right]$$
$$=C\frac{N_k}{N_k^*}\sigma_k^*\left[\frac{1}{4\sigma_k}\|\delta_1\|^2_{L_2(\Omega)}+\frac{1}{16\sigma_k}\|\sigma_2\|^2_{L_2(\Omega)}\right]$$
$$\leqslant C(N_0^*, N_0, \sigma_0^*, \sigma_0)(\|u_{k-1}-\tilde{u}_{k-1}\|_{L_2(\Omega)}+\|u_{k-2}-\tilde{u}_{k-2}\|_{L_2(\Omega)})^2,$$

where C stands for a general constant. This leads to (7).

Thus, the algorithm defined in § 2 with $\Pi(u_{k-2}, u_{k-1})$ defined in this section can give approximations with accuracy $O(h^\alpha)$ to solutions of (3) with $O(N)$ cost. By type 1 extrapolation from these data, an approximation to the solution of PDEs with order $O(h^\alpha)$ can be obtained where $\alpha=\min(\tau, 3)>2$.

References

[1] Huang Hong-ci, Liu Gui-yin, Successive Refinement Techniques for Solving Elliptic Boundary Problems, *Mathematica Numerica Sinica* (in Chinese), 1978, 28—35, 41—52.

[2] Lin Qun, Lü Tao, Shen Shu-min, Maximum Norm Estimate, Extrapolation and Optimal Points of Stresses for the Finite Element Methods on the Strongly Regular Triangulation, *Journal of Computational Mathematics*, 1: 4 (1983), 376—383.

[3] Lin Qun, Xu Jin-chao, Linear Finite Elements with High Accuracy, *Journal of Computational Mathematics*, **3**: 2 (1985), 115—133.
[4] Huang Hong-ci, Mu Mo, Han Wei-min, Extrapolation for Less Regular Solution of Elliptic Boundary Problems, The Third National Conference of Computational Mathematics, Beijing, 1985.
[5] Cai Zhi-qiang, Wang Neng-chao, Kan Li-shan, MGE Method for Elliptic Boundary Value Problem *Mathematica Numerica Sinica* (in Chinese), **8**: 1 (1986), 82—89.
[6] Lin Qun, A. Sehülher, Using Multigrid Techniques in Connection with Extrapolation Methods.