THE STABILITY ANALYSIS OF THE SOLUTIONS OF INVERSE EIGENVALUE PROBLEMS*

SUN JI-GUANG (孙继广)
(*Computing Center, Academia Sinica, Beijing, China*)

Abstract

This paper gives perturbation bounds of some solutions of the classical additive and multiplicative inverse eigenvalue problems for real symmetric matrices.

§ 1. Problems and Main Results

Throughout this paper we use the following notation. $S\mathbb{R}_0^{n\times n}$ is the set of all $n\times n$ real symmetric matrices with zero diagonal elements, and $S\mathbb{R}_1^{n\times n}$ the set of all $n\times n$ real symmetric matrices with unit diagonal elements. $\mathbb{R}^n$ denotes the set of all n-dimensional real column vectors. The norm $\|\ \|_1$ stands for both the vector 1-norm and the matrix 1-norm. The superscript T is for transpose. For an arbitrary $n\times n$ real symmetric matrix A with eigenvalues $\lambda_1\geqslant\cdots\geqslant\lambda_n$, the symbol $\mu(A)$ denotes the vector $(\lambda_1, \cdots, \lambda_n)^T\in\mathbb{R}^n$.

The following are the most common inverse eigenvalue problems:

Problem $\boldsymbol{A}(A, \lambda)$. Given $A=(a_{ij})\in S\mathbb{R}_0^{n\times n}$ and $\lambda=(\lambda_1, \cdots, \lambda_n)^T\in\mathbb{R}^n$, find $c=(c_1, \cdots, c_n)^T\in\mathbb{R}^n$ such that the eigenvalues of $A+\mathrm{diag}(c_1, \cdots, c_n)$ are $\lambda_1, \cdots, \lambda_n$.

Problem $\boldsymbol{M}(A, \lambda)$. Given a positive definite matrix $A=(a_{ij})\in S\mathbb{R}_1^{n\times n}$ and $\lambda=(\lambda_1, \cdots, \lambda_n)^T\in\mathbb{R}^n$ with $\lambda_i>0\ (i=1, \cdots, n)$, find $c=(c_1, \cdots, c_n)^T\in\mathbb{R}^n$ such that the eigenvalues of $\mathrm{diag}(c_1, \cdots, c_n)A$ are $\lambda_1, \cdots, \lambda_n$.

Problem $\boldsymbol{A}$ is the classical additive inverse eigenvalue problem and Problem $\boldsymbol{M}$ the multiplicative inverse eigenvalue problem. The solubility and the numbers of solutions $c\in\mathbb{R}^n$ as well as numerical methods for Problem $\boldsymbol{A}$ and Problem $\boldsymbol{M}$ have been studied (see [1], [2], [6] and the references contained therein). Nevertheless, to the best of the author's knowledge, the stability analysis of the solutions of Problem $\boldsymbol{A}$ and Problem $\boldsymbol{M}$ is not yet treated, and it is the subject of this paper.

Let

$$g_j=\sum_k|a_{jk}|, \quad j=1, \cdots, n \tag{1.1}$$

and

$$\mathscr{D}_\varepsilon=\{c=(c_1, \cdots, c_n)^T\in\mathbb{R}^n:\ \lambda_1+\varepsilon\geqslant c_1\geqslant c_2\geqslant\cdots\geqslant c_n\geqslant\lambda_n-\varepsilon\},$$

where $\varepsilon>0$ for Problem $\boldsymbol{A}$, and $\lambda_n>\varepsilon>0$ for Problem $\boldsymbol{M}$. The following theorems have been proved by Hadeler[2].

Theorem H-1. *If $A=(a_{ij})\in S\mathbb{R}_0^{n\times n}$ and $\lambda=(\lambda_1, \cdots, \lambda_n)^T\in\mathbb{R}^n$ satisfy $\lambda_1>\lambda_2>\cdots>\lambda_n$ and*

* Received August 5, 1985.

$$\lambda_j-\lambda_{j+1}>2\max\{g_j,\ g_{j+1}\},\quad j=1,\ \cdots,\ n-1,$$

then there exists a unique solution $c=(c_1,\ \cdots,\ c_n)^T$ *for Problem* $\boldsymbol{A}(A,\ \lambda)$ *in* $\mathscr{D}_s$, *and*

$$|c_j-\lambda_j|\leqslant g_j,\quad j=1,\ \cdots,\ n. \tag{1.2}$$

Theorem H-2. *If* $A=(a_{ij})\in S\mathbb{R}_1^{n\times n}$ *and* $\lambda=(\lambda_1,\cdots,\lambda_n)^T\in\mathbb{R}^n$ *satisfy* $\lambda_1>\lambda_2>\cdots>\lambda_n>0$ *and*

$$\lambda_j-\lambda_{j+1}>2\lambda_1\max\{g_j,\ g_{j+1}\},\quad j=1,\ \cdots,\ n-1,$$

then there exists a unique solution $c=(c_1,\ \cdots,\ c_n)^T$ *for Problem* $\boldsymbol{M}(A,\ \lambda)$ *in* $\mathscr{D}_s$, *and*

$$|c_j-\lambda_j|\leqslant\lambda_1 g_j,\quad j=1,\ \cdots,\ n. \tag{1.3}$$

On the basis of Hadeler's theorems we shall prove the following results.

Theorem 1. *Let* $A=(a_{ij}),\ \widetilde{A}=(\widetilde{a}_{ij})\in S\mathbb{R}_0^{n\times n},\ \lambda=(\lambda_1,\ \cdots,\ \lambda_n)^T,\ \widetilde{\lambda}=(\widetilde{\lambda}_1,\ \cdots,\ \widetilde{\lambda}_n)^T\in\mathbb{R}^n$. *Assume that*

$$\lambda_j-\lambda_{j+1}>2\max\{g_j,\ g_{j+1}\},\ \widetilde{\lambda}_j-\widetilde{\lambda}_{j+1}>2\max\{\widetilde{g}_j,\ \widetilde{g}_{j+1}\},\quad j=1,\ \cdots,\ n-1, \tag{1.4}$$

where g_j *is defined by* (1.1), *and*

$$\widetilde{g}_j=\sum_{k\neq j}|\widetilde{a}_{jk}|,\quad j=1,\ \cdots,\ n. \tag{1.5}$$

Suppose that $c=(c_1,\ \cdots,\ c_n)^T\in\mathscr{D}_s$ *and* $\widetilde{c}=(\widetilde{c}_1,\ \cdots,\ \widetilde{c}_n)^T\in\widetilde{\mathscr{D}}_s$ *are the solutions of Problem* $\boldsymbol{A}(A,\ \lambda)$ *and Problem* $\boldsymbol{A}(\widetilde{A},\ \widetilde{\lambda})$, *respectively, where*

$$\widetilde{\mathscr{D}}_s=\{c=(c_1,\ \cdots,\ c_n)^T\in\mathbb{R}^n:\ \widetilde{\lambda}_1+s\geqslant c_1\geqslant c_2\geqslant\cdots\geqslant c_n\geqslant\widetilde{\lambda}_n-s\},\quad s>0.$$

Then

$$\|\widetilde{c}-c\|_1<\frac{D}{\delta_A}(2\|\widetilde{a}-a\|_1+\|\widetilde{\lambda}-\lambda\|_1), \tag{1.6}$$

where

$$a=(a_{12},\ \cdots,\ a_{1n},\ a_{23},\ \cdots,\ a_{2n},\ \cdots,\ a_{n-1,n})^T, \tag{1.7}$$

$$\widetilde{a}=(\widetilde{a}_{12},\ \cdots,\ \widetilde{a}_{1n},\ \widetilde{a}_{23},\ \cdots,\ \widetilde{a}_{2n},\ \cdots,\ \widetilde{a}_{n-1,n})^T, \tag{1.8}$$

$$D=\max_{1\leqslant j\leqslant n-1}\max\{\lambda_j-\lambda_{j+1},\ \widetilde{\lambda}_j-\widetilde{\lambda}_{j+1}\}, \tag{1.9}$$

$$\delta_A=\min_i\delta_i'$$

and

$$\delta_i'=\min\{\min_{j\neq i}|\lambda_j-\lambda_i|-2g_i,\ \min_{j\neq i}|\widetilde{\lambda}_j-\widetilde{\lambda}_i|-2\widetilde{g}_i\},\quad i=1,\ \cdots,\ n.$$

Theorem 2. *Let* $A=(a_{ij}),\ \widetilde{A}=(\widetilde{a}_{ij})\in S\mathbb{R}_1^{n\times n},\ \lambda=(\lambda_1,\ \cdots,\ \lambda_n)^T,\ \lambda_1>\cdots>\lambda_n>0,\ \widetilde{\lambda}=(\widetilde{\lambda}_1,\ \cdots,\ \widetilde{\lambda}_n)^T,\ \widetilde{\lambda}_1>\cdots>\widetilde{\lambda}_n>0$. *Assume that*

$$\lambda_j-\lambda_{j+1}>2\lambda_1^*\max\{g_j,\ g_{j+1}\},\ \widetilde{\lambda}_j-\widetilde{\lambda}_{j+1}>2\lambda_1^*\max\{\widetilde{g}_j,\ \widetilde{g}_{j+1}\},\quad j=1,\ \cdots,\ n-1, \tag{1.10}$$

where g_j *and* $\widetilde{g}_j$ *are defined by* (1.1) *and* (1.5), *respectively, and*

$$\lambda_1^*=\max\{\lambda_1,\ \widetilde{\lambda}_1\}.$$

Suppose that $c=(c_1,\ \cdots,\ c_n)^T\in\mathscr{D}_s$ *and* $\widetilde{c}=(\widetilde{c}_1,\ \cdots,\ \widetilde{c}_n)^T\in\widetilde{\mathscr{D}}_s(\widetilde{\lambda}_n>s>0)$ *are the solutions of Problem* $\boldsymbol{M}(A,\ \lambda)$ *and Problem* $\boldsymbol{M}(\widetilde{A},\ \widetilde{\lambda})$, *respectively. Then*

$$\|\widetilde{c}-c\|_1<\frac{\lambda_1^*D}{\lambda_n^*\delta_M}(2\lambda_1^*\|\widetilde{a}-a\|_1+\|\widetilde{\lambda}-\lambda\|_1), \tag{1.11}$$

where a, $\widetilde{a}$ *and* D *are defined by* (1.7)—(1.9), *and*

$$\lambda_n^*=\min\{\lambda_n,\ \widetilde{\lambda}_n\},$$

$$\delta_M = \min_i \delta_i''$$

and

$$\delta_i'' = \min\{\min_{j\neq i} |\lambda_j - \lambda_i| - 2\lambda_1^* g_i,\ \min_{j\neq i} |\tilde{\lambda}_j - \tilde{\lambda}_i| - 2\lambda_1^* \tilde{g}_i\},\quad i=1, \cdots, n.$$

§ 2. Proof of Theorem 1

Let

$$C = \mathrm{diag}(c_1, \cdots, c_n),\quad \widetilde{C} = \mathrm{diag}(\tilde{c}_1, \cdots, \tilde{c}_n), \tag{2.1}$$

$$\lambda(t) = (\lambda_1(t), \cdots, \lambda_n(t))^T = (1-t)\lambda + t\tilde{\lambda}, \tag{2.2}$$

$$A(t) = (a_{ij}(t)) = (1-t)A + t\widetilde{A} \tag{2.3}$$

and

$$g_j(t) = \sum_{k\neq j} |a_{jk}(t)|,\quad j=1, \cdots, n, \tag{2.4}$$

where $0 \leqslant t \leqslant 1$. Observe that

$$g_j(t) = \sum_{k\neq j} |(1-t)a_{jk} + t\tilde{a}_{jk}| \leqslant (1-t)g_j + t\tilde{g}_j,\quad j=1, \cdots, n. \tag{2.5}$$

Therefore from (2.2) and (1.4) we have

$$\begin{aligned}\lambda_j(t) - \lambda_{j+1}(t) &= (1-t)(\lambda_j - \lambda_{j+1}) + t(\tilde{\lambda}_j - \tilde{\lambda}_{j+1}) \\ &> 2(1-t)\max\{g_j, g_{j+1}\} + 2t\max\{\tilde{g}_j, \tilde{g}_{j+1}\} \\ &\geqslant 2\max\{(1-t)g_j + t\tilde{g}_j, (1-t)g_{j+1} + t\tilde{g}_{j+1}\} \\ &\geqslant 2\max\{g_j(t), g_{j+1}(t)\},\quad j=1, \cdots, n-1,\ \forall t\in[0, 1]. \end{aligned} \tag{2.6}$$

Hence, there exists a unique solution $c(t) = (c_1(t), \cdots, c_n(t))^T$ of Problem $\boldsymbol{A}(A(t), \lambda(t))$ in the domain

$$\mathscr{D}_\varepsilon(t) = \{c = (c_1, \cdots, c_n)^T \in \mathbb{R}^n:\ \lambda_1(t) + \varepsilon \geqslant c_1 \geqslant c_2 \geqslant \cdots \geqslant c_n \geqslant \lambda_n(t) - \varepsilon\},\quad \varepsilon > 0$$

for an arbitrary point $t\in[0, 1]$. By the minimax property of eigenvalues of real symmetric matrices and Theorem H-1 (see (1.2)), $c(t)$ is also the unique solution of Problem $\boldsymbol{A}(A(t), \lambda(t))$ in the domain $c_1 > c_2 > \cdots > c_n$ of $\mathbb{R}^n$.

Set

$$C(t) = \mathrm{diag}(c_1(t), \cdots, c_n(t)),\quad 0 \leqslant t \leqslant 1.$$

According to the definition of $\mu(\cdot)$, we have

$$\mu(A(t) + C(t)) = \lambda(t),\quad 0 \leqslant t \leqslant 1.$$

Let

$$a(t) = (a_{12}(t), \cdots, a_{1n}(t), a_{23}(t), \cdots, a_{2n}(t), \cdots, a_{n-1,n}(t))^T,$$

$$\Phi(a(t), c(t)) = \mu(A(t) + C(t))$$

and

$$\phi(a(t), c(t), \lambda(t)) = \Phi(a(t), c(t)) - \lambda(t),\quad 0 \leqslant t \leqslant 1,$$

where $\Phi = (\Phi_1, \cdots, \Phi_n)^T$ and $\phi = (\phi_1, \cdots, \phi_n)^T$. From (2.6),

$$\lambda_1(t) > \lambda_2(t) > \cdots > \lambda_n(t).$$

Therefore by [4] we know that for an arbitrary fixed point $t\in[0, 1]$, $\Phi(a(t), c(t))$ is a real analytic function of elements of $a(t)$ and $c(t)$, and that if

$$A(t) + C(t) = U(t)\Lambda(t)U(t)^T$$

is a real orthogonal decomposition of $A(t) + C(t)$, where $U(t) = (u_{ij}(t))$ is an $n\times n$ real orthogonal matrix and $\Lambda(t) = \mathrm{diag}(\lambda_1(t), \cdots, \lambda_n(t))$, then the Jacobian matrix of the function ϕ at $c(t)$ is

$$\frac{\partial\phi(a(t),c(t),\lambda(t))}{\partial c(t)}=\frac{\partial\Phi(a(t),c(t))}{\partial c(t)}$$
$$=\begin{pmatrix}\frac{\partial\Phi_1(a(t),c(t))}{\partial c_1(t)} & \cdots & \frac{\partial\Phi_1(a(t),c(t))}{\partial c_n(t)}\\ \vdots & & \vdots\\ \frac{\partial\Phi_n(a(t),c(t))}{\partial c_1(t)} & \cdots & \frac{\partial\Phi_n(a(t),c(t))}{\partial c_n(t)}\end{pmatrix}$$
$$=(u_{ji}^2(t))=(w_{ij}(t))\equiv W(t). \tag{2.7}$$

According to the proof of Theorem 2 in [2], $\det W(t)>0$. Hence by the implicit function theorem from

$$\phi(a(t),c(t),\lambda(t))=0 \tag{2.8}$$

we get a unique real analytic solution

$$c(t)=p(a(t),\lambda(t))$$

at $(a(t)^T,\lambda(t)^T)^T$ for an arbitrary fixed $t\in[0,1]$.

Consequently it follows from (2.8) that

$$\frac{\partial c(t)}{\partial a(t)}=-\left(\frac{\partial\Phi(a(t),c(t))}{\partial c(t)}\right)^{-1}\cdot\frac{\partial\Phi(a(t),c(t))}{\partial a(t)}$$

and

$$\frac{\partial c(t)}{\partial\lambda(t)}=\left(\frac{\partial\Phi(a(t),c(t))}{\partial c(t)}\right)^{-1}.$$

Thus we obtain

$$\tilde{c}-c=\int_0^1 dc(t)=\int_0^1\left(\frac{\partial c(t)}{\partial a(t)}da(t)+\frac{\partial c(t)}{\partial\lambda(t)}d\lambda(t)\right)$$
$$=\int_0^1 W(t)^{-1}\left[-\frac{\partial\Phi(a(t),c(t))}{\partial a(t)}(\tilde{a}-a)+(\tilde{\lambda}-\lambda)\right]dt$$

and

$$\|\tilde{c}-c\|_1\leqslant\int_0^1\|W(t)^{-1}\|_1\left(\left\|\frac{\partial\Phi(a(t),c(t))}{\partial a(t)}\right\|_1\|\tilde{a}-a\|_1+\|\tilde{\lambda}-\lambda\|_1\right)dt, \tag{2.9}$$

where $W(t)$ is defined by (2.7).

According to the proof of Theorem 2 in [2] (see [2] p. 321), we have

$$2g_j^2(t)\geqslant\min_{k\neq j}(\lambda_k(t)-\lambda_j(t))^2\sum_{k\neq j}u_{jk}^2(t)$$

and

$$u_{jj}^2(t)-\sum_{k\neq j}u_{jk}^2(t)=1-2\sum_{k\neq j}u_{jk}^2(t)$$
$$\geqslant 1-\frac{(2g_j(t))^2}{\min\limits_{k\neq j}(\lambda_k(t)-\lambda_j(t))^2}\geqslant 1-\frac{2g_j(t)}{\min\limits_{k\neq j}|\lambda_k(t)-\lambda_j(t)|}$$
$$=\frac{\min\limits_{k\neq j}|\lambda_k(t)-\lambda_j(t)|-2g_j(t)}{\min\limits_{k\neq j}|\lambda_k(t)-\lambda_j(t)|},\quad j=1,\cdots,n.$$

Utilizing (2.2), (2.5) and (2.6) we can prove

$$\min_{k\neq j}|\lambda_k(t)-\lambda_j(t)|-2g_j(t)$$
$$>\min\{\min_{k\neq j}|\lambda_k-\lambda_j|-2g_j,\ \min_{k\neq j}|\tilde{\lambda}_k-\tilde{\lambda}_j|-2\tilde{g}_j\}=\delta_j' \tag{2.10}$$

and

$$\min_{k\neq j}|\lambda_k(t)-\lambda_j(t)|\leq\max_j\max\{\lambda_j-\lambda_{j+1},\ \tilde{\lambda}_j-\tilde{\lambda}_{j+1}\}=D; \tag{2.11}$$

thus

$$w_{jj}(t)-\sum_{k\neq j}w_{kj}(t)=u_{jj}^2(t)-\sum_{k\neq j}u_{jk}^2(t)>\frac{\delta_A}{D},\quad j=1,\ \cdots,\ n. \tag{2.12}$$

Hence (ref. [5], Corollary 1)

$$\|W(t)^{-1}\|_1<\frac{D}{\delta_A}. \tag{2.13}$$

By [4] (see [4] § 3) we have

$$\frac{\partial\Phi_k(a(t),\ c(t))}{\partial a_{ij}(t)}=2u_{ik}(t)u_{jk}(t),\quad k=1,\ \cdots,\ n,\ 1\leq i<j\leq n$$

and

$$\frac{\partial\Phi(a(t),\ c(t))}{\partial a(t)}$$

$$=2\begin{pmatrix} u_{11}(t)u_{21}(t)\cdots u_{11}(t)u_{n1}(t) & u_{21}(t)u_{31}(t)\cdots u_{21}(t)u_{n1}(t)\cdots u_{n-1,1}(t)u_{n1}(t) \\ u_{12}(t)u_{22}(t)\cdots u_{12}(t)u_{n2}(t) & u_{22}(t)u_{32}(t)\cdots u_{22}(t)u_{n2}(t)\cdots u_{n-1,2}(t)u_{n2}(t) \\ \vdots \quad\quad \vdots & \vdots \quad\quad \vdots \quad\quad \vdots \\ u_{1n}(t)u_{2n}(t)\cdots u_{1n}(t)u_{nn}(t) & u_{2n}(t)u_{3n}(t)\cdots u_{2n}(t)u_{nn}(t)\cdots u_{n-1,n}(t)u_{nn}(t) \end{pmatrix}$$

$$\equiv L(t). \tag{2.14}$$

Here $L(t)$ is an $n\times\frac{n(n-1)}{2}$ real matrix. Obviously, we have

$$\left\|\frac{\partial\Phi(a(t),\ c(t))}{\partial a(t)}\right\|_1\leq2. \tag{2.15}$$

Combining (2.13) and (2.15) with (2.9) we obtain the estimation (1.6). The proof of Theorem 1 is completed.

§ 3. Proof of Theorem 2

3.1. Let $A=(a_{ij})\in S\mathbb{R}_1^{n\times n}$ be a positive definite matrix, and $C=\mathrm{diag}(c_1,\ \cdots,\ c_n)$ with $c_i>0$, $i=1,\ \cdots,\ n$. Suppose that

$$\mu(CA)=\lambda,$$

where $\lambda=(\lambda_1,\ \cdots,\ \lambda_n)^T$ and $\lambda_1>\lambda_2>\cdots>\lambda_n>0$. Moreover, assume that

$$C^{\frac{1}{2}}AC^{\frac{1}{2}}=U\Lambda U^T \tag{3.1}$$

is a real orthogonal decomposition of $C^{\frac{1}{2}}AC^{\frac{1}{2}}$, where $\Lambda=\mathrm{diag}(\lambda_1,\ \cdots,\ \lambda_n)$, U is a real orthogonal matrix and

$$U=(u_1,\ \cdots,\ u_n),\quad u_j=(u_{1j},\ u_{2j},\ \cdots,\ u_{nj})^T.$$

If we consider the λ as a function of the elements of A and C, and write

$$\mu(CA)=\Psi(a,\ c)=(\Psi_1(a,\ c),\ \cdots,\ \Psi_n(a,\ c))^T,$$

where

$$a=(a_{12},\ \cdots,\ a_{1n},\ a_{23},\ \cdots,\ a_{2n},\ \cdots,\ a_{n-1,n})^T,\quad c=(c_1,\ \cdots,\ c_n)^T,$$

then we conclude that

$$\frac{\partial \Psi_i(a, c)}{\partial c_j} = \frac{\lambda_i u_{ji}^2}{c_j}, \quad i, j=1, \cdots, n. \tag{3.2}$$

The relations (3.2) can be proved as follows. Let

$$X=(x_1, \cdots, x_n)=C^{\frac{1}{2}}U, \quad Y=(y_1, \cdots, y_n)=C^{-\frac{1}{2}}U.$$

From (3.1)

$$Y(CA)X=\Lambda.$$

By Theorem 2.2 of [4] we have

$$\frac{\partial \Psi_i(a, c)}{\partial c_j} = y_i^T\left(\frac{\partial (CA)}{\partial c_j}\right)x_i$$

$$= u_i^T C^{-\frac{1}{2}} \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ a_{1j} \cdots a_{j-1,j} & 1 & a_{j,j+1} \cdots a_{jn} \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} C^{\frac{1}{2}} u_i, \quad 1 \leqslant i, j \leqslant n. \tag{3.3}$$

Observe that the decomposition (3.1) means that

$$AC^{\frac{1}{2}}U = C^{-\frac{1}{2}}U\Lambda.$$

Therefore

$$(a_{1j}, \cdots, a_{j-1,j}, 1, a_{j,j+1}, \cdots, a_{jn})C^{\frac{1}{2}}U = c_j^{-\frac{1}{2}}\lambda_i u_{ji}.$$

Substituting (3.3) we get

$$\frac{\partial \Psi_i(a, c)}{\partial c_j} = u_i^T C^{-\frac{1}{2}} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ c_j^{-\frac{1}{2}}\lambda_i u_{ji} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \frac{\lambda_i u_{ji}^2}{c_j},$$

i.e., the relations (3.2) are valid.

3.2. Suppose that $A=(a_{ij}) \in S\mathbb{R}_1^{n\times n}$ and $\lambda=(\lambda_1, \cdots, \lambda_n)^T \in \mathbb{R}^n$ satisfy the hypothesis of Theorem H-2, and $c=(c_1, \cdots, c_n)^T$ is the unique solution of Problem $M(A, \lambda)$ in $\mathscr{D}_s$. Set $C=\operatorname{diag}(c_1, \cdots, c_n)$, and assume that $C^{\frac{1}{2}}AC^{\frac{1}{2}}$ has a real orthogonal decomposition represented by (3.1). Let $W=(w_{ij})=(u_{ji}^2)$. Then we conclude that the matrix W is invertible and

$$\|W^{-1}\|_2 \leqslant \frac{D}{\delta_M^{(0)}}, \tag{3.4}$$

where D is defined by (1.10), and

$$\delta_M^{(0)} = \min_i \delta_i^{(0)},$$

$$\delta_i^{(0)} = \min_{j\neq i} |\lambda_j - \lambda_i| - 2\lambda_1 g_i, \quad i=1, \cdots, n.$$

The invertibility of W and inequality (3.4) can be proved as follows. Let e_i be the ith column of the $n\times n$ unit matrix, $i=1, \cdots, n$. From (1.1), (1.3) and $c_i \leqslant \lambda_1$ $(i=1, \cdots, n)$ it follows that

$$\|(C^{\frac{1}{2}}AC^{\frac{1}{2}} - \lambda_i I)e_i\|_2^2 = c_i\Big(\sum_{j\neq i}\sqrt{c_j}\,a_{ji}\Big)^2 + (c_i - \lambda_i)^2 \leqslant 2(\lambda_1 g_i)^2. \tag{3.5}$$

On the other hand, we have

$$\|(C^{\frac{1}{2}}AC^{\frac{1}{2}} - \lambda_i I)e_i\|_2^2 = \|U(A - \lambda_i I)U^T e_i\|_2^2 = \sum_{j\neq i}[(\lambda_j - \lambda_i)u_{ij}]^2$$

$$\geqslant \min_{j\neq i}(\lambda_j - \lambda_i)^2 \sum_{j\neq i} u_{ij}^2. \tag{3.6}$$

Combining (3.5) and (3.6) we get

$$\sum_{j\neq i} u_{ij}^2 \leqslant \frac{2(\lambda_1 g_i)^2}{\min\limits_{j\neq i}(\lambda_j - \lambda_i)^2}$$

and

$$u_{ii}^2 - \sum_{j\neq i} u_{ij}^2 = 1 - 2\sum_{j\neq i} u_{ij}^2 \geqslant 1 - \frac{(2\lambda_1 g_i)^2}{\min\limits_{j\neq i}(\lambda_j - \lambda_i)^2}$$

$$\geqslant 1 - \frac{2\lambda_1 g_i}{\min\limits_{j\neq i}|\lambda_j - \lambda_i|} = \frac{\min\limits_{j\neq i}|\lambda_j - \lambda_i| - 2\lambda_1 g_i}{\min\limits_{j\neq i}|\lambda_j - \lambda_i|}$$

$$\geqslant \frac{\delta_i^{(0)}}{D} \geqslant \frac{\delta_M^{(0)}}{D} > 0, \quad i=1, \cdots, n. \tag{3.7}$$

Hence by a result due to Varah[5] from (3.7) we know that the matrix W is invertible and the inequality (3.4) is valid.

3.3. Now we prove the inequality (1.11).

First we define C, $\widetilde{C}$, $\lambda(t)$, $A(t)$ and $g_j(t)$ $(j=1, \cdots, n)$ by (2.1)—(2.4). Utilizing the conditions (1.10) and inequalities (2.5) we get

$$\lambda_j(t) - \lambda_{j+1}(t) = (1-t)(\lambda_j - \lambda_{j+1}) + t(\tilde\lambda_j - \tilde\lambda_{j+1})$$

$$> 2\lambda_1^*[(1-t)\max\{g_j, g_{j+1}\} + t\max\{\tilde g_j, \tilde g_{j+1}\}]$$

$$\geqslant 2\lambda_1^*\max\{(1-t)g_j + t\tilde g_j, (1-t)g_{j+1} + t\tilde g_{j+1}\}$$

$$\geqslant 2\lambda_1^*\max\{g_j(t), g_{j+1}(t)\}, \quad j=1, \cdots, n-1. \tag{3.8}$$

Therefore there exists a unique solution $c(t) = (c_1(t), \cdots, c_n(t))^T$ of Problem $M(A(t), \lambda(t))$ in the domain

$$\mathscr{D}_\varepsilon(t) = \{c = (c_1, \cdots, c_n)^T \in \mathbb{R}^n: \lambda_1(t) + \varepsilon \geqslant c_1 \geqslant c_2 \geqslant \cdots \geqslant c_n \geqslant \lambda_n(t) - \varepsilon\}, \quad \lambda_n(t) > \varepsilon > 0$$

for an arbitrary point $t \in [0, 1]$. By the minimax property of eigenvalues of real symmetric matrices and Theorem H-2 (see (1.3)), $c(t)$ is also the unique solution of Problem $M(A(t), \lambda(t))$ in the domain $c_1 > c_2 > \cdots > c_n$ of $\mathbb{R}^n$.

Set

$$C(t) = \mathrm{diag}(c_1(t), \cdots, c_n(t)), \quad 0 \leqslant t \leqslant 1.$$

According to the definition of $\mu(\cdot)$, we have

$$\mu(C(t)A(t))=\lambda(t).$$

Let

$$a(t)=(a_{12}(t),\cdots,a_{1n}(t),a_{23}(t),\cdots,a_{2n}(t),a_{n-1,n}(t))^T,$$
$$\Psi(a(t),c(t))=\mu(C(t)A(t))$$

and

$$\psi(a(t),c(t),\lambda(t))=\Psi(a(t),c(t))-\lambda(t),\quad 0\leqslant t\leqslant 1,$$

where $\Psi=(\Psi_1,\cdots,\Psi_n)^T$ and $\psi=(\psi_1,\cdots,\psi_n)^T$. From (3.8)

$$\lambda_1(t)>\lambda_2(t)>\cdots>\lambda_n(t).$$

Therefore by [4] we know that for an arbitrary fixed point $t\in[0,1]$, $\Psi(a(t),c(t))$ is a real analytic function of elements of $a(t)$ and $c(t)$, and that if

$$C(t)^{\frac{1}{2}}A(t)C(t)^{\frac{1}{2}}=U(t)\Lambda(t)U(t)^T \tag{3.9}$$

is a real orthogonal decomposition of $C(t)^{\frac{1}{2}}A(t)C(t)^{\frac{1}{2}}$, where $U(t)=(u_{ij}(t))$ is an $n\times n$ real orthogonal matrix and $\Lambda(t)=\mathrm{diag}(\lambda_1(t),\cdots,\lambda_n(t))$, then from (3.2) the Jacobian matrix of the function ψ at $c(t)$ is

$$\frac{\partial\psi(a(t),c(t),\lambda(t))}{\partial c(t)}=\frac{\partial\Psi(a(t),c(t))}{\partial c(t)}=\Lambda(t)W(t)C(t)^{-1}. \tag{3.10}$$

Here $W(t)=(u_{ji}^2(t))$. According to 3.2 of this section $W(t)$ is invertible, and thus $\frac{\partial\psi(a(t),c(t),\lambda(t))}{\partial c(t)}$ is nonsingular. Hence by the implicit function theorem from

$$\psi(a(t),c(t),\lambda(t))=0 \tag{3.11}$$

we get a unique real analytic solution

$$c(t)=q(a(t),\lambda(t))$$

at $(a(t)^T,\lambda(t)^T)^T$ for an arbitrary fixed $t\in[0,1]$.

Consequently, it follows from (3.11) that

$$\frac{\partial c(t)}{\partial a(t)}=-\left(\frac{\partial\Psi(a(t),c(t))}{\partial c(t)}\right)^{-1}\cdot\frac{\partial\Psi(a(t),c(t))}{\partial a(t)}$$

and

$$\frac{\partial c(t)}{\partial\lambda(t)}=\left(\frac{\partial\Psi(a(t),c(t))}{\partial c(t)}\right)^{-1}.$$

Thus we obtain

$$\tilde c-c=\int_0^1 dc(t)=\int_0^1 C(t)W(t)^{-1}\Lambda(t)^{-1}\left[-\frac{\partial\Psi(a(t),c(t))}{\partial a(t)}(\tilde a-a)+(\tilde\lambda-\lambda)\right]dt$$

and

$$\|\tilde c-c\|_1\leqslant\int_0^1\|C(t)W(t)^{-1}\Lambda(t)^{-1}\|_1\left(\left\|\frac{\partial\Psi(a(t),c(t))}{\partial a(t)}\right\|_1\|\tilde a-a\|_1+\|\tilde\lambda-\lambda\|_1\right)dt. \tag{3.12}$$

According to 3.1 of this section we get

$$u_{ii}^2(t)-\sum_{j\neq i}u_{ij}^2(t)\geqslant 1-\frac{2\lambda_1(t)g_i(t)}{\min\limits_{j\neq i}|\lambda_j(t)-\lambda_i(t)|},\quad i=1,\cdots,n.$$

Utilizing an argument similar to that applied to inequalities (2.12) we can prove

$$u_{ii}^2(t)-\sum_{j\neq i}u_{ij}^2(t)>\frac{\delta_M}{D},\quad i=1, \cdots, n.$$

Hence (ref. [5])

$$\|W(t)^{-1}\|_1<\frac{D}{\delta_M},$$

and thus

$$\|C(t)W(t)^{-1}\Lambda(t)^{-1}\|_1\leqslant\frac{\lambda_1^*D}{\lambda_n^*\delta_M}. \tag{3.13}$$

By [4] (see [4] § 3) we have

$$\frac{\partial\Psi_k(a(t), c(t))}{\partial a_{ij}(t)}=2\sqrt{c_ic_j}\,u_{ik}(t)u_{jk}(t),\quad k=1, \cdots, n,\ 1\leqslant i<j\leqslant n$$

and

$$\frac{\partial\Psi(a(t), c(t))}{\partial a(t)}=L(t)\operatorname{diag}(\sqrt{c_1c_2}, \cdots, \sqrt{c_1c_n}, \sqrt{c_2c_3}, \cdots, \sqrt{c_2c_n}, \cdots, \sqrt{c_{n-1}c_n})$$
$$\equiv M(t),$$

where $L(t)$ takes the form of (2.14). Obviously, we have

$$\left\|\frac{\partial\Psi(a(t), c(t))}{\partial a(t)}\right\|_1\leqslant 2\max_{1\leqslant i<j\leqslant n}\sqrt{c_ic_j}\leqslant 2\lambda_1^*. \tag{3.14}$$

Combining (3.13) and (3.14) with (3.12) we obtain the estimation (1.11). The proof of Theorem 2 is completed.

References

[1] S. Friedland, Inverse eigenvalue problems, *Linear Algebra and Appl.*, **17** (1977), 15—51.

[2] K. P. Hadeler, Existenz-und Eindeutigkeitssätze für inverse Eigenwertaufgaben mit Hilfe des topologischen Abbildungsgrades, *Arch. Rat. Mech. and Anal.*, **42** (1971), 317—322.

[3] Ole H. Hald, Inverse eigenvalue problems for Jacobi matrices, *Linear Algebra and Appl.*, **14** (1976), 63—85.

[4] J. G. Sun, Eigenvalues and eigenvectors of a matrix dependent on several parameters, *J. Comp. Math.*, **3** (1985), 351—364.

[5] J. M. Varah, A lower bound for the smallest singular value of a matrix, *Linear Algebra and Appl.*, **11** (1975), 3—5.

[6] Q. Ye, An iterative method for solving inverse eigenvalue problems, submitted to Math. Numer. Sinica.