# ON MODIFIED HERMITE-FEJÉR INTERPOLATION OMITTING DERIVATIVES*1)

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## § 1. Introduction

Let us consider the Hermite-Fejér interpolation

$$H_n(f,x)=\sum_{k=1}^{n} f(x_k)h_{kn}(x), \tag{1.1}$$

on the interval $[-1, 1]$ for a function $f\in C\ [-1, 1]$ where

$$-1\leqslant x_{nn}<\cdots<x_{1n}\leqslant 1, \quad n=1, 2, \cdots,$$

$$W_n(x)=\prod_{k=1}^{n}(x-x_{kn}),$$

$$l_{kn}(x)=W_n(x)/[W'_n(x_{kn})(x-x_{kn})], \quad k=1, \cdots, n,$$

$$h_{kn}(x)=[1-W''_n(x_{kn})(x-x_{kn})/W'_n(x_{kn})]l^2_{kn}(x), \quad k=1, \cdots, n.$$

It is well-known that for zeros of Chebyshev polynomial $T_n(x)$

$$x_{kn}=\cos\theta_{kn}=\cos(2k-1)\pi/(2n), \quad k=1, \cdots, n, \tag{1.2}$$

according to a classical result of L. Fejér[1] $H_n(f, x)$ converges uniformly to $f(x)$. In 1960, P. Turán suggested that perhaps omission of derivatives at a "few" exceptional points would not damage the convergence property of the modified Hermite-Fejér polynomial $H^*_{\mu(n)}(f, x)$ with the nodes (1.2). In [2], P. Turán proved that $H^*_{\mu(n)}(f,x)$ does not converge uniformly in general. Later, K. Kumar and K. K. Mathur[3] considered the following question:

Is there any matrix of nodes for which the modified Hermite-Fejér interpolation $H^*_{\mu(n)}(f, x)$ given by

$$H^*_{\mu(n)}(f, x)=H_n(f, x)+(x-x_\mu)l^2_\mu(x)\,W'^2_n(x_\mu)\sum_{k=1}^{n} f(x_k)\frac{W''_n(x_k)}{W'^3_n(x_k)}, \tag{1.3}$$

satisfying the properties

$$H^*_{\mu(n)}(f, x_k)=f(x_k), \quad k=1, \cdots, n,$$

$$H^{*\prime}_{\mu(n)}(f, x_k)=0, \quad 1\leqslant k\leqslant n,\ k\neq 0,$$

converges uniformly to every $f\in C\ [-1, 1]$. They claimed an affirmative answer for the interpolation $H^*_{\mu(n)}(f, x)$ constructed on the point-systems

$$\{\cos(2k-1)\pi/(2n+1)\}_{k=1}^{n+1}, \tag{1.4}$$

$$\{\cos 2k\pi/(2n+1)\}_{k=0}^{n}, \tag{1.4$'$}$$

$$\{\cos(k-1)\pi/(n-1)\}_{k=0}^{n}. \tag{1.5}$$

* Received May 15, 1985. The Chinese version was received July 5, 1984.

1) Projects supported by the Science Fund of the Chinese Academy of Sciences.

But, their result was incorrect. In fact, even $H^*_{1(n)}(f_0, 1)$ with the nodes $\{\cos(2k-1)\pi/(2n+1)\}_{k=1}^{n+1}$ does not converge to $f_0(1)$, where $f_0(x)=x$.

On the other hand, P. Turán[2] proved thet uniform convergence of $H^*_{\mu(n)}(f, x)$ with the nodes (1.2) in $[-1, 1]$ holds if and only if

$$\int_{-1}^{1}\frac{xf(x)}{\sqrt{1-x^2}}dx=0. \tag{1.6}$$

Condition (1.6) is related to $f$. In the present paper the author considers the following question:

What are the necessary and sufficient conditions which ensure that the uniform convergence of $H^*_{\mu(n)}(f, x)$ still holds for every $f\in C[-1, 1]$ when a derivative out of the points (1.4) and (1.5) is omitted.

## § 2. Main Result

**Theorem 2.1.** *For the interpolation $H^*_{\mu(n)}(f, x)$ constructed on the pointsystem (1.4) uniform convergence to every $f\in C[-1, 1]$ holds if and only if*

$$n-\mu(n)=O(1). \tag{2.1}$$

*Proof.* Denote by $H_n(f, x)$ the Hermite-Fejér operator based on the nodes $\{\cos(2k-1)\pi/(2n+1)\}_{k=1}^{n+1}$. From (1.3) we have

$$H^*_{\mu(n)}(f, x)=H_n(f, x)+J_n(x), \tag{2.2}$$

$$J_n(x)=\frac{(1+x)^2P_n^{(-\frac{1}{2}, \frac{1}{2})}(x))^2}{x-x_\mu}\Big[\frac{2}{(2n+1)^2}\sum_{k=1}^{n}\frac{f(x_k)}{1+x_k}-\frac{2n(n+1)}{3(2n+1)^2}f(-1)\Big],$$

where $P_n^{(-\frac{1}{2}, \frac{1}{2})}(x)=\cos(2n+1)\frac{\theta}{2}\Big/\cos\frac{\theta}{2}(x=\cos\theta)$. To prove (2.1) is necessary, suppose that $H^*_{\mu(n)}(f, x)$ converges uniformly to every $f\in C[-1, 1]$. On using Theorem 1 of [4], i.e., $\lim_{n\to\infty}H_n(f, x)=f(x)$ uniformly for every $f\in C[-1, 1]$, we have that for every $f\in C[-1, 1]$

$$\lim_{n\to\infty}J_n(x)=0$$

holds uniformly. Particularly, when $x^*=\cos\theta^*$, $\theta^*=\theta_\mu-\pi/[2(2n+1)]$ and $f(x)=\Omega(1+x)$ where $\Omega(x)$ satisfies the following conditions:

$$\begin{aligned}&\text{(i) } \Omega(x)\in C[0, 2] \text{ and } \Omega(x) \text{ is nondecreasing,}\\ &\text{(ii) } \Omega(x)\geqslant 0\ (x\geqslant 0) \text{ and } \Omega(0)=0;\end{aligned} \tag{2.3}$$

noting that

$$\sum_{k=1}^{n}1/(1+x_k)=n(n+1)/3,$$

we have that

$$\lim_{n\to\infty}J_n(x^*)=\lim_{n\to\infty}\left\{\frac{\cos^2\theta^*/2}{\sin\frac{1}{2}(\theta_\mu-\theta^*)\sin\frac{1}{2}(\theta_\mu+\theta^*)}\cdot\frac{2}{(2n+1)^2}\sum_{k=1}^{n}\frac{\Omega(1+x_k)}{1+x_k}\right\}=0 \tag{2.4}$$

holds. From tne monotonicity of $\Omega(x)$ we obtain

$$\frac{2}{(2n+1)^2}\sum_{k=1}^{n}\frac{\Omega(1+x_k)}{1+x_k}\geqslant\frac{2}{(2n+1)\pi^2}\int_{\frac{1}{2n+1}}^{1/2}\frac{\Omega(2x^2)}{x^2}\,dx$$

$$\geqslant\frac{2}{(2n+1)\pi^2}\sum_{k=2}^{2n}\Omega\left(\frac{2}{(k+1)^2}\right). \tag{2.5}$$

On the other hand, it is easy to see that

$$\left|\sin\frac{1}{2}(\theta_\mu+\theta^*)\right|\sim\sin\theta_\mu,\quad \cos^2\theta^*/2\sim\cos^2\theta_\mu/2,\quad \mu\neq n+1$$

hence

$$\left|\cos^2\frac{\theta^*}{2}\Big/\left[\sin\frac{1}{2}(\theta_\mu-\theta^*)\sin\frac{1}{2}(\theta_\mu+\theta^*)\right]\right|\geqslant c\cdot n\cos\frac{1}{2}\theta_\mu\geqslant c(n-\mu+2),$$
$$1\leqslant\mu\leqslant n+1. \tag{2.6}$$

From (2.4)—(2.6), we have for every $\Omega(x)$ satisfying condition (2.3)

$$\lim_{n\to\infty}\frac{n-\mu}{n}\sum_{k=1}^{2n}\Omega\left(\frac{2}{(k+1)^2}\right)=0. \tag{2.7}$$

If $n-\mu(n)\neq O(1)$, since $n-\mu(n)$ is monotone, $n-\mu(n):=N(n)\to\infty\,(n\to\infty)$. Define the following functions $N(x)$ and $\Omega_0(x)$:

$$N(x):=\begin{cases}N(n), & x=n\geqslant 3,\\ N(3), & 0\leqslant x\leqslant 3,\\ \text{linear, otherwise,}\end{cases}$$

and

$$\Omega_0(x):=\begin{cases}N^{-1}(x^{-\frac{1}{2}}), & x>0,\\ 0, & x=0.\end{cases}$$

Obviously, $\Omega_0(x)$ satisfies (2.3). Then

$$|J_n(x^*)|\geqslant c\,\frac{N(n)}{n}\sum_{k=2}^{2n}\Omega_0\left(\frac{2}{(k+1)^2}\right)\geqslant cN(n)\Omega_0(1/n^2)\geqslant c>0.$$

This contradicts (2.7), which completes the proof of necessity.

Now, assume that $n-\mu(n)=O(1)$. Obviously,

$$\frac{2}{(2n+1)^2}\sum_{k=1}^{n}\frac{f(x_k)}{1+x_k}-\frac{2n(n+1)}{3(2n+1)^2}f(-1)=O\left(\frac{1}{n}\right)\sum_{k=1}^{n}\omega\left(f,\frac{1}{k^2}\right). \tag{2.8}$$

Since

$$\cos\frac{1}{2}\theta=\cos\frac{1}{2}(\theta\pm\theta_\mu)\cos\frac{1}{2}\theta_\mu\pm\sin\frac{1}{2}(\theta\pm\theta_\mu)\sin\frac{1}{2}\theta_\mu,$$

we have

$$\frac{(1+x)^2(P_n^{(-\frac{1}{2},\frac{1}{2})}(x))^2}{|x-x_\mu|}=\frac{2\cos^2\frac{\theta}{2}\cos^2(2n+1)\frac{\theta}{2}}{\left|\sin\frac{1}{2}(\theta-\theta_\mu)\sin\frac{1}{2}(\theta+\theta_\mu)\right|}=O(1). \tag{2.9}$$

Combining (2.8) and (2.9) and noting Theorem 1 of [4], we see

$$\lim_{n\to\infty}H^*_{\mu(n)}(f,x)=f(x)$$

holds uniformly. This completes the proof.

Similarly, we can prove

**Theorem 2.2.** *For the interpolation process* $Q^*_{\mu(n)}(f, x)$ *based on the point-system* (1.5), *uniform convergence for every* $f \in C[-1, 1]$ *holds if and only if*

$$\mu(n) = O(1) \text{ or } n - \mu(n) = O(1).$$

## References

[1] L. Fejér, *Math. Zeit.*, **32** (1930), 426—457.
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[4] V. Kumar *Publ. Math. Debrecen*, **24** (1977), 30—37.