ON ERROR ESTIMATE OF THE BOUNDARY ELEMENT METHOD FOR PARABOLIC EQUATIONS IN A TIME-DEPENDENT INTERVAL*

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Abstract

This paper discusses the direct boundary element method for parabolic equations in a time-dependent interval. An optimal estimate of the error in maximum norm for the boundary element collocation scheme is given.

§ 1. Introduction

Compared with the domain methods such as the finite difference method or the finite element method, the boundary element method reduces the dimensions of the problem by one, so that the amount of computational work can be greatly decreased. In recent years, therefore, some authors studied its applications to numerical solution of parabolic equations and moving boundary problems (e.g. [1]—[3]). However, little work on mathematical analysis of the convergence of the method has been done. The only published work, to the author's knowledge, is by K. Onishi ([4]). But, as pointed out by the author in [5], his proof is based on a wrong estimate of matrix norm and thus is incorrect. In [5], the author proved the uniform convergence of the boundary element method and gave an optimal error estimate in maximum norm for the one-dimensional heat equation, using the method of matrix analysis which is not, however, applicable to problems in a two-dimensional or time-dependent domain.

In this paper we give an optimal estimate of the error for the boundary element collocation scheme for heat equation in a time-dependent interval, using the theory of operator analysis. The two-dimensional case will be discussed in another paper.

§ 2. Parabolic Equation in a Time-dependent Interval

For definiteness, we consider the following heat equation:

$$k\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}=0,\quad 0<x<S(t),\ 0<t\leqslant T<\infty, \tag{2.1}$$

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BCs.

$$\frac{\partial u}{\partial x}(0, t)=q_1(t), \quad \frac{\partial u}{\partial x}(S(t), t)=q_2(t), \quad 0<t\leqslant T, \tag{2.2}$$

IC.
$$u(x, 0)=u_0(x), \quad 0\leqslant x\leqslant S(0) \quad (S(0)=L>0), \tag{2.3}$$

where $k>0$ is a constant, $q_i(t)$ $(i=1, 2)$ are bounded, $u_0(x)$ is Lipschitz continuous, $S(t)$ has continuous first derivative $\dot{S}(t)$ and is assumed, without loss of generality, to be a nondecreasing function of t.

§ 3. Boundary Integral Equation

The fundamental solution of (2-1) is

$$u^*(x, \xi; t, \tau)=\frac{1}{2\sqrt{\pi k(t-\tau)}}\exp\left[-\frac{(x-\xi)^2}{4k(t-\tau)}\right], \quad t>\tau. \tag{3.1}$$

Let

$$q^*(x, \xi; t, \tau)=\frac{\partial u^*}{\partial \xi}=\frac{(x-\xi)}{4\sqrt{\pi}\,(k(t-\tau))^{3/2}}\exp\left[-\frac{(x-\xi)^2}{4k(t-\tau)}\right], \quad t>\tau, \tag{3.2}$$

$$g_1(t)=2\left\{-k\int_0^t q_1(\tau)u^*(0, 0; t, \tau)d\tau+k\int_0^t q_2(\tau)u^*(0, S(\tau); t, \tau)d\tau \right.$$
$$\left. +\int_0^L u_0(\xi)u^*(0, \xi; t, 0)d\xi\right\}, \tag{3.3a}$$

$$g_2(t)=2\left\{-k\int_0^t q_1(\tau)u^*(S(t), 0; t, \tau)d\tau+k\int_0^t q_2(\tau)u^*(S(t), S(\tau); t, \tau)d\tau \right.$$
$$\left. +\int_0^L u_0(\xi)\, u^*(S(t), \xi; t, 0)d\xi\right\}. \tag{3.3b}$$

The boundary integral equation corresponding to (2.1)—(2.3) is ([1])

$$\frac{1}{2}u(0, t)-k\int_0^t q^*(0, 0; t, \tau)u(0, \tau)d\tau+k\int_0^t q^*(0, S(\tau), t, \tau)u(S(\tau), \tau)d\tau$$
$$-\int_0^t u^*(0, S(\tau), t, \tau)\dot{S}(\tau)u(S(\tau), \tau)d\tau=\frac{1}{2}g_1(t), \tag{3.4a}$$

$$\frac{1}{2}u(S(t), t)-k\int_0^t q^*(S(t), 0; t, \tau)u(0, \tau)d\tau$$
$$+k\int_0^t q^*(S(t), S(\tau); t, \tau)u(S(\tau), \tau)d\tau$$
$$-\int_0^t u^*(S(t), S(\tau); t, \tau)\dot{S}(\tau)u(S(\tau), \tau)d\tau=\frac{1}{2}g_2(t), \tag{3.4b}$$

Define the column vectors $U(t)=(u_1(t), u_2(t))^T$ with $u_1(t)=u(0, t)$, $u_2(t)=u(S(t), t)$, $G(t)=(g_1(t), g_2(t))^T$ and the matrix $K(t, \tau)=(k_{ij}(t, \tau))_{2\times 2}$ s.t.

$$\begin{cases} k_{ij}=0, \ 1\leqslant i, j\leqslant 2, \ 0\leqslant t<\tau\leqslant T; \\ k_{11}=q^*(0, 0; t, \tau), \quad k_{12}=u^*(0, S(\tau); t, \tau)\dot{S}(\tau)-q^*(0, S(\tau); t, \tau), \\ k_{21}=q^*(S(t), 0; t, \tau), \quad k_{22}=u^*(S(t), S(\tau); t, \tau)\dot{S}(\tau)-q^*(S(t), S(\tau); t, \tau), \\ \qquad 0\leqslant \tau<t\leqslant T. \end{cases} \tag{3.5}$$

Let $\lambda=2k$. The boundary integral equation can, then, be written in the form

$$U(t)-\lambda\int_0^T K(t,\tau)U(\tau)d\tau=G(t). \tag{3.6}$$

Let $R[0, T]$ be the space of Riemann integrable functions. Define the normed linear space $X=(R[0,T])^2$ with maximum norm $\|\ \|_\infty$ s.t. $\forall V(t)=(v_1(t), v_2(t))^T \in X$,

$$\|V\|_\infty=\max_{i=1,2}\max_{t\in[0,T]}|v_i(t)|. \tag{37}$$

Define also the integral operator K: $X\to X$ s.t. $\forall V(t)\in X$,

$$(KV)(t)=\int_0^T K(t,\tau)V(\tau)d\tau. \tag{3.8}$$

It is easy to check that the norm of K is

$$\|K\|_\infty=\max_{i=1,2}\max_{t\in[0,T]}\left\{\sum_{j=1}^{2}\int_0^T|k_{ij}(t,\tau)|d\tau\right\}. \tag{3.9}$$

We see that $k_{11}(t,\tau)=q^*(0,0;t,\tau)=0$,

$$k_{12}(t,\tau)=\left[\frac{S(\tau)}{4\sqrt{\pi}\,(k(t-\tau))^{3/2}}+\frac{\dot S(\tau)}{2k\sqrt{\pi k(t-\tau)}}\right]\exp\left[-\frac{S(\tau)^2}{4k(t-\tau)}\right],\quad t>\tau.$$

By the assumption, $S(\tau)>L>0$. Hence $k_{12}\to 0$ as $t-\tau\to 0$. This implies that $k_{12}(t,\tau)$ is continuous in $0\leqslant t,\ \tau\leqslant T$. Similarly, $k_{21}(t,\tau)$ is also continuous in $0\leqslant t,\ \tau\leqslant T$. Moreover, we have $k_{22}(t,\tau)=H(t,\tau)/\sqrt{t-\tau}$ with

$$H(t,\tau)=\left[\frac{\dot S(\tau)}{2k\sqrt{\pi k}}-\frac{S(t)-S(\tau)}{4\sqrt{\pi}\,k^{3/2}(t-\tau)}\right]\exp\left[-\frac{(S(t)-S(\tau))^2}{4k(t-\tau)}\right],\quad t>\tau.$$

When $t-\tau\to 0$, $H(t,\tau)\to\dot S(t)/4\sqrt{\pi k}$. This implies that $H(t,\tau)$ is continuous in $0\leqslant\tau\leqslant t\leqslant T$. We conclude that $K(t,\tau)$ is weakly singular. It is not difficult to verify that operator K is compact.

Now (3-6) can be written in the operator form

$$(I-\lambda K)U=G. \tag{3.10}$$

It is easy to verify that $\lambda\|K\|_\infty<1$. This implies that $I-\lambda K$ has a bounded inverse.

§ 4. Boundary Element Collocation Scheme

To define the boundary element discretization of (3.6), we divide the time interval $[0, T]$ into N equal subintervals $J^n=[t^{n-1}, t^n]$ $(1\leqslant n\leqslant N)$ with length $h=T/N$. Let $S_h(t)$ be the piecewise polynomial interpolant of degree m of $S(t)$ such that $S_h\left(t^{n-1}+\frac{j}{m}h\right)=S\left(t^{n-1}+\frac{j}{m}h\right)(0\leqslant j\leqslant m,\ 1\leqslant n\leqslant N)$ and the restriction of $S_h(t)$ to each subinterval J^n is a polynomial of degree $m\geqslant 1$. Then we define the space R_N^r of piecewise polynomials of degree $r\geqslant 0$ as follows.

1) $r=0$. R_N^0 is the space of piecewise constants s.t. The restriction of each function in R_N^0 to each subinterval $\tilde{J}^n=(t^{n-1}, t^n]$ $(1\leqslant n\leqslant N)$ is constant. We take $\{t_i=t^i,\ 1\leqslant i\leqslant N\}$ as the set of nodal points and to be associated with each node t_i a function

$$\phi_i(t)=\begin{cases}1 & \text{if } t\in\tilde{J}^n, \text{ or } t=0 \text{ and } i=1,\\ 0 & \text{otherwise.}\end{cases} \tag{4.1}$$

The functions $\{\phi_i(t),\ 1\leqslant i\leqslant N\}$ constitute a basis of R_N^0. Apparently, any function $v(t)\in R_N^0$ has a unique expansion

$$v(t)=\sum_{i=1}^{N} v(t_i)\phi_i(t). \tag{4.2}$$

2) $r\geqslant 1$, $R_N^r=\{v(t)\in C^0[0,T];\ v|_{J^n}\in P_r(J^n),\ 1\leqslant n\leqslant N\}$. We define the nodal points $t_j^n=t^{n-1}+\frac{j}{m}h$, $0\leqslant j\leqslant r$, $1\leqslant n\leqslant N$ and to be associated with each node t_j^n (note that $t_r^n=t_0^{n+1}=t^n$) a function $\phi_j^n(t)\in R_N^r$ s.t. It takes the value 1 at node t_j^n and the value 0 at all other nodes. We number all the nodes $\{t_j^n\}$ in the following order;

$$t_0^1,\ \cdots,\ t_r^1,\ t_1^2,\ \cdots,\ t_r^2,\ \cdots\cdots,\ t_1^N,\ \cdots,\ t_r^N. \tag{4.3}$$

The nodal functions $\{\phi_j^n\}$ are numbered in the same order. Then t_j^n (or $\phi_j^n(t)$) is the i-th node t_i (or nodal function $\phi_i(t)$) with $i=(n-1)r+j$. The set $\{\phi_i(t),\ 0\leqslant i\leqslant Nr\}$ is a basis of R_N^r, and any function $v(t)\in R_N^r$ has a unique expansion

$$v(t)=\sum_{i=0}^{Nr} v(t_i)\ \phi_i(t). \tag{4.4}$$

Note. The explicit expression of functions $\{\phi_j^n\}$ is the following:

$$\phi_j^n(t)=\begin{cases}\prod\limits_{\substack{i=0\\ i\neq j}}^{r}\dfrac{(t-t_i^n)}{(t_j^n-t_i^n)} & \text{in } J^n, \\ 0 & \text{otherwise,}\end{cases}\qquad j\neq 0,\ r;$$

$$\phi_r^n(t)=\phi_0^{n+1}(t)=\begin{cases}\prod\limits_{i=0}^{r-1}\dfrac{(t-t_i^n)}{(t_r^n-t_i^n)} & \text{in } J^n, \\ \prod\limits_{i=1}^{r}\dfrac{(t-t_i^{n+1})}{(t_0^{n+1}-t_i^{n+1})} & \text{in } J^{n+1}, \\ 0, & \text{otherwise.}\end{cases} \tag{4.5}$$

Let $\{t_i,\ 1\leqslant i\leqslant I\}$ be the set of all nodal points of R_N^r, $r\geqslant 0$. We define the piecewise polynomial interpolant $v^I(t)\in R_N^r$ of a function $v(t)\in R[0,T]$ by

$$v^I(t)=\sum_{i=1}^{I} v(t_i)\phi_i(t). \tag{4.6}$$

Note that if $v(t)$ is discontinuous at $t=t_i$, we take in (4.6) $v(t_i)=\lim\limits_{t\to t_i-0} v(t)$.

In order to approximate the initial value $u_0(\xi)$, we divide the space interval $[0, L]$ into M equal subintervals with length $\Delta x=L/M$. Then define, similarly to the above, the space Y_M^r of piecewise polynomials of degree r and the corresponding interpolant $u_0^I(\xi)\in Y_M^r$ of $u_0(\xi)$.

Let $\widetilde{X}_N^r=(R_N^r)^2\subset X$. Then any $\widetilde{V}(t)\in\widetilde{X}_N^r$ has a unique expansion

$$\widetilde{V}(t)=\sum_{i=1}^{I}\phi_i(t)\widetilde{V}(t_i). \tag{4.7}$$

Now the boundary element collocation scheme for (3.6) can be defined as follows.

Find $\widetilde{U}(t)\in\widetilde{X}_N^r$ such that

$$\widetilde{U}(t_i)-\lambda\int_0^T K_h(t_i,\tau)\widetilde{U}(\tau)d\tau=\widetilde{G}(t_i),\quad i=1,\ \cdots,\ I, \tag{4.8}$$

where $K_h(t_i,\tau)$ is obtained after $S(t)$, $S(\tau)$ and $\dot{S}(\tau)$ in (3.5) are replaced by $S_h(t)$, $S_h(\tau)$ and $\dot{S}_h(\tau)$ respectively, and $\widetilde{G}(t_i)$ is obtained after $S(t)$, $S(\tau)$, $\dot{S}(\tau)$,

$u(\tau)$ and $q_j(\tau)$, $j=1, 2$, in (3.3) are replaced by $S_h(t)$, $S_h(\tau)$, $\dot{S}_h(\tau)$, $u^I(\xi)$ and $q_j^I(\tau)$, $j=1, 2$, respectively. Let $K_h^I(t, \tau)=\sum_{i=1}^{I}\phi_i(t)\,K_h(t_i, \tau)$, $\widetilde{G}(t)=\sum_{i=1}^{I}\phi_i(t)\,\widetilde{G}(t_i)$. We can define the integral equation

$$\widetilde{U}(t)-\lambda\int_0^T K_h^I(t, \tau)\widetilde{U}(\tau)d\tau=\widetilde{G}(t), \tag{4.9}$$

since, evidently, $\forall V(t)\in X$, $\int_0^T K_h^I(t, \tau)V(\tau)d\tau\in\widetilde{X}_N^r$.

Lemma 4.1. *If $\widetilde{U}(t)\in\widetilde{X}_N^r$ is the solution of (4.9), then it satisfies (4.8). Conversely, if $\widetilde{U}(t)\in\widetilde{X}_N^r$ is the solution of (4.8), then it satisfies (4.9). Therefore (4.8) and (4.9) are equivalent.*

Proof. Using (4.7), we obtain from (4.9)

$$\sum_{i=1}^{I}\phi_i(t)\left\{\widetilde{U}(t_i)-\lambda\int_0^T K_h(t_i, \tau)\widetilde{U}(\tau)d\tau\right\}=\sum_{i=1}^{I}\phi_i(t)\widetilde{G}(t_i).$$

This yields (4.8). Conversely, multiplying both sides of (4.8) by $\phi_i(t)$ and summing up from $i=1$ to $i=I$, we obtain (4.9). Q.E.D.

Define the integral operator K_h^I: $X\to\widetilde{X}_N^r$ s.t. $\forall V(t)\in X$,

$$(K_h^I V)(t)=\int_0^T K_h^I(t, \tau)V(\tau)d\tau.$$

Since the range of K_h^I is finite-dimensional, K_h^I is, of course, compact. (4-9) can be rewritten in the operator form

$$(I-\lambda K_h^I)\widetilde{U}=\widetilde{G}. \tag{4.10}$$

For later use, we define the projection (interpolation) operator

$$P_N^r: X\to\widetilde{X}_N^r \text{ s.t. } \forall V\in X,\ (P_N^r V)(t)=V^I(t)=\sum_{i=1}^{I}\phi_i(t)V(t_i). \tag{4.11}$$

If $V(t)$ is discontinuous at $t=t_i$, we take in (4.11), similarly to (4.6), $V(t_i)=\lim_{t\to t_i-0}V(t)$. Let $K^I(t, \tau)=\sum_{i=1}^{I}\phi_i(t)K(t_i, \tau)$. We define also the integral operator K^I: $X\to\widetilde{X}_N^r$ s.t. $\forall V(t)\in X$, $(K^I V)(t)=\int_0^T K^I(t, \tau)V(\tau)d\tau$.

It is clear that

$$K^I=P_N^r K. \tag{4.12}$$

§ 5. Error Estimate for the Boundary Element Collocation Scheme

Now we start out to analyse the error of the boundary element collocation solution, i.e. the solution of (4.9). First, we prove three lemmas related to projection P_N^r.

Lemma 5.1.

1) $\qquad \|P_N^r\|_\infty=1$ *if* $r=0, 1$; 2) $\|P_N^r\|_\infty\leqslant L_r$ *if* $r\geqslant 2$, $\qquad$ (5.1)

where L_r is the Lebesque constant associated with polynomial interpolation of degree r. Note that $L_r=1$ if $r=0, 1$.

Proof. 1) Let $r=0, 1$. $\|P_N^r V\|_\infty=\left\|\sum_{i=1}^{I}\phi_i(t)V(t_i)\right\|_\infty\leqslant\|V\|_\infty\max_{t\in[0,T]}\sum_{i=1}^{I}|\phi_i(t)|=\|V\|_\infty$. But for $V\in\widetilde{X}_N^r$, $P_N^r V=V$. Hence $\|P_N^r\|_\infty=1$.

2) Let $\|V\|_\infty=1$. If $r\geqslant 2$,

$$\|P_N^r V\|_\infty=\max_{i=1,2}\max_{1\leqslant n\leqslant N}\max_{t\in J^n}\left|\sum_{j=0}^r \phi_j^n(t)v_i(t_j^n)\right|\leqslant\max_{1\leqslant n\leqslant N}\max_{t\in J^n}\sum_{j=0}^r|\phi_j^n(t)|$$

$$=\max_{s\in[0,1]}\sum_{j=0}^r\left|\prod_{\substack{i=0\\ i\neq j}}^r\frac{(s-i/r)}{(j-i)/r}\right|=L_r.$$

Thus $\|P_N^r\|_\infty\leqslant L_r$. Q.E.D.

Lemma 5.2. *Let $V(t)\in(C^0[0,T])^2$. Define its modulus of continuity $\omega(V,h)$ by*

$$\omega(V,h)=\max_{i=1,2}\max_{\substack{|t_1-t_2|\leqslant h\\ 0\leqslant t_1,t_2\leqslant T}}|v_i(t_1)-v_i(t_2)|. \tag{5.2}$$

Then,

1) $$\|V-P_N^rV\|_\infty\leqslant\omega(V,h)\quad if\ r=0,1; \tag{5.3a}$$

2) $$\|V-P_N^rV\|_\infty\leqslant(1+L_r)\omega(V,h)\quad if\ r\geqslant 2. \tag{5.3b}$$

Proof. 1) $\|V-P_N^0V\|_\infty=\max_{i=1,2}\max_{1\leqslant n\leqslant N}\max_{t\in J^n}|v_i(t)-v_i(t^n)|\leqslant\omega(V,h)$.

$$\|V-P_N^1V\|_\infty=\max_{i=1,2}\max_{1\leqslant n\leqslant N}\max_{t\in J^n}|v_i(t)-v_i^I(t)|.$$

When $r=1$, by the mean value theorem, there exists, for any $t\in J^n$, $\tau_i(t)\in J^n$ s.t. $v_i^I(t)=v_i(\tau_i(t))$. Therefore,

$$\max_{t\in J^n}|v_i(t)-v_i^I(t)|\leqslant\max_{t\in J^n}|v_i(t)-v_i(\tau_i(t))|\leqslant\max_{|t_1-t_2|\leqslant h}|v_i(t_1)-v_i(t_2)|.$$

This implies $\|V-P_N^1V\|_\infty\leqslant\omega(V,h)$.

2) Let $r\geqslant 2$. We have $P_N^r(P_N^1V)=P_N^1V$, since $P_N^1V\in\widetilde{X}_N^r$.

$$\|V-P_N^rV\|_\infty\leqslant\|V-P_N^1V\|_\infty+\|P_N^r(V-P_N^1V)\|_\infty$$
$$\leqslant(1+\|P_N^r\|_\infty)\|V-P_N^1V\|_\infty\leqslant(1+L_r)\omega(V,h)$$

(here we have used Lemma 5.1). Q.E.D.

Lemma 5.3.

1) a) $$\|K-K^I\|_\infty\leqslant\Omega(K,h)\quad if\ r=0,1; \tag{5.4a}$$

b) $$\|K-K^I\|_\infty\leqslant(1+L_r)\Omega(K,h)\quad if\ r\geqslant 2, \tag{5.4b}$$

where

$$\Omega(K,h)=\max_{i=1,2}\max_{\substack{|t_1-t_2|\leqslant h\\ 0\leqslant t_1,t_2\leqslant T}}\left\{\sum_{j=1}^2\int_0^T|k_{ij}(t_1,\tau)-k_{ij}(t_2,\tau)|d\tau\right\}. \tag{5.5}$$

2) $$\Omega(K,h)\leqslant C\sqrt{h},\ as\ h\to 0. \tag{5.6}$$

We use hereafter the letter C to denote various positive constants independent of h and Δx.

Remark. Using (5.6), it is easy to verify that $\forall V\in X$, $KV\in(C^0[0,T])^2$ and that K is compact.

Proof. 1) By Lemma 5.2, $\|KV-K^IV\|_\infty=\|KV-P_N^rKV\|_\infty\leqslant\omega(KV,h)$, if $r=0,1$. But

$$\omega(KV,h)=\max_{i=1,2}\max_{\substack{|t_1-t_2|\leqslant h\\ 0\leqslant t_1,t_2\leqslant T}}\left|\sum_{j=1}^2\int_0^T(k_{ij}(t_1,\tau)-k_{ij}(t_2,\tau))v_j(\tau)d\tau\right|\leqslant\Omega(K,h)\|V\|_\infty.$$

This yields (5.4a). (5.4b) follows similarly.

2) We prove (5.6) by estimating each integral $I_{ij}=\int_0^T |k_{ij}(t_1, \tau)-k_{ij}(t_2, \tau)|d\tau$ in (5.5) with $t_2<t_1$.

A) Since $k_{11}(t, \tau)=0$, we have $I_{11}=0$. (5.7)

B) It is easy to check that $\frac{\partial k_{12}(t, \tau)}{\partial t}$ and $\frac{\partial k_{21}(t, \tau)}{\partial t}$ are continuous in $0\leqslant t,\ \tau\leqslant T$. Therefore,

$$I_{12}\leqslant C(t_1-t_2)\leqslant Ch,\ I_{21}\leqslant C(t_1-t_2)\leqslant Ch. \tag{5.8}$$

C) $$I_{22}=\int_0^{t_2}\left|\frac{H(t_1, \tau)}{\sqrt{t_1-\tau}}-\frac{H(t_2, \tau)}{\sqrt{t_2-\tau}}\right|d\tau+\int_{t_2}^{t_1}\frac{|H(t_1, \tau)|}{\sqrt{t_1-\tau}}d\tau\equiv I_1+I_2.$$

$H(t, \tau)$ is continuous in $0\leqslant\tau\leqslant t\leqslant T$. Hence it is bounded. We have

$$I_2\leqslant C\int_{t_2}^{t_1}\frac{1}{\sqrt{t_1-\tau}}d\tau\leqslant C\sqrt{t_1-t_2}\leqslant C\sqrt{h}, \tag{5.9}$$

$$I_1\leqslant\int_0^{t_2}\frac{|H(t_1, \tau)-H(t_2, \tau)|}{\sqrt{t_1-\tau}}d\tau+\int_0^{t_2}|H(t_2, \tau)|\left[\frac{1}{\sqrt{t_2-\tau}}-\frac{1}{\sqrt{t_1-\tau}}\right]d\tau\equiv I_1'+I_1'',$$

$$I_1''\leqslant C\left[\sqrt{t_1-t_2}-(\sqrt{t_1}-\sqrt{t_2})\right]\leqslant C\sqrt{t_1-t_2}\leqslant C\sqrt{h}, \tag{5.10}$$

$$\begin{aligned}
&|H(t_1, \tau)-H(t_2, \tau)|\\
&\leqslant C\left|\frac{S(t_1)-S(\tau)}{t_1-\tau}\exp\left[-\frac{(S(t_1)-S(\tau))^2}{4k(t_1-\tau)}\right]\right.\\
&\quad\left.-\frac{S(t_2)-S(\tau)}{t_2-\tau}\exp\left[-\frac{(S(t_2)-S(\tau))^2}{4k(t_2-\tau)}\right]\right|\\
&\quad+C\left|\exp\left[-\frac{(S(t_1)-S(\tau))^2}{4k(t_1-\tau)}\right]-\exp\left[-\frac{(S(t_2)-S(\tau))^2}{4k(t_2-\tau)}\right]\right|\\
&\leqslant C\left|\frac{(S(t_1)-S(\tau))}{t_1-\tau}-\frac{(S(t_2)-S(\tau))}{t_2-\tau}\right|\\
&\quad+C\left|\exp\left[-\frac{(S(t_1)-S(\tau))^2}{4k(t_1-\tau)}\right]-\exp\left[-\frac{(S(t_2)-S(\tau))^2}{4k(t_2-\tau)}\right]\right|\\
&\leqslant C\left\{\frac{S(t_1)-S(t_2)}{t_1-\tau}+|S(t_2)-S(\tau)|\left[\frac{1}{t_2-\tau}-\frac{1}{t_1-\tau}\right]\right.\\
&\quad\left.+\left|\frac{(S(t_1)-S(\tau))^2}{t_1-\tau}-\frac{(S(t_2)-S(\tau))^2}{t_2-\tau}\right|\right\}\\
&\leqslant C\left\{\frac{t_1-t_2}{t_1-\tau}+1-\frac{t_2-\tau}{t_1-\tau}+\frac{S(t_1)-S(\tau)+S(t_2)-S(\tau)}{t_1-\tau}(S(t_1)-S(t_2))\right.\\
&\quad\left.+(S(t_2)-S(\tau))^2\left[\frac{1}{t_2-\tau}-\frac{1}{t_1-\tau}\right]\right\}\\
&<C\left[\frac{1}{t_1-\tau}+1\right](t_1-t_2).
\end{aligned}$$

Hence

$$\begin{aligned}
I_1'&\leqslant C(t_1-t_2)\int_0^{t_2}\left[\frac{1}{\sqrt{t_1-\tau}}+\frac{1}{(t_1-\tau)^{3/2}}\right]d\tau\\
&=2C(t_1-t_2)\left[\sqrt{t_1}-\sqrt{t_1-t_2}+\frac{1}{\sqrt{t_1-t_2}}-\frac{1}{\sqrt{t_1}}\right]\\
&\leqslant C(t_1-t_2)\left[\sqrt{t_1}+\frac{1}{\sqrt{t_1-t_2}}\right]\leqslant C\sqrt{t_1-t_2}\leqslant C\sqrt{h}.
\end{aligned} \tag{5.11}$$

(5.9)—(5.11) yield

$$I_{22}\leqslant C\sqrt{h}. \tag{5.12}$$

Combining (5.7), (5.8) with (5.12), we obtain (5.6). Q.E.D.

Next, we prove two lemmas related to approximation to the boundary.

Lemma 5.4. *If $S(t)\in C^{m+1}[0, T]$, then*

$$\|K^I-K_h^I\|_\infty\leqslant Ch^m, \quad as\ h\to 0. \tag{5.13}$$

Proof.

$$\|K^I-K_h^I\|_\infty=\max_{n=1,2}\max_{t\in[0,T]}\Big[\sum_{j=1}^2\int_0^T\Big|\sum_{i=1}^I\phi_i(t)(k_{nj}(t_i,\tau)-k_{hnj}(t_i,\tau))\Big|d\tau\Big]$$

$$\leqslant L_r\cdot\max_{n=1,2}\max_{1<i<I}\Big[\sum_{j=1}^2\int_0^T|k_{nj}(t_i,\tau)-k_{hnj}(t_i,\tau)|d\tau\Big]. \tag{5.14}$$

We can prove (5.13) by estimating each integral

$$I_{nj}=\int_0^T|k_{nj}(t_i,\tau)-k_{hnj}(t_i,\tau)|d\tau.$$

1) Since $k_{11}(t_i,\tau)=k_{h11}(t_i,\tau)=0$, we have $I_{11}=0$. (5.15)

2) $$I_{12}\leqslant\int_0^{t_i}\frac{1}{4\sqrt{\pi}\,(k(t_i-\tau))^{3/2}}\Big|S(\tau)\exp\Big[-\frac{S(\tau)^2}{4k(t_i-\tau)}\Big]-S_h(\tau)\exp\Big[-\frac{S_h(\tau)^2}{4k(t_i-\tau)}\Big]\Big|d\tau$$
$$+\int_0^{t_i}\frac{1}{2k\sqrt{\pi k(t_i-\tau)}}\Big|\dot S(\tau)\exp\Big[-\frac{S(\tau)^2}{4k(t_i-\tau)}\Big]-\dot S_h(\tau)\exp\Big[-\frac{S_h(\tau)^2}{4k(t_i-\tau)}\Big]\Big|d\tau\equiv I_1+I_2,$$

$$I_1\leqslant\int_0^{t_i}\frac{|S(\tau)-S_h(\tau)|}{(t_i-\tau)^{3/2}}\exp\Big[-\frac{S(\tau)^2}{4k(t_i-\tau)}\Big]d\tau$$
$$+C\int_0^{t_i}\frac{|S_h(\tau)|}{(t_i-\tau)^{3/2}}\Big|\exp\Big[-\frac{S(\tau)^2}{4k(t_i-\tau)}\Big]-\exp\Big[-\frac{S_h(\tau)^2}{4k(t_i-\tau)}\Big]\Big|d\tau\equiv I_1'+I_1''.$$

Since $S(t)\in C^{m+1}[0, T]$, we have ([6]) $\max\limits_{\tau\in[0,T]}|S(\tau)-S_h(\tau)|\leqslant Ch^{m+1}$. Hence

$$I_1'\leqslant Ch^{m+1}. \tag{5.16}$$

By the mean value theorem,

$$\Big|\exp\Big[-\frac{S(\tau)^2}{4k(t_i-\tau)}\Big]-\exp\Big[-\frac{S_h(\tau)^2}{4k(t_i-\tau)}\Big]\Big|$$
$$\leqslant\frac{\theta(\tau)}{2k(t_i-\tau)}\exp\Big[-\frac{\theta(\tau)^2}{4k(t_i-\tau)}\Big]\cdot|S(\tau)-S_h(\tau)|,$$

where $\min(S(\tau), S_h(\tau))\leqslant\theta(\tau)\leqslant\max(S(\tau), S_h(\tau))$ and $\min\limits_{\tau\in[0,T]}\theta(\tau)>0$, since $S(t)\geqslant L>0$. It follows that

$$I_1''\leqslant Ch^{m+1}. \tag{5.17}$$

Therefore

$$I_1\leqslant Ch^{m+1}. \tag{5.18}$$

Since $\max\limits_{\tau\in[0,T]}|\dot S(\tau)-\dot S_h(\tau)|\leqslant Ch^m$ ([6]), we can similarly deduce

$$I_2\leqslant Ch^m. \tag{5.19}$$

Combination of (5.18) and (5.19) gives

$$I_{12}\leqslant Ch^m. \tag{5.20}$$

Similarly, we have

$$I_{21}\leqslant Ch^{m+1}. \tag{5.21}$$

3) $$I_{22}\leqslant C\int_0^{t_i}\frac{1}{\sqrt{t_i-\tau}}\Big|\dot S(\tau)\exp\Big[-\frac{(S(t_i)-S(\tau))^2}{4k(t_i-\tau)}\Big]-\dot S_h(\tau)\exp\Big[-\frac{(S_h(t_i)-S_h(\tau))^2}{4k(t_i-\tau)}\Big]\Big|d\tau$$
$$+C\int_0^{t_i}\frac{1}{(t_i-\tau)^{3/2}}\Big|(S(t_i)-S(\tau))\exp\Big[-\frac{(S(t_i)-S(\tau))^2}{4k(t_i-\tau)}\Big]-(S_h(t_i)-S_h(\tau))\exp\Big[-\frac{(S_h(t_i)-S_h(\tau))^2}{4k(t_i-\tau)}\Big]\Big|d\tau\equiv I_3+I_4,$$

$$I_4\leqslant C\int_0^{t_i}\frac{|(S(t_i)-S(\tau))-(S_h(t_i)-S_h(\tau))|}{(t_i-\tau)^{3/2}}d\tau$$
$$+C\int_0^{t_i}\frac{|S_h(t_i)-S_h(\tau)|}{(t_i-\tau)^{3/2}}\Big|\exp\Big[-\frac{(S(t_i)-S(\tau))^2}{4k(t_i-\tau)}\Big]-\exp\Big[-\frac{(S_h(t_i)-S_h(\tau))^2}{4k(t_i-\tau)}\Big]\Big|d\tau\equiv I_4'+I_4''.$$

By an argument similar to 2), we can deduce

$$I_3\leqslant Ch^m,\quad I_4''\leqslant Ch^{m+1}. \tag{5.22}$$

Unless $t_i=0$ when $I_{22}=0$, we have $t_i\geqslant h$ for $r=0$, 1 or $t_i\geqslant\frac{h}{r}$ for $r\geqslant2$. Then an integration by parts yields

$$I_4'\leqslant\frac{C}{\sqrt{t_i}}|S(t_i)-S_h(t_i)|+C\int_0^{t_i}\frac{|\dot S(\tau)-\dot S_h(\tau)|}{\sqrt{t_i-\tau}}d\tau\leqslant Ch^{m+\frac12}+Ch^m\leqslant Ch^m. \tag{5.23}$$

Combination of (5.22)—(5.23) gives

$$I_{22}\leqslant Ch^m. \tag{5.24}$$

Finally, collecting all estimates for I_{nj}, $1\leqslant n,\ j\leqslant 2$, we obtain (5.13). Q.E.D.

Lemma 5.5. *Let* $G^I(t)=P_N^rG(t)=\sum_{i=1}^I\phi_i(t)G(t_i)$, $G_h^I=P_N^rG_h(t)=\sum_{i=1}^I\phi_i(t)G_h(t_i)$, *where* $G_h(t)$ *is obtained after* $S(t)$, $S(\tau)$ *and* $\dot S(\tau)$ *in* (3.3) *are replaced by* $S_h(t)$, $S_h(\tau)$ *and* $\dot S_h(\tau)$, *respectively. If* $S(t)\in C^{m+1}[0,T]$, *then,*

$$\|G^I-G_h^I\|_\infty\leqslant Ch^{m+\frac12},\quad as\ h\to0. \tag{5.25}$$

Proof. $\|G^I-G_h^I\|_\infty=\Big\|\sum_{i=1}^I\phi_i(t)(G(t_i)-G_h(t_i))\Big\|_\infty\leqslant L_r\cdot\max_{1\leqslant i\leqslant I}\max_{j=1,2}|g_j(t_i)-g_{jh}(t_i)|$,

$$I_1=|g_1(t_i)-g_{1h}(t_i)|$$
$$\leqslant2k\int_0^{t_i}|q_2(\tau)|\frac{1}{2\sqrt{\pi k(t_i-\tau)}}\Big|\exp\Big[-\frac{S(\tau)^2}{4k(t_i-\tau)}\Big]-\exp\Big[-\frac{S_h(\tau)^2}{4k(t_i-\tau)}\Big]\Big|d\tau,$$

$$I_2=|g_2(t_i)-g_{2h}(t_i)|$$
$$\leqslant2k\int_0^{t_i}|q_1(\tau)|\frac{1}{2\sqrt{\pi k(t_i-\tau)}}\Big|\exp\Big[-\frac{S(t_i)^2}{4k(t_i-\tau)}\Big]-\exp\Big[-\frac{S_h(t_i)^2}{4k(t_i-\tau)}\Big]\Big|d\tau$$

$$+2k\int_0^{t_i}|q_2(\tau)|\frac{1}{2\sqrt{\pi k(t_i-\tau)}}\Big|\exp\Big[-\frac{(S(t_i)-S(\tau))^2}{4k(t_i-\tau)}\Big]$$
$$-\exp\Big[-\frac{(S_h(t_i)-S_h(\tau))^2}{4k(t_i-\tau)}\Big]\Big|d\tau+2\Big|\int_0^L u_0(\xi)\frac{1}{4\sqrt{\pi kt_i}}\exp\Big[-\frac{(S(t_i)-\xi)^2}{4kt_i}\Big]d\xi$$
$$-\int_0^L u_0(\xi)\frac{1}{2\sqrt{\pi kt_i}}\exp\Big[-\frac{(S_h(t_i)-\xi)^2}{4kt_i}\Big]d\xi\Big|\equiv I_2'+I_2''+I_2'''.$$

Following the argument in the proof of Lemma 5.4, we can deduce

$$I_1\leqslant Ch^{m+1},\quad I_2'\leqslant Ch^{m+1},\quad I_2''\leqslant Ch^{m+1}. \tag{5.26}$$

Unless $t_i=0$ when $I_2'''=0$, we have $t_i\geqslant h$ for $r=0$, 1, or $t_i\geqslant\frac{h}{r}$ for $r\geqslant 2$. After changing variables in the integrals, we obtain

$$I_2'''=\frac{1}{\sqrt{\pi}}\Big|\int_{\eta_1}^{\eta_2}u_0(S(t_i)-2\sqrt{kt_i}\,\eta)\exp(-\eta^2)d\eta$$
$$-\int_{\eta_3}^{\eta_4}u_0(S_h(t_i)-2\sqrt{kt_i}\,\eta)\exp(-\eta^2)d\eta\Big|,$$

where

$$\eta_1=\frac{S(t_i)-L}{2\sqrt{kt_i}},\quad \eta_2=\frac{S(t_i)}{2\sqrt{kt_i}},\quad \eta_3=\frac{S_h(t_i)-L}{2\sqrt{kt_i}}\quad\text{and}\quad \eta_4=\frac{S_h(t_i)}{2\sqrt{kt_i}}$$

Assuming $S(t_i)\geqslant S_h(t_i)$ (the case of $S(t_i)<S_h(t_i)$ can be treated similarly), we have

$$I_2'''=\frac{1}{\sqrt{\pi}}\Big|\int_{\eta_4}^{\eta_2}u_0(S(t_i)-2\sqrt{kt_i}\,\eta)\exp(-\eta^2)d\eta-\int_{\eta_3}^{\eta_1}u_0(S_h(t_i)-2\sqrt{kt_i}\,\eta)\exp(-\eta^2)d\eta$$
$$+\int_{\eta_1}^{\eta_4}\{u_0(S(t_i)-2\sqrt{kt_i}\,\eta)-u_0(S_h(t_i)-2\sqrt{kt_i}\,\eta)\}\exp(-\eta^2)d\eta\Big|$$
$$\leqslant C(|\eta_2-\eta_4|+|\eta_1-\eta_3|+|S(t_i)-S_h(t_i)|)\leqslant C\Big[\frac{1}{\sqrt{t_i}}+1\Big]|S(t_i)-S_h(t_i)|$$
$$\leqslant C\Big[1+\frac{1}{\sqrt{h}}\Big]h^{m+1}\leqslant Ch^{m+\frac{1}{2}}. \tag{5.27}$$

Combination of (5.26) and (5.27) gives (5.25). Q.E.D.

The lemma below is concerned with approximation to the initial-boundary data.

Lemma 5.6. *Assume that* $q_j(t)\in C^{r+1}[0,T]$, $j=1,2$, *and that* $u_0(\xi)\in C^{r+1}[0,L]$. *Then*

$$\|G_h^I-\widetilde{G}\|_\infty\leqslant C(h^{r+1}+(\Delta x)^{r+1}). \tag{5.28}$$

Proof. $\|G_h^I-\widetilde{G}\|_\infty=\Big\|\sum_{i=1}^{I}\phi_i(t)(G_h(t_i)-\widetilde{G}(t_i))\Big\|_\infty\leqslant L_r\cdot\max\limits_{1<i<I}\max\limits_{j=1,2}|g_{jh}(t_i)-\tilde{g}_j(t_i)|$.

By the assumption, we have the following interpolation errors ([6]):

$$\max_{\tau\in[0,T]}|q_j(\tau)-q_j^I(\tau)|\leqslant Ch^{r+1},\ j=1,2;\quad \max_{\xi\in[0,L]}|u_0(\xi)-u_0^I(\xi)|\leqslant C(\Delta x)^{r+1}. \tag{5.29}$$

Hence we have, for any $t\in[0,T]$,

$$|g_{1h}(t)-\tilde{g}_1(t)|\leqslant 2k\int_0^t|q_1(\tau)-q_1^I(\tau)|u^*(0,0;t,\tau)d\tau$$
$$+2k\int_0^t|q_2(\tau)-q_2^I(\tau)|u^*(0,S_h(\tau);t,\tau)d\tau$$
$$+2\int_0^L|u_0(\xi)-u_0^I(\xi)|u^*(0,\xi;t,0)d\xi\leqslant C(h^{r+1}+(\Delta x)^{r+1}).$$

Similarly, $|g_{2h}(t)-\tilde{g}_2(t)|\leqslant C(h^{r+1}+(\Delta x)^{r+1})$. Q.E.D.

(5.28) follows immediately.

The following theorem is quoted from Theorem 4.34 of [7].

Theorem 5.1. *Let X be a normed linear space with norm $\|\ \|$. K and K_m are two bounded linear operators mapping X into X. Let $T=I-\lambda K$, $T_m=I-\lambda K_m$. Assume that T^{-1} exists with $q=\|T^{-1}\|\|T-T_m\|<1$ and that K and K_m are compact. Then T_m^{-1} exists and*

$$\|T_m^{-1}\|\leqslant\|T^{-1}\|/(1-q). \tag{5.30}$$

Now we are ready to derive the error estimate for the boundary element collocation scheme. We have the following result:

Theorem 5.2. *Assume 1) $S(t)\in C^{m+1}[0, T]$, $q_j(t)\in C^{r+1}[0, T]$, $j=1, 2$ and $u_0(x)\in C^{r+1}[0, L]$; 2) the exact solution $U(t)$ of boundary integral equation (3.6) belongs to $(C^{r+1}[0, T])^2$. Let $\tilde{U}(t)$ be the solution of boundary element collocation equation (4.9) (or, equivalently, (4.8)). Then, there exists a constant C independent of h and Δx such that, when $h\to 0$ and $\Delta x\to 0$,*

$$\|U-\tilde{U}\|_\infty\leqslant C(h^m+h^{r+1}+(\Delta x)^{r+1}). \tag{5.31}$$

Proof. Taking in (3.6) $t=t_i$, $i=1, \cdots, I$, successively, we get

$$U(t_i)-\lambda\int_0^T K(t_i, \tau)U(\tau)d\tau=G(t_i), \quad i=1, \cdots, I.$$

This leads to

$$U(t_i)-\lambda\int_0^T K_h(t_i, \tau)U^I(\tau)d\tau=\lambda\int_0^T\{K(t_i, \tau)-K_h(t_i, \tau)\}U(\tau)d\tau$$
$$+\lambda\int_0^T K_h(t_i, \tau)\{U(\tau)-U^I(\tau)\}d\tau+G(t_i). \tag{5.32}$$

Multiplying both sides of (5.32) with $\phi_i(t)$ and summing up from $i=1$ to $i=I$, we obtain, in operator form,

$$(I-\lambda K_h^I)U^I=\lambda(K^I-K_h^I)U+\lambda K_h^I(U-U^I)+G^I. \tag{5.33}$$

Subtracting (4.10), which is the operator form of (4.9), from (5.33), we get

$$(I-\lambda K_h^I)(U^I-\tilde{U})=\lambda(K^I-K_h^I)U+\lambda K_h^I(U-U^I)+(G^I-\tilde{G}). \tag{5.34}$$

Let $T=I-\lambda K$, $T_N=I-\lambda K_h^I$. By Lemmas 5.3 and 5.4,

$$\|K-K_h^I\|_\infty\leqslant\|K-K^I\|_\infty+\|K^I-K_h^I\|_\infty\to 0, \quad \text{as } h\to 0.$$

Hence

$$1>q=\|T^{-1}\|_\infty\|T-T_N\|_\infty=|\lambda|\,\|T^{-1}\|_\infty\|K-K_h^I\|_\infty\to 0, \quad \text{as } h\to 0.$$

It has been shown in sections 3 and 4 that K and K_h^I are compact. So we can apply Theorem 5.1 to obtain

$$\|T_N^{-1}\|_\infty\leqslant\|T^{-1}\|_\infty/(1-q)\leqslant C, \quad \text{as } h\to 0.$$

Furthermore, it is obvious that, when $h\to 0$, the operators $\{K_h^I\}$ are uniformly bounded, i.e. $\|K_h^I\|_\infty\leqslant C$. We deduce from (5–34)

$$\|U^I-\tilde{U}\|_\infty\leqslant\|T_N^{-1}\|_\infty\{|\lambda|\,\|K^I-K_h^I\|_\infty\|U\|_\infty+|\lambda|\,\|K_h^I\|_\infty\|U-U^I\|_\infty+\|G^I-\tilde{G}\|_\infty\}$$
$$\leqslant C\{\|K^I-K_h^I\|_\infty+\|U-U^I\|_\infty+\|G^I-\tilde{G}\|_\infty\}. \tag{5.35}$$

By the triangular inequality $\|U-\tilde{U}\|_\infty\leqslant\|U-U^I\|_\infty+\|U^I-\tilde{U}\|_\infty$, we get

$$\|U-\tilde{U}\|_\infty \leqslant C\{\|K^I-K_h^I\|_\infty+\|U-U^I\|_\infty+\|G^I-\tilde{G}\|_\infty\}. \tag{5.36}$$

By assumption 2), we have ([6])

$$\|U-U^I\|_\infty \leqslant \max_{j=1,2}\max_{t\in[0,T]} |u_j(t)-u_j^I(t)| \leqslant Ch^{r+1}. \tag{5.37}$$

Application of Lemmas 5.4—5.5 gives

$$\|K^I-K_h^I\|_\infty \leqslant Ch^m \tag{5.38}$$

and

$$\|G^I-\tilde{G}\|_\infty \leqslant \|G^I-G_h^I\|_\infty+\|G_h^I-\tilde{G}\|_\infty \leqslant C\{h^{m+\frac{1}{2}}+h^{r+1}+(\Delta x)^{r+1}\}. \tag{5.39}$$

Finally, combining (5.37) with (5.39), we obtain (5.31). Q.E.D.

Remark. 1) For the problems in a fixed interval where $S(t)$ $S_h(t)\equiv L$, we obtain from (5.31) $\|U-\tilde{U}\|_\infty \leqslant C(h^{r+1}+(\Delta x)^{r+1})$, which coincides with the result in [5].

2) It is seen from (5.31) that the accuracy of the approximate solution is limited by the accuracy of approximation to the boundary. Hence, the most reasonable choice for boundary elements is to take $r=m-1$.

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