APPLICATION OF THE REGULARIZATION METHOD TO THE NUMERICAL SOLUTION OF ABEL'S INTEGRAL EQUATION (II)*

HAO SHOU (郝 寿)
(*Hebei Institute of Technology, Tianjin, China*)

§1

The main purpose of this paper is to use the regularization method to solve the following integral equation of the Abel type

$$A_f=2\int_p^\infty \frac{rf(r)}{\sqrt{r^2-p^2}}\,dr=g(p), \tag{1}$$

which is of great importance in many applications[1].

Suppose that the function $f_T(r)$ having a continuous first derivative and compact support $[0, T]$ is a solution of equation (1) with right-hand side $g_T(p)$, i.e.,

$$A_{f_T}=2\int_p^T \frac{rf(r)}{\sqrt{r^2-p^2}}\,dr=g_T(p)$$

and is yet to be found.

There are two cases to be considered:

Case I. The position of the right end point of the compact support $[0, T]$ is given exactly in advance.

Case II. The position is known only approximately.

The problem of solving Abel's integral equation

$$A_z=\int_0^x \frac{z(s)}{(x-s)^a}\,ds=u(x)$$

has been studied in [2]. In Case I in exactly the same way one can easily see that the analogous problem of determining the solution $f(r)$ of the Abel type integral equation (1) in the space $C[0, T]$ from the initial data $g(p)$ in the space $L_2[0, T]$ is not well-posed on the pair of spaces (C, L_2) ([3] p. 16 and [2]) and that the problem of constructing approximate solutions can be solved in accordance with the method described in [2].

In Case II we are thus forced to adopt a somewhat different approach to solve problem (1) for $f_T(r)$. In the following we shall treat this problem in detail.

§2

In Case II because of the ambiguity of the position of the right end point we

* Received April 24, 1985.

prefer to study problem (1) on the pair of spaces $(\bar{C}, L_2)$, where

$$L_2 = L_2[0, \bar{T}],$$

$$\bar{C} = \bar{C}[0, \bar{T}] = \{f(r): f(r) \text{ is continuous on } [0, \bar{T}] \text{ and has compact support } [0, \xi],\ 0<\xi\leqslant T,\ T<\bar{T}\},$$

$$\|f\|_{\bar{C}} = \max|f(r)|.$$

The problem of determining the solution $f(r)$ from the initial data $g(p)$, like the problem considered in [2], is not well-posed on $(\bar{C}, L_2)$. For, in the first place, the set $A\bar{C}$ does not coincide with L_2. Secondly, the inverse operator A^{-1} is not continuous.

Furthermore, it should be noted that the reciprocity formula for $f(r)$ holds[1]:

$$f(r) = \frac{-1}{\pi r}\frac{d}{dr}\int_r^\infty \frac{pg(p)}{\sqrt{p^2-r^2}}dp.$$

Below, following[4], we shall employ the regularization method for the Abel type equation (1) to construct a regularizing operator that provides a stable method for determining approximate solutions. For this purpose we consider the functional $M^\alpha[f, g]$ defined on $\bar{C}_1[0, T]$:

$$M^\alpha[f, g] = \|Af-g\|^2_{L_2} + \alpha\int_0^T [f^2(r)+f'(r)^2]dr$$

$$= \int_0^{\bar{T}}\left[2\int_p^T \frac{rf(r)}{\sqrt{r^2-p^2}}dr - g(p)\right]^2 dp + \alpha\int_0^{\bar{T}}[f^2(r)+f'(r)^2]dr,$$

$$\bar{C}_1 = \bar{C}_1[0, \bar{T}] = \{f(r): f(r)\in\bar{C},\ f(r) \text{ has a continuous derivative}\}.$$

Theorem 1. *For every function $g\in L_2$ and every positive parameter α, there exists a unique function $f_\alpha\in\bar{C}_1$ for which the functional $M^\alpha[f, g]$ attains its greatest lower bound, that is*

$$M^\alpha[f_\alpha, g] = \inf M^\alpha[f, g].$$

Proof. 1) This is a variational problem with free boundaries; the left and right end points of the unknown curve $f_\alpha(r)$ are on lines $r=0$ and $p=0$ respectively. Thus, we obtain after simple calculation the first variation δM^α of the functional M^α:

$$\delta M^\alpha = 4\int_0^\xi\left\{\int_0^r \frac{r}{\sqrt{r^2-p^2}}\left[2\int_p^\xi \frac{tf(t)}{\sqrt{t^2-p^2}}dt - g(p)\right]dp\right\}h(r)dr$$

$$+2\alpha\int_0^\xi [f(r)-f''(r)]h(r)dr + 2\alpha f'(r)h(r)\Big|_{r=0}^{r=\xi},$$

and hence the function $f_\alpha(r)$ should be determined by the Euler integro-differential equation

$$\alpha L[f] = 4\int_0^r \frac{r}{\sqrt{r^2-p^2}}\left[\int_p^\xi \frac{tf(t)}{\sqrt{t^2-p^2}}dt\right]dp - 2\int_0^r \frac{r}{\sqrt{r^2-p^2}}g(p)dp,\quad L[f]=f''-f \tag{2}$$

and the boundary conditions

$$f'(0)=0,\quad f'(\xi)=0,\quad f(\xi)=0. \tag{3}$$

2) Under given boundary conditions (3) the associated homogeneous equation

$$\alpha L[f] = 4\int_0^r \frac{r}{\sqrt{r^2-p^2}}\left[\int_p^\xi \frac{tf(t)}{\sqrt{t^2-p^2}}dt\right]dp, \tag{4}$$

cannot possess a nontrivial solution. For, were $f(r)$ such a solution, then, multiplying (4) by $f(r)$, and integrating with respect to r, we should get the following equality

$$-\alpha\int_0^{\ell}[f'(r)^2+f^2(r)]dr=4\int_0^{\ell}\left[\int_p^{\ell}\frac{rf(r)}{\sqrt{r^2-p^2}}dr\right]^2dp.$$

This would contradict the hypothesis that α is positive.

3) By means of Green's function $G(r, \xi)$ for the differential operator $L[f]$ under conditions (3), finding the desired solution $f(r)$ of equation (2) under (3) is equivalent to solving the following integral equation

$$\alpha f(r)=4\int_0^{\ell}G(r,\zeta)\left[\int_0^{\zeta}\frac{\zeta}{\sqrt{\zeta^2-p^2}}\int_p^{\ell}\frac{tf(t)dt}{\sqrt{t^2-p^2}}dp\right]d\zeta$$

$$-2\int_0^{\ell}G(r,\zeta)\left[\int_0^{\zeta}\frac{\zeta}{\sqrt{\zeta^2-p^2}}g(p)dp\right]d\zeta. \qquad (5)$$

From 2) the associated homogeneous integral equation (5) has only a trivial solution, and hence the inhomogeneous equation possesses a uniquely determined solution $f_\alpha(r)$.

Thus, from 1), 2) and 3) the result of Theorem 1 follows.

From this theorem it follows that an operator $R(g, \alpha)$ into $\bar{C}_1$ is defined on the set (g, α):

$$f_\alpha=R(g,\alpha),$$

where $g\in L_2$ and $\alpha>0$.

Furthermore, for the regularization parameter, we select $\alpha(\delta)=\delta^2$.

Then, we can show that $R(g, \alpha(\delta))$ is a regularizing operator for equation (1) and hence that the function

$$f_{\alpha(\delta)}=R(g_\delta,\alpha(\delta)),$$

can be taken as an approximate solution of equation (1).

Theorem 2. *Let* $f_T(r)$ *denote a solution of equation* (1) *with right-hand side* $g=g_T$, *that is*

$$Af_T=2\int_p^T\frac{rf_T(r)}{\sqrt{r^2-p^2}}dr=g_T(p).$$

Then, for any positive number ε, *there exists a number* $\delta(\varepsilon)$ *such that the inclusion* $g\in L_2$ *and the inequality*

$$\|g_\delta-g_T\|_{L_2}\leqslant\delta\leqslant\delta(\varepsilon)$$

imply

$$\|f_\alpha-f_T\|_{\bar{C}}<\varepsilon,$$

where

$$f_\alpha=R(g_\delta,\alpha(\delta)).$$

Proof. Since the functional $M^\alpha[f, g]$ attains its minimum when $f=f_\alpha$, we have

$$M^{\alpha(\delta)}[f_{\alpha(\delta)},g_\delta]\leqslant M^{\alpha(\delta)}[f_T,g_\delta].$$

Therefore,

$$\|Af_{\alpha(\delta)}-g_\delta\|^2_{L_2}+\alpha(\delta)\int_0^T[f^2_{\alpha(\delta)}(r)+f'_{\alpha(\delta)}(r)^2]dr$$

$$\leqslant\|Af_T-g_\delta\|^2_{L_2}+\alpha(\delta)\int_0^T[f^2_T(r)+f'_T(r)^2]dr\leqslant\delta^2d,$$

$$d=1+\int_0^{\bar{T}}[f_T^2(r)+f'_T(r)^2]\,dr.$$

Consequently,

$$\int_0^{\bar{T}}[f^2_{\alpha(\delta)}(r)+f'_{\alpha(\delta)}(r)^2]\,dr\leqslant d$$

and

$$\|Af_{\alpha(\delta)}-g_\delta\|_{L_2}\leqslant\delta\sqrt{d}.$$

Thus, the functions f_T and $f_{\alpha(\delta)}$ belong to the compact subset M of the space $\bar{C}[0, \bar{T}]$:

$$M=\left\{f(r):\int_0^{\bar{T}}[f^2(r)+f'(r)^2]\,dr\leqslant d\right\}$$

and hence

$$g_T(p)=Af_T\in AM,\quad g_{\alpha(\delta)}(p)=Af_{\alpha(\delta)}\in AM.$$

Because of the continuity of the inverse operator A^{-1} on AM, for every $\varepsilon>0$ there exists a number $\eta(\varepsilon)$ such that for $\|g_{\alpha(\delta)}-g_T\|_{L_2}\leqslant\eta(\varepsilon)$

$$\|f_{\alpha(\delta)}-f_T\|_{\bar{C}}<\varepsilon.$$

Furthermore, since

$$\|g_{\alpha(\delta)}-g_T\|_{L_2}\leqslant\|g_{\alpha(\delta)}-g_\delta\|_{L_2}+\|g_\delta-g_T\|\leqslant\delta(1+\sqrt{d}),$$

we may choose

$$\delta(\varepsilon)=\frac{\eta(\varepsilon)}{1+\sqrt{d}}.$$

Consequently, for $\delta\leqslant\delta(\varepsilon)$, we have

$$\|g_{\alpha(\delta)}-g_T\|_{L_2}\leqslant\eta(\varepsilon),$$

and hence

$$\|f_{\alpha(\delta)}-f_T\|_{\bar{C}}<\varepsilon.$$

This completes the proof of the theorem.

§ 3

In this section we shall use the finite-difference method for the numerical solution of equation (1) to provide a computational algorithm that can easily be realized on a computer. For this purpose we consider the functional

$$M_h^\alpha[f^h, g^h]=\sum_{i=1}^n h\Big[2\sum_{j=i}^n a_{i,j}f_j-g_i\Big]^2+\alpha\sum_{j=0}^n h\Big[\frac{f_{j+1}-f_j}{h}\Big]^2+\alpha\sum_{j=1}^n hf_j^2,$$

where $f^h=(f_0, f_1, \cdots, f_{n+1})$ and $g^h=(g_0, g_1, \cdots, g_{n+1})$ are difference functions defined on uniform grids ω_r^h and ω_p^h respectively:

$$\omega_r^h=\{r_j: r_j=jh,\ j=0, \cdots, n+1\},\ \omega_p^h=\{p_i: p_i=ih,\ i=0, \cdots, n+1\},\ h=\frac{\bar{T}}{n+1},$$

and

$$\underset{(i=1, \cdots, n)}{a_{i,j}}=\begin{cases}0, & j<i,\\ \displaystyle\int_{r_i}^{r_{i+1}}\frac{r}{\sqrt{r^2-p_i^2}}\Big(1-\frac{r-r_i}{h}\Big)dr, & j=i,\\ \displaystyle\int_{r_j}^{r_{j+1}}\frac{r}{\sqrt{r^2-p_i^2}}\Big(1-\frac{r-r_j}{h}\Big)dr+\int_{r_{j-1}}^{r_j}\frac{r}{\sqrt{r^2-p_i^2}}\,\frac{r-r_{j-1}}{h}\,dr, & i+1\leqslant j\leqslant n-1,\\ \displaystyle\int_{r_n}^{r_{n+1}}\frac{r}{\sqrt{r^2-p_i^2}}\,dr+\int_{r_{n-1}}^{r}\frac{r}{\sqrt{r^2-p_i^2}}\,\frac{r-r_{n-1}}{h}\,dr, & j=n.\end{cases}$$

Theorem 3. *For every difference function g^h and every positive parameter α, there exists a unique function f_α^h for which the functional $M_h^\alpha[f^h, g^h]$ attains its greatest lower bound:*

$$M_h^\alpha[f_\alpha^h, g^h]=\inf M_h^\alpha[f^h, g^h].$$

Proof. The proof is analogous to that of Theorem 1.

1) The function f_α^h should be determined by the Euler equation

$$\alpha L^h[f^h]=4\sum_{i=1}^{j} a_{i,j}\Big[\sum_{l=i}^{n} a_{i,l} f_l\Big]-2\sum_{i=1}^{j} a_{i,j} g_i, \quad i=1, 2, \cdots, n, \tag{6}$$

$$L^h[f^h]=\frac{f_{j+1}-2f_j+f_{j-1}}{h^2}-f_j,$$

and the boundary conditions

$$f_1=f_0, \quad f_{n+1}=f_n. \tag{7}$$

2) Under conditions (7) the homogeneous equation (6) possesses only a trivial solution, and hence the inhomogeneous equation (6) has one and only one solution.

This completes the proof of the theorem.

Theorem 4. *Let $f_T(r)$ denote a solution of equation (1) with right-hand side $g_T(p)$:*

$$Af_T=2\int_p^T \frac{rf(r)}{\sqrt{r^2-p^2}}\,dr=g_T(p).$$

Then for every positive number ε there exist $\delta(\varepsilon)$ and $h(\varepsilon)$ such that for $\delta\leqslant\delta(\varepsilon)$, $h\leqslant h(\varepsilon)$ the inequality

$$\sum_{i=1}^{n} h[g_{\delta,i}-g_{T,i}]^2\leqslant\delta^2,$$

implies the inequality

$$|f_{\alpha(\delta),j}-f_{T,j}|<\varepsilon,$$

where

$$g_{T,i}=g_T(p_i), \quad f_{T,j}=f_T(r_j)$$

and $f_{\alpha(\delta)}^h$ is the minimizer of functional $M_h^{\alpha(\delta)}[f^h, g_\delta^h]$.

Proof. Since $f_{\alpha(\delta)}^h$ is the minimizer of $M_h^{\alpha(\delta)}$, we have

$$M_h^{\alpha(\delta)}[f_{\alpha(\delta)}^h, g_\delta^h]\leqslant M_h^{\alpha(\delta)}[f_T^h, g_\delta^h]$$

or

$$\sum_{i=1}^{n} h\Big[2\sum_{j=i}^{n} a_{i,j} f_{\alpha(\delta),j}-g_{\delta,i}\Big]^2+\alpha(\delta)\sum_{j=0}^{n} h\Big[\frac{f_{\alpha(\delta),j+1}-f_{\alpha(\delta),j}}{h}\Big]^2+\alpha(\delta)\sum_{j=1}^{n} hf_{\alpha(\delta),j}^2$$

$$\leqslant\sum_{i=1}^{n} h\Big[2\sum_{j=i}^{n} a_{i,j} f_{T,j}-g_{\delta,i}\Big]^2+\alpha(\delta)\sum_{j=0}^{n} h\Big[\frac{f_{T,j+1}-f_{T,j}}{h}\Big]^2+\alpha(\delta)\sum_{j=1}^{n} hf_{T,j}^2$$

$$=\sum_{i=1}^{n} h\Big\{2\sum_{j=i}^{n} f_{T,j}\int_{r_j}^{r_{j+1}}\frac{r}{\sqrt{r^2-p_i^2}}\Big(1-\frac{r-r_j}{h}\Big)dr$$

$$+2\sum_{j=i}^{n} f_{T,j+1}\int_{r_j}^{r_{j+1}}\frac{r}{\sqrt{r^2-p_i^2}}\,\frac{r-r_j}{h}\,dr-g_{\delta,i}\Big\}^2$$

$$+\alpha(\delta)\int_0^T[f_T^2(r)+f_T'(r)^2]dr+\xi \quad (\xi\to 0,\ h\to 0)$$

$$=\sum_{i=1}^{n} h\Big\{2\sum_{j=i}^{n}\int_{r_j}^{r_{j+1}}\frac{r}{\sqrt{r^2-p_i^2}}f_{T,j}dr+2\sum_{j=i}^{n}\int_{r_j}^{r_{j+1}}\frac{r}{\sqrt{r^2-p_i^2}}\,\frac{f_{T,j+1}-f_{T,j}}{h}(r-r_j)\,dr-g_{\delta,i}\Big\}^2$$

$$+\alpha(\delta)\int_0^{\bar{T}}[f_T^2(r)+f'_T(r)^2]dr+\xi$$

$$=\sum_{i=1}^n h\left\{2\sum_{j=i}^n\int_{r_j}^{r_{j+1}}\frac{r}{\sqrt{r^2-p_i^2}}f_T(r)dr-g_{\delta,i}\right\}^2+\xi_1(\xi_1\to 0,\ h\to 0)$$

$$+\alpha(\delta)\int_0^{\bar{T}}[f_T^2(r)+f'_T(r)^2]dr+\xi$$

$$=\sum_{i=1}^n h[g_{T,i}-g_{\delta,i}]^2+\alpha(\delta)\int_0^{\bar{T}}[f_T^2(r)+f'_T(r)^2]dr+\xi_1+\xi$$

$$\leqslant\delta^2+\delta^2\int_0^{\bar{T}}[f_T^2(r)+f'_T(r)^2]dr+\delta^2=\delta^2 d\quad(h\leqslant h(\varepsilon)),$$

$$d=2+\int_0^{\bar{T}}[f_T^2(r)+f'_T(r)^2]dr.$$

Therefore

$$\sum_{j=0}^n h\left[\frac{f_{\alpha(\delta),j+1}-f_{\alpha(\delta),j}}{h}\right]^2+\sum_{j=1}^n hf^2_{\alpha(\delta),j}\leqslant d,$$

$$\sum_{i=1}^n h\left[2\sum_{j=i}^n a_{i,j}f_{\alpha(\delta),j}-g_{\delta,i}\right]^2\leqslant\delta^2 d.$$

Thus, the functions

$$f_T^h(r)=f_{T,j}+\frac{f_{T,j+1}-f_{T,j}}{h}(r-r_j),\quad f^h_{\alpha(\delta)}(r)=f_{\alpha(\delta),j}+\frac{f_{\alpha(\delta),j+1}-f_{\alpha(\delta),j}}{h}(r-r_j),$$

$$r\in[r_j,\ r_{j+1}],\quad j=0,\ 1,\ \cdots,\ n$$

belong to the compact subset M_h of the space $\bar{C}$:

$$M_h=\left\{f(r):\int_0^{\bar{T}}[f^2(r)+f'(r)^2]dr\leqslant 3d\right\},$$

and hence

$$Af_T^h(r)\in AM_h,\quad Af^h_{\alpha(\delta)}(r)\in AM_h.$$

It follows from the continuity of A^{-1} on AM_h that for $\varepsilon>0$, there exists $\eta(\varepsilon)$ such that for $\|Af_T^h-Af^h_{\alpha(\delta)}\|_{L_2}\leqslant\eta(\varepsilon)$

$$\|f_T^h(r)-f^h_{\alpha(\delta)}\|_{\bar{C}}<\varepsilon$$

or

$$|f_{T,j}-f_{\alpha(\delta),j}|<\varepsilon.$$

Since

$$\|Af_T^h-Af^h_{\alpha(\delta)}\|^2_{L_2}\leqslant\delta^2(2d+3),$$

as in Theorem 2 we may choose

$$\delta(\varepsilon)=\frac{\eta(\varepsilon)}{\sqrt{2d+3}}.$$

Consequently, for $\delta\leqslant\delta(\varepsilon)$ we have

$$\|Af_T^h-Af^h_{\alpha(\delta)}\|\leqslant\eta(\varepsilon),$$

and hence

$$|f_{T,j}-f_{\alpha(\delta),j}|<\varepsilon.$$

Thus, Theorem 4 is proven.

References

[1] Feng Kang, Inverse Problems in Mathematical Physics, 1983.
[2] Hao Shou, Application of the regularization method to the numerical solution of Abel's integral equation, *JCM*, **3**: 1 (1985), 27—34.
[3] А. Н. Тихонов, В. Я. Арсенин, Методы Решения Некорректных Задач, 1979.
[4] А. Н. Тихонов, О. Решении Некорректно Поставленных Задач, *ДАН СССР*, 151, 3, 1963.