NUMERICAL SOLUTION FOR THE STEFAN PROBLEM WITH CERTAIN SINGULARITIES*1)

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§ 1. Introduction

In this paper the Stefan problem refers to the heat flow problem of the materials which undergo phase change. As is well-known, it is an important example of the free boundary problem. In recent years, there have been a great number of numerical methods for the Stefan problem: difference methods in which one establishes difference schemes for the given original problem; enthalpy methods in which one gets rid of the free boundary through introducing enthalpy, a function of temperature; variational methods, finite element methods, etc. For the survey of these methods, we refer the readers to [1], [2].

However, most of these numerical methods do not deal with the discontinuity of the temperature at the origin (0, 0) or at some point $(b, 0)$ $(b>0)$.

Due to the discontinuity, the derivatives of temperature with respect to t and x must be very large in the neighbourhood of the point of discontinuity. R. Bonnet and P. Jamet[3] suggested a third-order method which permits the discontinuity in the t direction. They computed the Stefan problem with initial discontinuity. Later, according to the idea of [3], Chin Hsien Li[4] suggested another method which is applicable to the problem whose free boundary is in an implicit form. The result given in [3] is good if t is not very small, but near the point of discontinuity, the error is not very small. This is shown by our computation given in § 7.

One of the authors has developed the singularity-separating difference method, by which very accurate numerical results on discontinuous solutions of quasilinear hyperbolic systems have been obtained[5]. Moreover, based on the analytical property of solution of flow in the region where a shock wave passes through a "strong explosion" center, a quite effectual method for this problem has been proposed in [6] and very good results have been achieved. These experiences show that it is possible to establish a numerical method with high accuracy for the problems with certain singularities by considering the properties of solutions.

A similar idea was proposed by L. Fox in [1]. He pointed out that the combination of numerical and analytical methods is the best way to solve the problem. Especially for the problem with singularity, this kind of method not only avoids the disorder caused by the singularity, but also can clearly describe the state of solution near the singularity.

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According to these experiences and ideas, we suggest a new method which could give very accurate results everywhere. The feature of this difference method is that by using the analytical solution given in [7], we introduce a new coordinate system under which the derivatives are not large before the schemes are established. In this paper we give the method and some numerical results, and compare it with other two methods. Error comparison shows that our method is better than the method in [3] and the Galerkin method[8].

§ 2. Formulation of the Problem

Consider the ice-water system

$$c_1(x, t)\frac{\partial u_1}{\partial t}=\frac{\partial}{\partial x}\left(k_1(x, t)\frac{\partial u_1}{\partial x}\right), \quad \text{in } \Omega_1, \tag{2.1}$$

$$c_2(x, t)\frac{\partial u_2}{\partial t}=\frac{\partial}{\partial x}\left(k_2(x, t)\frac{\partial u_2}{\partial x}\right), \quad \text{in } \Omega_2 \tag{2.2}$$

with boundary conditions

$$u_1(0, t)=g(t), \tag{2.3}$$

$$u_2(Q, t)=h(t), \tag{2.4}$$

free boundary conditions on the phasechange line $x=b+\xi(t)$

$$u_1(b+\xi(t), t)=u_2(b+\xi(t), t)=0, \tag{2.5}$$

$$\lambda\frac{d\xi}{dt}=k_1(x, t)\left.\frac{\partial u_1}{\partial x}\right|_{x=b+\xi(t)}-k_2(x, t)\left.\frac{\partial u_2}{\partial x}\right|_{x=b+\xi(t)}, \tag{2.6}$$

and initial condition

$$\begin{cases} u_1(x, 0)=\mu(x), & 0<x<b, & (2.7)\\ u_2(x, 0)=f(x), & b<x<Q, & (2.8)\end{cases}$$

where

$\Omega_1=\{(x, t): 0<x<b+\xi(t), t>0\}$ is the solid state region;

$\Omega_2=\{(x, t): b+\xi(t)<x<Q, t>0\}$ is the liquid state region;

and λ is the latent heat of phasechange. Moreover, we make the following assumptions:

Assumption A. $c_i(x, t)$, $k_i(x, t)$ $(i=1, 2)$, $g(t)$, $h(t)$, $\mu(x)$, $f(x)$ are all continuous functions.

Assumption B. $\mu(b)<0<f(b)$ (if $b=0$, we assume $g(0)<0<f(0)$ instead).

We refer to (2.1)—(2.8) as System (SI); when $b=0$, we refer to (2.1)—(2.6), (2.8) as System (S).

For the sake of later reference, we list some notation here.

$$c_i=c_i(b, 0),\ k_i=k_i(b, 0), \quad i=1, 2,$$

$$g=g(0),\ \mu=\mu(b),\ f=f(b).$$

§ 3. Systems (CSI) and (CS)

Suppose we have the following equations and conditions:

$$c_1\frac{\partial u_{10}}{\partial t}=k_1\frac{\partial^2 u_{10}}{\partial x^2}, \quad \text{in } \Omega_{10}, \tag{3.1}$$

$$c_2 \frac{\partial u_{20}}{\partial t} = k_2 \frac{\partial^2 u_{20}}{\partial x^2}, \quad \text{in } \Omega_{20}, \tag{3.2}$$

$$u_{10}(0, t) = g, \tag{3.3}$$

$$u_{10}(b+\xi_0(t), t) = u_{20}(b+\xi_0(t), t) = 0, \tag{3.4}$$

$$\lambda \frac{d\xi_0}{dt} = k_1 \frac{\partial u_1}{\partial x}\bigg|_{x=b+\xi_0(t)} - k_2 \frac{\partial u_{20}}{\partial x}\bigg|_{x=b+\xi_0(t)}, \tag{3.5}$$

$$u_{10}(x, 0) = \mu, \tag{3.6}$$

$$u_{20}(x, 0) = f, \tag{3.7}$$

where $\Omega_{10} = \{(x, t): -\infty < x < b+\xi_0(t), t > 0\}$ (if $b=0$, $\Omega_{10} = \{(x, t): 0 < x < \xi_0(t), t > 0\}$), $\Omega_{20} = \{(x, t): b+\xi_0(t) < x < \infty, t > 0\}$.

In what follows, (3.1), (3.2), (3.4)—(3.7) are called System (CSI), and (3.1)—(3.5), (3.7) with $b=0$ are called System (CS).

Noticing that System (SI) has the same parameters $k_i(x, t)$, $c_i(x, t)$ $(i=1, 2)$, $\mu(x)$, $f(x)$ as System (CSI) has as (x, t) tend to $(b, 0)$, we can describe the state of solution of System (SI) in the neighbourhood of $(b, 0)$ with the help of the analytical solution of System (CSI). Obviously, there exists a similar relation between System (S) and System (CS). This fact makes it possible to overcome the difficulties caused by the singularity.

Now we seek the analytical solutions of Systems (CSI) and (CS).

§ 4. Solutions of Systems (CSI) and (CS)

We shall describe in detail the procedure of looking for the analytical solution of System (CS).

We consider the heat flow equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \tag{4.1}$$

a being a constant. Under the similarity transformation

$$\begin{cases} x' = kx, \\ t' = k^2 t, \end{cases} \tag{4.2}$$

k being a constant, the form of the equation is unchanged. Therefore

$$u(x, t) = u(kx, k^2 t),$$

k being any constant.

In other words, along the curve $\frac{x}{\sqrt{t}} = \text{constant}$ $u(x, t)$ is a constant. This means that $u(x, t)$ is a function of the variable $z = \frac{cx}{\sqrt{t}}$, c being any constant. Let $c = 1/2a$ and put $u(x, t) = \psi(z)$ into (4.1). We have

$$\psi'(z) \frac{-x}{2a \cdot 2t^{3/2}} = \frac{a^2}{4a^2 t} \psi''(z).$$

This is an ODE, whose general solution is

$$\psi(z) = A + B\int_0^z e^{-\xi^2} d\xi, \tag{4.3}$$

A, B being constants.

The solution of the problem with initial and boundary conditions which are unchanged under (4.2) also has the form (4.3).

We turn back to System (CS). Let

$$a_1^2=k_1/c_1, \quad a_2^2=k_2/c_2$$

and assume its solution is of the form

$$u_{10}(x,\ t)=A_1+B_1\Phi\left(\frac{x}{2a_1\sqrt{t}}\right),$$

$$u_{20}(x,\ t)=A_2+B_2\Phi\left(\frac{x}{2a_2\sqrt{t}}\right),$$

where A_i, B_i $(i=1,\ 2)$ are constants and $\Phi(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-\xi^2}d\xi$.

Firstly, we determine the form of phasechange line $x=\xi_0(t)$. Let $\varphi(t)=\frac{\xi_0(t)}{\sqrt{t}}$, and put $\varphi(t)$ into (3.4). We obtain

$$A_1+B_1\Phi\left(\frac{\varphi(t)}{2a_1}\right)=A_2+B_2\Phi\left(\frac{\varphi(t)}{2a_2}\right).$$

Then differentiating both sides with respect to t, we have

$$\varphi'(t)\left(\frac{B_1}{a_1}e^{-\left(\frac{\varphi(t)}{2a_1}\right)^2}-\frac{B_2}{a_2}e^{-\left(\frac{\varphi(t)}{2a_2}\right)^2}\right)=0,$$

which means either

$$\varphi'(t)=0 \quad \text{or} \quad \varphi(t)=\left\{\frac{\ln\left(\frac{B_1a_2}{B_2a_1}\right)}{\left(\frac{1}{2a_1}\right)^2-\left(\frac{1}{2a_2}\right)^2}\right\}^{1/2}.$$

In any case, $\varphi(t)\equiv\alpha$ (constant) follows, i.e.,

$$\xi_0(t)=\alpha\sqrt{t}.$$

Secondly, we determine the constants A_i, B_i $(i=1,\ 2)$ in $u_{10}(x,\ t)$, $u_{20}(x,\ t)$. Using (3.3), (3.4), (3.7) and noticing $\Phi(\infty)=1$, we get

$$A_1=g, \quad B_1=-\frac{g}{\Phi\left(\frac{\alpha}{2a_1}\right)},$$

$$A_2=-\frac{f\Phi\left(\frac{\alpha}{2a_2}\right)}{1-\Phi\left(\frac{\alpha}{2a_2}\right)}, \qquad B_2=\frac{f}{1-\Phi\left(\frac{\alpha}{2a_2}\right)}.$$

Finally, the constant α can be determined in the following way. From (3.5), we get the identity:

$$\frac{k_1ge^{-\frac{\alpha^2}{4a_1^2}}}{a_1\Phi\left(\frac{\alpha}{2a_1}\right)}+\frac{k_2fe^{-\frac{\alpha^2}{4a_2^2}}}{a_2\left\{1-\Phi\left(\frac{\alpha}{2a_2}\right)\right\}}=-\lambda\alpha\cdot\frac{\sqrt{\pi}}{2}$$

or equivalently, α is a root of equation $F(\alpha)=0$, where

$$F(\alpha)\equiv\frac{k_1 g e^{-\alpha^2/4a_1^2}}{a_1\Phi\left(\frac{\alpha}{2a_1}\right)}+\frac{k_2 f e^{-\alpha^2/4a_2^2}}{a_2\left\{1-\Phi\left(\frac{\alpha}{2a_2}\right)\right\}}+\lambda\alpha\cdot\frac{\sqrt{\pi}}{2}.$$

The function $F(\alpha)$ has the properties:

1. When α changes from 0 to ∞, $F(\alpha)$ changes from $-\infty$ to ∞;
2. $F(\alpha)$ is a strictly increasing function.

Property 1 is easy to get from the definition of $F(\alpha)$. Let

$$f_1(\alpha)=-\frac{e^{-\alpha^2}}{\Phi(\alpha)},$$

$$f_2(\alpha)=\frac{e^{-\alpha^2}}{1-\Phi(\alpha)}.$$

Clearly, if $f_1(\alpha)$, $f_2(\alpha)$ are strictly increasing functions of α, then $F(\alpha)$ also is an increasing function.

Differentiating $f_1(\alpha)$, $f_2(\alpha)$, we have

$$f_1'(\alpha)=\frac{2}{\sqrt{\pi}}\cdot\frac{2\alpha e^{-\alpha^2}\int_0^\alpha e^{-\xi^2}d\xi+(e^{-\alpha^2})^2}{\Phi^2(\alpha)}>0,$$

$$f_2'(\alpha)=\frac{2}{\sqrt{\pi}}\cdot\frac{\left(e^{-\alpha^2}-2\alpha\int_\alpha^\infty e^{-\xi^2}d\xi\right)}{(1-\Phi(\alpha))^2}\cdot e^{-\alpha^2}$$

$$>\frac{2}{\sqrt{\pi}}\cdot e^{-\alpha^2}\cdot\frac{e^{-\alpha^2}-\int_\alpha^\infty 2\xi e^{-\xi^2}d\xi}{(1-\Phi(\alpha))^2}=0.$$

By using these two relations, Property 2 can be proved easily.

According to these two properties, we know that $F(\alpha)=0$ has a unique root in the interval $(0,\infty)$. In practical computation, it can be determined by a certain numerical method.

We can use the same procedure to seek the analytical solution of System (CSI). The results are listed below:

$$\tilde{u}_{10}(x,t)=\frac{\mu}{\tilde{\Phi}\left(\frac{\alpha}{2a_1}\right)}\left\{\tilde{\Phi}\left(\frac{\alpha}{2a_1}\right)-\tilde{\Phi}\left(\frac{x-b}{2a_1\sqrt{t}}\right)\right\},$$

$$\tilde{u}_{20}(x,t)=\frac{f}{1-\tilde{\Phi}\left(\frac{\alpha}{2a_2}\right)}\left\{\tilde{\Phi}\left(\frac{x-b}{2a_2\sqrt{t}}\right)-\tilde{\Phi}\left(\frac{\alpha}{2a_2}\right)\right\},$$

where $\tilde{\Phi}(z)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{z}e^{-\xi^2}d\xi$ and the phasechange line is

$$x=b+\tilde{\xi}_0(t).$$

Here $\tilde{\xi}_0(t)=\alpha\sqrt{t}$ and α is the unique root of $\tilde{F}(\alpha)=0$, where

$$\tilde{F}(\alpha)=\lambda\sqrt{\pi}\,\alpha+\frac{k_1\mu}{a_1}\cdot\frac{e^{-\left(\frac{\alpha}{2a_1}\right)^2}}{\tilde{\Phi}\left(\frac{\alpha}{2a_1}\right)}+\frac{k_2 f}{a_2}\cdot\frac{e^{-\left(\frac{\alpha}{2a_2}\right)^2}}{1-\tilde{\Phi}\left(\frac{\alpha}{2a_2}\right)}.$$

§ 5. Analysis on Systems (S) and (SI)

With the help of the solutions of Systems (CSI) and (CS), we can describe the state of the solution of Systems (SI), (S) in the neighbourhood of $(b, 0)$.

The methods for Systems (SI) and (S) are essentially the same, though formally different. For clearness and simplicity, we shall only describe in detail the method for System (S), i.e., for the case $b=0$.

We begin with System (CS). The family of curves $\{B\xi_0(t): 0<B<1\}$ has the properties:

1. It covers Ω_{10}.
2. All curves intersect at and only at $t=0$ and $x=0$.
3. Along each curve, u_{10} is equal to a constant C_B. When B changes from 0 to 1, C_B changes from g to 0.

For System (S), the family of curves $\{B\xi(t): 0<B<1\}$ has the properties 1 (Ω_{10} must be replaced by Ω_1) and 2, but generally it does not have property 3. Now we assume

$$\lim_{t\to 0}\frac{\xi(t)}{\xi_0(t)}=1, \tag{5.1}$$

and

$$\lim_{t\to 0}\frac{u_1(B\xi(t), t)}{u_{10}(B\xi_0(t), t)}=1. \tag{5.2}$$

Therefore,

$$\lim_{t\to 0} u_1(B\xi(t), t)=\lim_{t\to 0} u_{10}(B\xi_0(t), t)=C_B.$$

This means that System (S) has the property:

3'. Along every curve $x=B\xi(t)$, the limit of $u_1(x, t)$ at $t=0$ is equal to $u_{10}(B\xi_0(t), t)=C_B$.

The properties 1, 2, 3' show us the fact: Because the limits of u_1 at $t=0$ along different curves are different and these curves knot together at the origin, the temperature must change violently in a small region near the origin. In order to overcome the difficulty in computation caused by this phenomenon, we make a coordinate transformation such that the curves will not go together as t tends 0. This can be realized in the following way.

We choose a coordinate transformation $A_1: (x, t)\to(y, t)$ such that the family of curves $\{B\xi(t): 0<B<1\}$ is transformed into that of parallel lines $\{y=B: 0<B<1\}$. Let $v_1(y, t)=u_1(x(y, t), t)$. Clearly, the value of $v_1(y, 0)$ may be computed by using the formula

$$v_1(B, 0)=\lim_{t\to 0} u_1(B\xi(t), t)=\lim_{t\to 0} u_{10}(B\xi_0(t), t)=C_B.$$

We have analysed the problem in Ω_1. Now we turn to Ω_2.

The family of curves $\{B\xi(t): 1<B<\infty\}$ has the properties:

1. It covers Ω_2 and the region $\{(x, t): Q\leqslant x<\infty, t>0\}$.
2. All curves intersect at and only at $t=0$ and $x=0$.

3'. Along each curve $x=B\xi(t)$, the limit of $u_2(x, t)$ at $t=0$ is equal to $u_{20}(B\xi_0(t), t)=C_B$.

Now some problems occur. Because the limits of u_2 at $t=0$ along different curves are different and these curves have a knot at $t=0$ and $x=0$, the gradient of $u_2(x, t)$ must be very large near the origin. If we deal with Ω_2 as we do with Ω_1, i.e., transform $\{B\xi(t): 1<B<\infty\}$ to $\{y=B: 1<B<\infty\}$ in a new coordinate system, we shall face two problems:

1. Because the length of the interval in the y direction is infinite, it is unfeasible for practical computation.

2. The line on which the initial values should be given becomes the line $y=\infty$.

We suggest a method to cope with these two problems.

Take a positive number M such that $|C_M-f(0)|$ is sufficiently small. This means that $|u_2(M\xi(0), 0)-f(0)|$ is sufficiently small. From the definition of C_M, the number M which satisfies the condition above does exist. We introduce such a coordinate transformation A_2: $(x, t)\to(y, t)$ that the family of curves $\{B\xi(t): 1<B<M\}$ is transformed into the family of parallel lines $\{y=B: 1<B<M\}$, and the complementary set of $\{B\xi(t): 1<B<M\}$ in Ω_2 into the set $\{(y, t): M\le y<M+1, t>0\}$. Let $v_2(y, t)=u_2(x(y, t), t)$. Clearly, the value of $v_2(y, 0)$ may be computed by using the formula

$$\begin{cases} v_2(B, 0)=\lim\limits_{t\to 0} u_2(B\xi(t), t)=u_{20}(B\xi_0(t), t), & \text{if } 1<B<M, \\ v_2(B, 0)=\varphi(B), & \text{if } M\le B<M+1, \end{cases}$$

where $\varphi(y)=f(x(y, 0))$.

Now we formulate the procedure that we have described.

Define the transformation:

$$(*)\qquad \begin{cases} y=\begin{cases} \dfrac{x}{\xi(t)}, & 0<x<M\xi(t), \\ \dfrac{x-M\xi(t)}{Q-M\xi(t)}+M, & M\xi(t)<x<Q, \end{cases} \\ t=t. \end{cases}$$

Clearly, between the functions in two coordinate systems there are the following relations:

$$u_1(x, t)=v_1(y, t), \quad u_2(x, t)=v_2(y, t),$$

$$f(x)=\varphi(y), \quad g(t)=g(t), \quad h(t)=h(t),$$

$$\frac{\partial u}{\partial t}=\frac{\partial v}{\partial y}\frac{\partial y}{\partial t}+\frac{\partial v}{\partial t},$$

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial v}{\partial t}\frac{\partial t}{\partial x}=\frac{\partial v}{\partial y}\frac{\partial y}{\partial x},$$

$$\frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right)=\frac{\partial y}{\partial x}\cdot\frac{\partial}{\partial y}\left(k\frac{\partial y}{\partial x}\frac{\partial v}{\partial y}\right),$$

where

$$\frac{\partial y}{\partial t}=\begin{cases} -\dfrac{\xi'(t)}{\xi(t)}y, & 0<x<M\xi(t), \\ \dfrac{M\xi'(t)(y-M-1)}{Q-M\xi(t)}, & M\xi(t)<x<Q, \end{cases}$$

$$\frac{\partial y}{\partial x}=\begin{cases}\frac{1}{\xi(t)}, & 0<x<M\xi(t),\\ \frac{1}{Q-M\xi(t)}, & M\xi(t)<x<Q.\end{cases}$$

And under the new coordinate system, the phase change line is $y=1$. Therefore, under the new coordinate system the problem related to System (S) can be described as System (S′), which consists of the equations

$$c_1(y,t)\frac{\partial v_1}{\partial t}=\frac{1}{\xi(t)}\frac{\partial}{\partial y}\left(\frac{k_1(y,t)}{\xi(t)}\cdot\frac{\partial v_1}{\partial y}\right)+c_1(y,t)\frac{\xi'(t)}{\xi(t)}y\frac{\partial v_1}{\partial y},\quad 0<y<1,\ t>0,$$

$$c_2(y,t)\frac{\partial v_2}{\partial t}=\frac{1}{\xi(t)}\frac{\partial}{\partial y}\left(\frac{k_2(y,t)}{\xi(t)}\cdot\frac{\partial v_2}{\partial y}\right)+c_2(y,t)\frac{\xi'(t)}{\xi(t)}y\frac{\partial v_2}{\partial y},\quad 1<y<M,\ t>0,$$

$$c_2(y,t)\frac{\partial v_2}{\partial t}=\frac{1}{Q-M\xi(t)}\frac{\partial}{\partial y}\left(\frac{k_2(y,t)}{Q-M\xi(t)}\frac{\partial v_2}{\partial y}\right)$$
$$-c_2(y,t)\frac{M\xi'(t)(y-M-1)}{Q-M\xi(t)}\frac{\partial v_2}{\partial y},\quad M\leqslant y<M+1,\ t>0,$$

the initial value

$$\begin{cases}v_1(y,0)=u_{10}(y\xi_0(t),t)=C_y, & 0<y<1,\\ v_2(y,0)=u_{20}(y\xi_0(t),t)=C_y, & 1<y<M,\\ v_2(y,0)=\varphi(y), & M\leqslant y<M+1,\end{cases}$$

the boundary conditions

$$v_1(0,t)=g(t),\quad t>0,$$
$$v_2(M+1,t)=h(t),\quad t>0,$$

and the free boundary conditions

$$v_1(1,t)=v_2(1,t)=0,$$
$$\lambda\frac{d\xi}{dt}=\frac{1}{\xi(t)}\left\{k_1(y,t)\frac{\partial v_1}{\partial y}\bigg|_{y=1}-k_2(y,t)\frac{\partial v_2}{\partial y}\bigg|_{y=1}\right\}.$$

What is left is to establish difference schemes for System (S′) and to get numerical solution. We shall discuss these problems in the next section.

A similar procedure can be applied to analyse System (SI). In this case, instead of using the solution of System (CS), we use the solution of System (CSI) to "connect" two parts of the discontinuous initial value of System (SI).

We take a number M such that $|\tilde{u}_{10}(b-M\tilde{\xi}_0(t),t)-\mu(b)|$ and $|f(b)-\tilde{u}_{20}(b+M\tilde{\xi}_0(t),t)|$ are sufficiently small. And we try to find a coordinate transformation, which.

1. Turns the family of curves $\{b+B_1\xi(t):\ -M<B_1<1\}$ into that of parallel lines $\{y=2M+B_1:\ -M<B_1<1\}$;

2. Turns the complementary set of $\{b+B_1\xi(t):\ -M<B_1<1\}$ in Ω_1 into the strip region $\{(y,t):\ 0<y<M,\ t>0\}$;

3. Turns the family of curves $\{b+B_2\xi(t):\ 1<B_2<M\}$ into that of parallel lines $\{y=2M+B_2:\ 1<B_2<M\}$; and

4. Turns the complementary set of $\{b+B_2\xi(t):\ 1<B_2<M\}$ in Ω_2 into the strip region $\{(y,t):\ 3M<y<4M,\ t>0\}$.

This is the transformation we want:

$$(\mathrm{I}^*)\qquad \begin{cases} y=\begin{cases} \dfrac{Mx}{b-M\xi(t)}, & 0<x<b-M\xi(t), \\ \dfrac{x-b}{\xi(t)}+2M, & b-M\xi(t)<x<b+M\xi(t), \\ M\dfrac{x-(b+M\xi(t))}{Q-(b+M\xi(t))}+3M, & b+M\xi(t)<x<Q, \end{cases} \\ t=t. \end{cases}$$

Through the transformation the phase change line $x=b+\xi(t)$ becomes the straight line $y=2M+1$, and the curved lines $x=b-M\xi(t)$ and $x=b+M\xi(t)$ are turned into $y=M$ and $y=3M$. Obviously, the analytical solution of System (CSI) can be used to provide a part of the initial value for the problem under the new coordinate system. Letting

$$v_1(y,t)=u_1(x(y,t),t),$$
$$v_2(y,t)=u_2(x(y,t),t),$$

and noticing

$$\varphi(y)=f(x(y,0)),$$
$$\psi(y)=\mu(x(y,0)),$$

we can rewrite System (SI) as System (SI′), which consists of the equations

$$c_1(y,t)\frac{\partial v_1}{\partial t}=\frac{M}{b-M\xi(t)}\cdot\frac{\partial}{\partial y}\left(\frac{Mk_1(y,t)}{b-M\xi(t)}\cdot\frac{\partial v_1}{\partial y}\right)$$
$$-c_1(y,t)\frac{M\xi'(t)}{b-M\xi(t)}y\frac{\partial v_1}{\partial y},\quad 0<y<M,\ t>0,$$

$$c_1(y,t)\frac{\partial v_1}{\partial t}=\frac{1}{\xi(t)}\frac{\partial}{\partial y}\left(\frac{k_1(y,t)}{\xi(t)}\frac{\partial v_1}{\partial y}\right)$$
$$+c_1(y,t)\frac{\xi'(t)}{\xi(t)}(y-2M)\frac{\partial v_1}{\partial y},\quad M<y<2M+1,\ t>0,$$

$$c_2(y,t)\frac{\partial v_2}{\partial t}=\frac{1}{\xi(t)}\frac{\partial}{\partial y}\left(\frac{k_2(y,t)}{\xi(t)}\frac{\partial v_2}{\partial y}\right)$$
$$+c_2(y,t)\frac{\xi'(t)}{\xi(t)}(y-2M)\frac{\partial v_2}{\partial y},\quad 2M+1<y<3M,\quad t>0,$$

$$c_2(y,t)\frac{\partial v_2}{\partial t}=\frac{M}{Q-(b+M\xi(t))}\frac{\partial}{\partial y}\left(\frac{Mk_2(y,t)}{Q-(b+M\xi(t))}\cdot\frac{\partial v_2}{\partial y}\right)$$
$$-c_2(y,t)\frac{M\xi'(t)}{Q-(b+M\xi(t))}(y-4M)\frac{\partial v_2}{\partial y},\quad 3M<y<4M,\ t>0;$$

the initial value

$$\begin{cases} v_1(y,0)=\psi(y), & 0<y<M, \\ v_1(y,0)=u_{10}(b+(y-2M)\xi_0(t),t), & M\leqslant y<2M+1, \\ v_2(y,0)=u_{20}(b+(y-2M)\xi_0(t),t), & 2M+1<y\leqslant 3M, \\ v_2(y,0)=\varphi(y), & 3M<y<4M; \end{cases}$$

the boundary value conditions

$$v_1(0,t)=g(t),\quad t>0,$$
$$v_2(4M,t)=h(t),\quad t>0;$$

and the free boundary conditions

$$v_1(2M+1,\ t)=v_2(2M+1,\ t)=0,$$

$$\lambda\frac{d\xi}{dt}=\frac{1}{\xi(t)}\left\{k_1(y,\ t)\frac{\partial v_1}{\partial y}\bigg|_{y=2M+1}-k_2(y,\ t)\frac{\partial v_2}{\partial y}\bigg|_{y=2M+1}\right\}.$$

It is easy to establish difference schemes for System (SI′) and solve this problem.

§ 6. Difference Schemes

We only establish difference schemes for System (S′). Those for System (SI′) are similar.

In this and the next sections, we use the following notation. Δt: time step; Δy: space step; y_i: $i\Delta y$; F_i^n: approximate value of function $F(y,\ t)$ at $(i\Delta y,\ n\Delta t)$; $\xi^{n+\frac{1}{2}}=\frac{1}{2}(\xi^n+\xi^{n+1})$.

System (S′) is discretized in the following way. The equations are approximated by

$$c_{1i}^{n+\frac{1}{2}}\frac{v_{1i}^{n+1}-v_{1i}^n}{\Delta t}=\left(\frac{1}{\xi^{n+\frac{1}{2}}}\right)^2\{\Delta_-(k_{1i}^{n+1}\theta\Delta_+v_{1i}^{n+1})+\Delta_-(k_{1i}^n(1-\theta)\Delta_-v_{1i}^n)\}$$
$$+c_{1i}^{n+\frac{1}{2}}\frac{\xi^{n+1}-\xi^n}{\xi^{n+\frac{1}{2}}\Delta t}y_i(\theta\Delta_0v_{1i}^{n+1}+(1-\theta)\Delta_0v_{1i}^n),$$
$$i=1,\ \cdots,\ N_1-1,\ N_1\Delta y=1, \tag{6.1}$$

$$c_{2i}^{n+\frac{1}{2}}\frac{v_{2i}^{n+1}-v_{2i}^n}{\Delta t}=\left(\frac{1}{\xi^{n+\frac{1}{2}}}\right)^2\{\Delta_-(k_{2i}^{n+1}\theta\Delta_+v_{2i}^{n+1})+\Delta_-(k_{2i}^n(1-\theta)\Delta_+v_{2i}^n)\}$$
$$+c_{2i}^{n+\frac{1}{2}}\frac{\xi^{n+1}-\xi^n}{\xi^{n+\frac{1}{2}}\Delta t}(1+y_i)\{\theta\Delta_0v_{2i}^{n+1}+(1-\theta)\Delta_0v_{2i}^n\},$$
$$i=1,\ \cdots,\ N_2-1,\ N_2\Delta y=M-1, \tag{6.2}$$

$$c_{2i}^{n+\frac{1}{2}}\frac{v_{2i}^{n+1}-v_{2i}^n}{\Delta t}=\left(\frac{1}{Q-M\xi^{n+\frac{1}{2}}}\right)^2\{\Delta_-(k_{2i}^{n+1}\theta\Delta_+v_{2i}^{n+1})+\Delta_-(k_{2i}^n(1-\theta)\Delta_+v_{2i}^n)\}$$
$$-c_{2i}^{n+\frac{1}{2}}\frac{M(\xi^{n+1}-\xi^n)}{\Delta t(Q-M\xi^{n+\frac{1}{2}})}(1+y_i-M-1)(\theta\Delta_0v_{2i}^{n+1}+(1-\theta)\Delta_0v_{2i}^n),$$
$$i=N_2+1,\ \cdots,\ N_2+N_3-1,\ N_3\Delta y=1, \tag{6.3}$$

where $0<\theta<1$. The boundary conditions and the initial value are rewritten as follows:

$$v_{1,0}^{n+1}=g(t^{n+1}), \tag{6.4}$$
$$v_{2,N_2+N_3}^{n+1}=h(t^{n+1}), \tag{6.5}$$
$$v_{1i}^0=u_{10}(i\Delta y\xi_0(t),\ t),\quad i=0,\ 1,\ \cdots,\ N_1, \tag{6.6}$$
$$v_{2i}^0=u_{2,0}((1+i\Delta y)\xi_0(t),\ t),\quad i=0,\ 1,\ \cdots,\ N_2, \tag{6.7}$$
$$v_{2i}^0=\varphi(1+i\Delta y),\quad i=N_2+1,\ \cdots,\ N_2+N_3. \tag{6.8}$$

And the continuity condition on the phase change line is unchanged:

$$v_{1,N_1}^{n+1}=v_{2,0}^{n+1}=0. \tag{6.9}$$

If $\xi^{n+\frac{1}{2}}$ and v_{1i}^n of the previous level are known, then (6.1), (6.4), (6.9) can be solved independently. However, (6.2), (6.3), (6.5), (6.9) still form an underdetermined system which cannot be solved uniquely. One additional condition should be added. This condition should show the continuity of $\frac{\partial u_2}{\partial x}$ on the curve $x=M\xi(t)$ in the (x, t) plane, i.e., the continuity of $\frac{\partial v_2}{\partial y}\cdot\frac{\partial y}{\partial x}$ on the straight line $y=M$

$$\frac{1}{\xi(t)}\frac{\partial v_2(M_-, t)}{\partial y}=\frac{1}{Q-M\xi(t)}\frac{\partial v_2(M_+, t)}{\partial y}.$$

Its discrete form is

$$\frac{1}{\xi^{n+1}}\frac{v_{2,N_2}^{n+1}-v_{2,N_2-1}^{n+1}}{\Delta y}=\frac{1}{Q-M\xi^{n+1}}\frac{v_{2,N_2+1}^{n+1}-v_{2,N_2}^{n+1}}{\Delta y}. \tag{6.10}$$

Now we describe the method of simultaneously solving ξ^{n+1}, $v_{1,i}^{n+1}$, $v_{2,i}^{n+1}$.

Suppose v_{1i}^n, v_{2i}^n, ξ^n are known. The method we use here is the same as that in [9]. It is a predictor-corrector method which is of second order accuracy.

Define

$$R(t^n, \Delta t)=\frac{1}{\xi^n\lambda}\left\{k_{1,N_1}^n\frac{v_{1,N_1-2}^n-4v_{1,N_1-1}^n+3v_{1,N_1}^n}{2\Delta y}+k_{2,N_1}^n\frac{v_{2,N_1+2}^n-4v_{2,N_1+1}^n+3v_{2,N_1}^n}{2\Delta y}\right\},$$

and

$$\xi_{(0)}^{n+1}=\xi^n+\Delta tR(t^n, \Delta t).$$

We replace ξ^{n+1} in difference schemes (6.1)—(6.3) by $\xi_{(0)}^{n+1}$. Then solve the problem and get approximate solution v_{1i}^{n+1}, v_{2i}^{n+1}. For distinction, they are denoted by $v_{1,i(0)}^{n+1}$, $v_{2,i(0)}^{n+1}$.

Then we define

$$R^{(0)}(t^{n+1}, t)=\frac{1}{\xi_{(0)}^{n+1}\lambda}\left\{k_{1,N_1}^{n+1}\frac{v_{1,N_1-2(0)}^{n+1}-4v_{1,N_1-1(0)}^{n+1}+3v_{1,N_1(0)}^{n+1}}{2\Delta y}\right.$$
$$\left.+k_{2,N_1}^{n+1}\frac{v_{2,N_1+2(0)}^{n+1}-4v_{2,N_1+1(0)}^{n+1}+3v_{2,N_1(0)}^{n+1}}{2\Delta y}\right\},$$

and

$$\xi^{n+1}=\xi^n+\frac{1}{2}\Delta t\{R(t^n, \Delta t)+R^{(0)}(t^{n+1}, \Delta t)\}.$$

Finally, solving the above system of linear algebraic equations once again, we obtain $v_{1,i}^{n+1}$, $v_{2,i}^{n+1}$.

Remark 1. In (6.1)—(6.3), instead of the same spatial step Δy, one can use different spatial steps, say Δy_1, Δy_2, Δy_3 in (6.1), (6.2), (6.3).

Remark 2. Though $\xi(t)\leqslant Q$ is always true, $M\xi(t)$ is perhaps greater than Q. Therefore at a certain time $t=t_0$, the transformation $(*)$ should be replaced by transformation $(**)$:

$$(**)\qquad \begin{cases} z=\begin{cases}\dfrac{x}{\xi(t)}, & 0<x<\xi(t),\\ \dfrac{x-\xi(t)}{Q-\xi(t)}+1, & \xi(t)<x<Q,\end{cases}\\ t=t,\end{cases}$$

and taking the difference solution at $t=t_0$ as initial value, we continue the computation with the method in [9].

§ 7. Numerical Results

This section includes: 1. numerical examples; 2. error comparisons among our method, the method in [3] and the Galerkin method[8].

1. Numerical Examples

For simplicity, $k_i(x, t)$, $c_i(x, t)$ $(i=1, 2)$ are taken as constants.

Problem 1. For this problem, $b=0$, $k_1=3.38$, $k_2=1.85$, $c_1=500.0$, $c_2=730.0$, $\lambda=34000$, and the length of the interval in the x direction $Q=3.0$.

The initial value condition is $f(x)=x+10$, the left boundary condition is $g(t)=-t-10$, and the right boundary condition is $h(t)=t+Q+10$.

In computation we take $M=41$, $Q=0.5$, $\Delta y_1=0.025$, $\Delta y_2=1.0$, $\Delta y_3=0.05$, $\Delta t=0.05$.

We give three temperature curves for $t=0.05$, 0.5, 1.0 respectively in Fig. 1, which shows that the temperature changes rapidly near the phase change point. The curve $x=\xi(t)$ is given in Fig. 2. The fact that the curve looks like a curve of function $x=\alpha\sqrt{t}$ at the neighbourhood of $t=0$ coincides with our assumption (5.1).

Problem 2. In this problem $b=1$, and Q, λ, k_i, c_i $(i=1, 2)$ are the same as in Problem 1.

The initial value is $\mu(x)=x-11.0$ and $f(x)=x+9$, the left boundary condition is $g(t)=-t-11.0$, and the right boundary condition is $h(t)=t+Q+9$.

In our computation, we take $M=300$, $\theta=0.5$, $\Delta y_1=6.0$, $\Delta y_2=3.01$, $\Delta y_3=2.99$, $\Delta y_4=0.05$, $\Delta t=0.05$.

The phase change curve $x=1+\xi(t)$ is shown in Fig. 3. The temperature curves for $t=0.05$ and $t=1$ are shown in Fig. 4, and the parts near the phase change point are amplified in Fig. 5.

2. Error Comparisons

Problem 3. In this problem b, Q, λ, k_i, c_i $(i=1, 2)$ are the same as in Problem 1, but the initial value and boundary conditions are different. Here the initial value is $f(x)=10$, the left boundary condition is $g(t)=-10$ and the right boundary condition is

$$h(t)=\frac{10}{1-\Phi\left(\frac{\alpha}{2a_2}\right)}\left(\Phi\left(\frac{Q}{2a_2\sqrt{t}}\right)-\Phi\left(\frac{\alpha}{2a_2}\right)\right),$$

where $a_2=(k_2/c_2)^{1/2}$ and the definitions of $\Phi(x)$ and α are the same as in § 4.

This problem possesses an analytical solution

$$u_1(x, t)=\frac{-10}{\Phi\left(\frac{\alpha}{2a_1}\right)}\left(\Phi\left(\frac{\alpha}{2a_1}\right)-\Phi\left(\frac{x}{2a_1\sqrt{t}}\right)\right),$$

$$u_2(x, t)=\frac{10}{1-\Phi\left(\frac{\alpha}{2a_2}\right)}\left(\Phi\left(\frac{x}{2a_2\sqrt{t}}\right)-\Phi\left(\frac{\alpha}{2a_2}\right)\right),$$

where $a_1=(k_1/c_1)^{1/2}$.

In Figs. 6 and 7, the dot-and-dash lines represent the analytical solution.

We take $M=41$, $\theta=0.5$, $\Delta y_1=0.025$, $\Delta y_2=1.0$, $\Delta y_3=0.05$, $\Delta t=0.05$.

When our method is used, 120 net points in the y direction are taken. At $t=0.05$, the maximum error which occurs near 8°C is 0.07325 and at $t=0.2$ the maximum error which occurs near 5°C is 0.07186. These points, almost all with maximum errors, are denoted by "$\cdot$" in Figs. 6 and 7. The CPU time spent from $t=0$ to 0.2 is only 9.96 sec, a part of which (about 5.5 sec.) is spent in computing the initial value.

For the method in [3], we take 20 net points on the left-hand side of the phase change point and 100 net points on the right-hand side. The maximum error is 0.48054 when $t=0.5$, and 0.30772 when $t=0.2$. Both occur near 10°C and these points, almost all with maximum errors, are denoted by "$\times$" in Figs. 6 and 7. The CPU time spent from $t=0$ to 0.2 is 53.5 sec. In the other section of the curve, both "$\cdot$" and "$\times$" are too close to the true solution to denote.

Problem 4. For this problem $b=1$ and Q, λ, k_i, c_i ($i=1, 2$) are the same as in Problem 1.

The initial conditions are $\mu(x)=-10$, $f(x)=10$, the left boundary condition is

$$g(t)=\frac{-10}{\tilde{\Phi}\left(\frac{\alpha}{2a_1}\right)}\left(\tilde{\Phi}\left(\frac{\alpha}{2a_1}\right)-\tilde{\Phi}\left(\frac{-b}{2a_1\sqrt{t}}\right)\right)$$

and the right boundary condition is

$$h(t)=\frac{10}{1-\tilde{\Phi}\left(\frac{\alpha}{2a_2}\right)}\left(\tilde{\Phi}\left(\frac{Q-b}{2a_2\sqrt{t}}\right)-\tilde{\Phi}\left(\frac{\alpha}{2a_2}\right)\right).$$

This problem also has an analytical solution:

$$u_1(x, t)=\frac{-10}{\tilde{\Phi}\left(\frac{\alpha}{2a_1}\right)}\left(\tilde{\Phi}\left(\frac{\alpha}{2a_1}\right)-\tilde{\Phi}\left(\frac{x-b}{2a_1\sqrt{t}}\right)\right),$$

$$u_2(x, t)=\frac{10}{1-\tilde{\Phi}\left(\frac{\alpha}{2a_2}\right)}\left(\tilde{\Phi}\left(\frac{x-b}{2a_2\sqrt{t}}\right)-\tilde{\Phi}\left(\frac{\alpha}{2a_2}\right)\right).$$

In Figs. 8 and 9, the dot-and-dash lines represent the analytical solution.

In computation, we take $M=300$, $\theta=0.5$, $\Delta y_1=6.0$, $\Delta y_2=3.01$, $\Delta y_3=2.99$, $\Delta y_4=0.05$, $\Delta t=0.00625$.

When our method is used, 270 net points are taken in the y direction. The maximum error is 0.08095 when $t=0.00625$ and 0.07622 when $t=0.025$. Both occur near -10°C. The points, almost all with maximum errors, are denoted by "$\cdot$" in Figs. 8 and 9. The CPU time spent from $t=0$ to 0.2 is 11.41 sec., a part of which (about 5.84 sec.) is spent in computing the initial value.

When the method in [3] is used, on each side of the phase change point 140 net points are taken. The short-dash lines in Figs. 8 and 9 represent approximate solution computed by the method in [3]. When $t=0.00625$, the maximum error is 0.70643 and when $t=0.025$, it is 0.55357. Both occur near 10°C. The CPU time spent from $t=0$ to 0.2 is 2 min. 13.13 sec.

Table 1

k \ E \ M	I	k \ E \ M	I
24	0.0505	60	0.0397
28	0.0485	64	0.0389
32	0.0470	68	0.0382
36	0.0459	72	0.0376
40	0.0449	76	0.0370
44	0.0435	80	0.0364
48	0.0424	84	0.0360
52	0.0414	88	0.0355
56	0.0402	92	0.0351

Table 2

k \ E \ M	I	II	III
1	0.0810	0.7957	1.983
4	0.0762	0.6291	0.657
8	0.0660	0.2772	0.390
12	0.0604	0.1448	0.299
16	0.0565	0.0864	0.268
20	0.0531	0.0568	0.245

$\Delta t=0.00625$, $t=k\Delta t$,

E——maximum error, k——number of steps, M——methods,

I——our method, II——the method of [3], III——Galerkin method.

When the Galerkin method[8] is used, on each side of the phase change point, 130 net points are taken. The long-dash lines in Figs. 8 and 9 represent approximate solution computed by the Galerkin method. The maximum error is 1.98252 when $t=0.00625$ and 0.65687 when $t=0.025$. Both occur near 10°C.

Table 1 lists the maximum errors of our method at several time levels.

Table 2 shows the maximum error comparison among our method, the method in [3] and the Galerkin method.

We also make error comparison between our method and the method of [3] in L^1 sense in a certain interval when $t=0.00625$. Taking the net points which are nearest to $b-M\xi(t)$ and $\xi(t)$ as the left and the right end points of the interval, and the other net points as the knot points, we make a cubic spline function $S_3^{(1)}(x)$. Comparing $S_3^{(1)}(x)$ with the true solution in L^1 sense, we get the error $E^{(1)}$. Similarly, taking the net points which are nearest to $\xi(t)$ and $b+M\xi(t)$ as the left and the right end points of the interval, we get spline $S_3^{(2)}(x)$ and error $E^{(2)}$.

For our method $E^{(1)}+E^{(2)}=3.0\times10^{-3}$, and for the method of [3] $E^{(1)}+E^{(2)}=3.8\times10^{-2}$.

From these figures and tables, we can see that for the problems here, the accuracy of the method in [3] is higher than that of the Galerkin method and our method is much more accurate than both the method in [3] and the Galerkin method. Moreover, much less CPU time is needed for our method than for the method in [3].

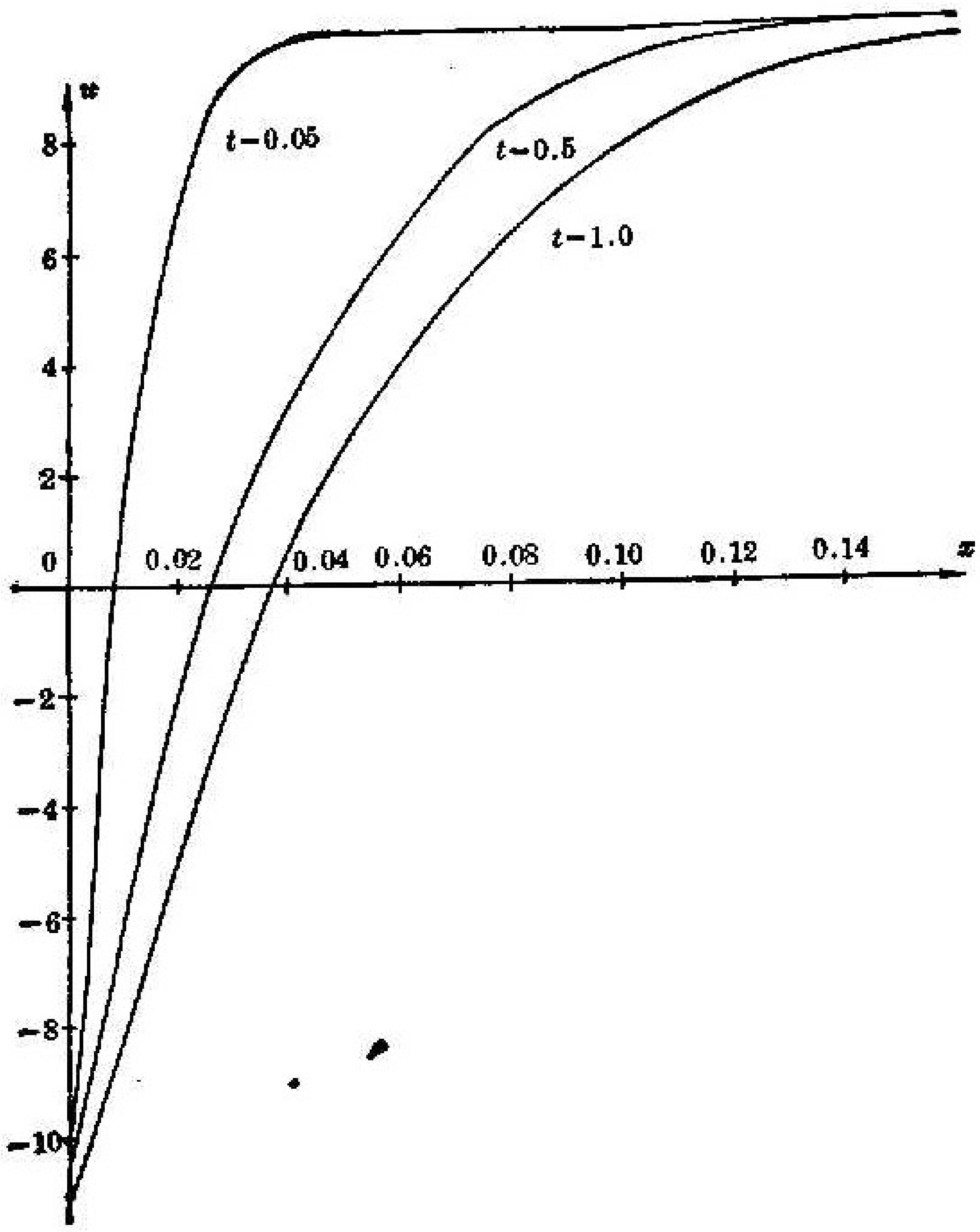

Fig. 1

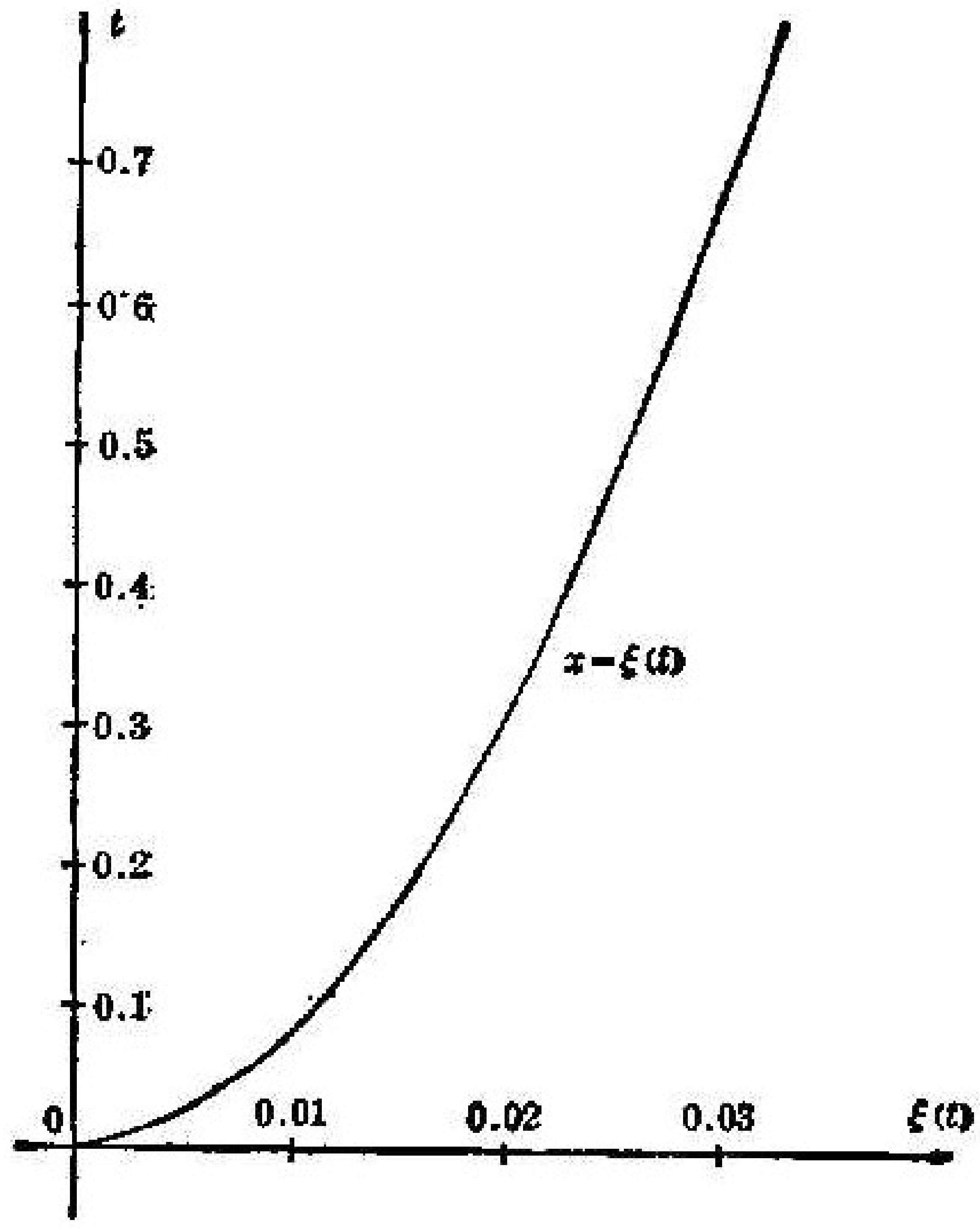

Fig. 2

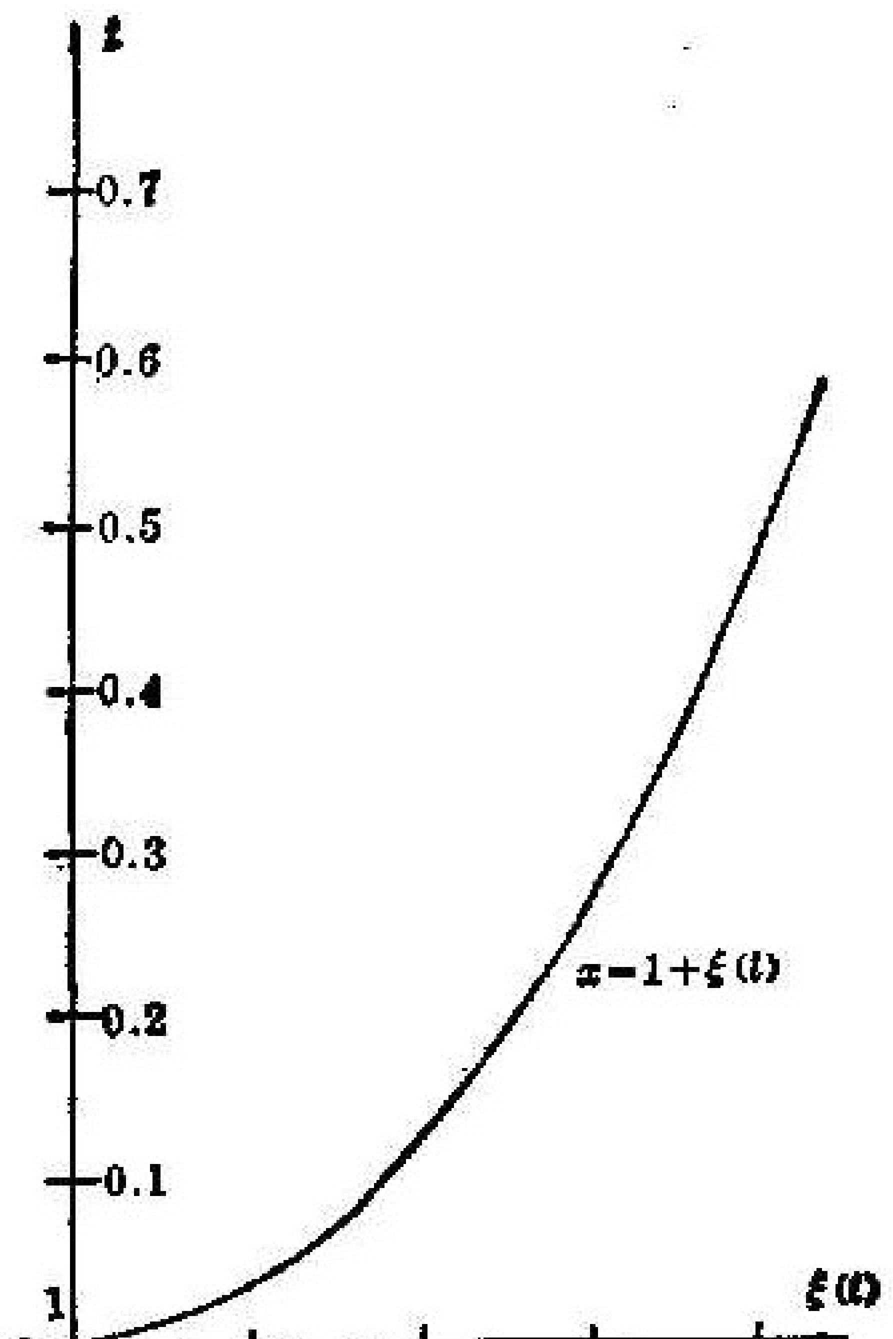

Fig. 3

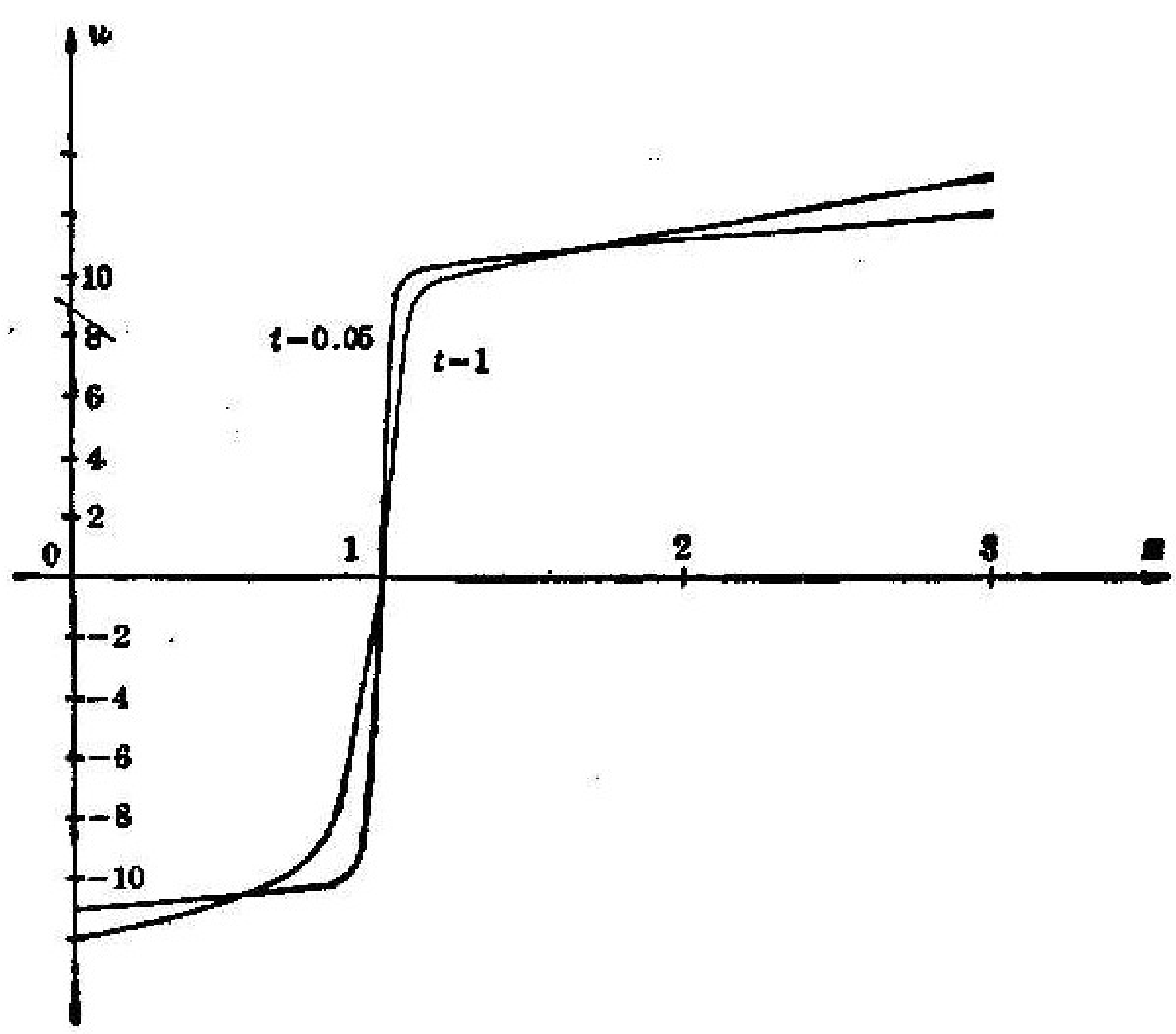

Fig. 4

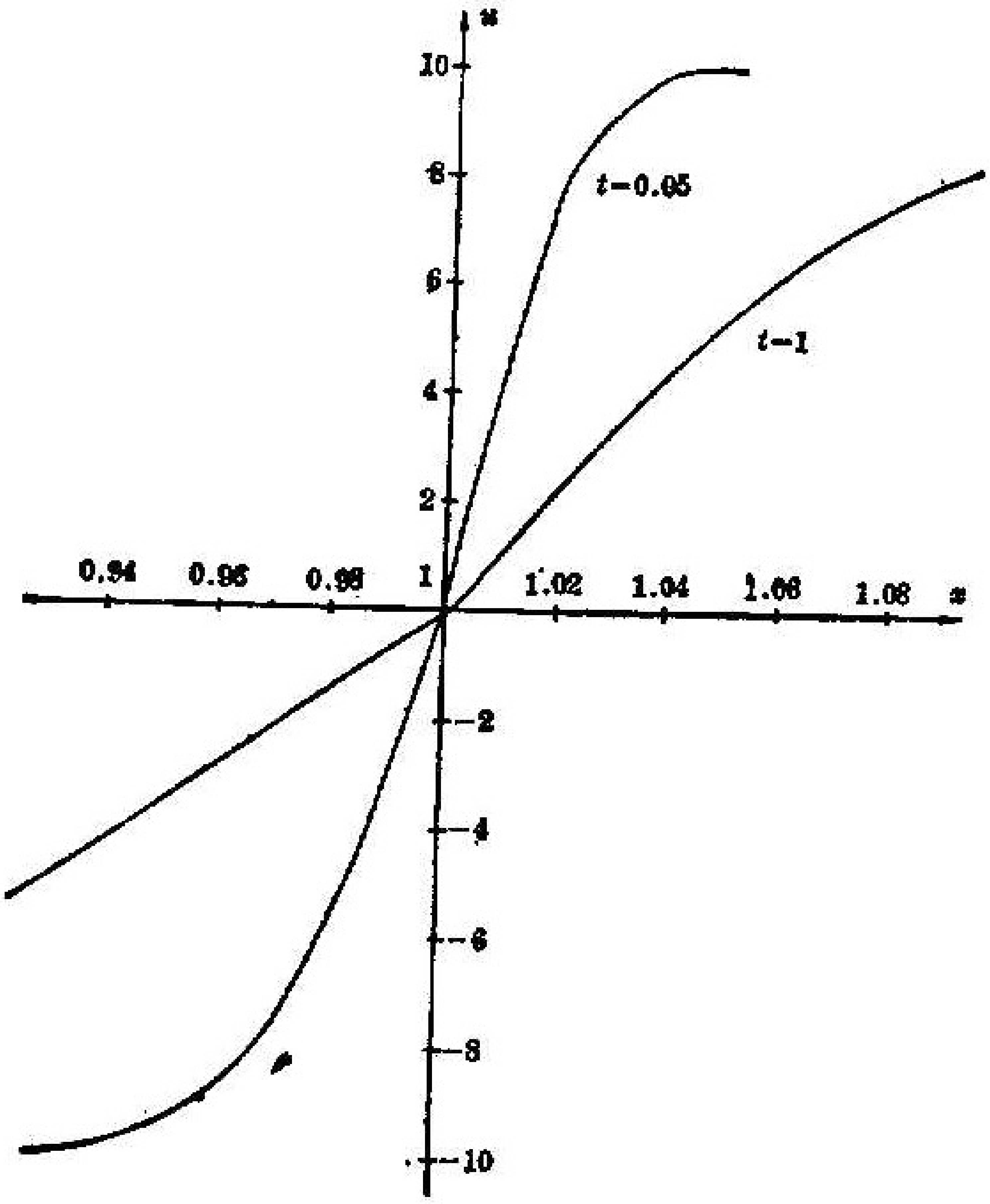

Fig. 5

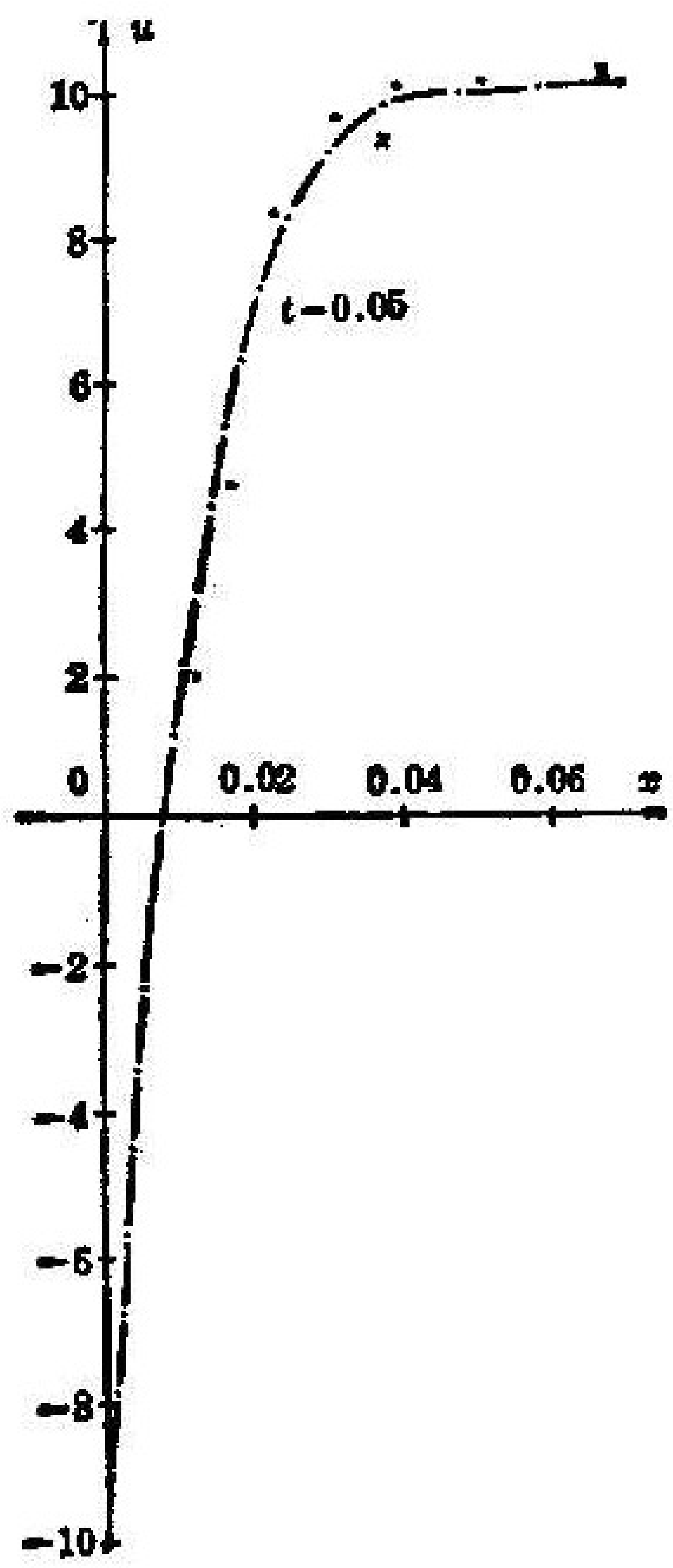

Fig. 6

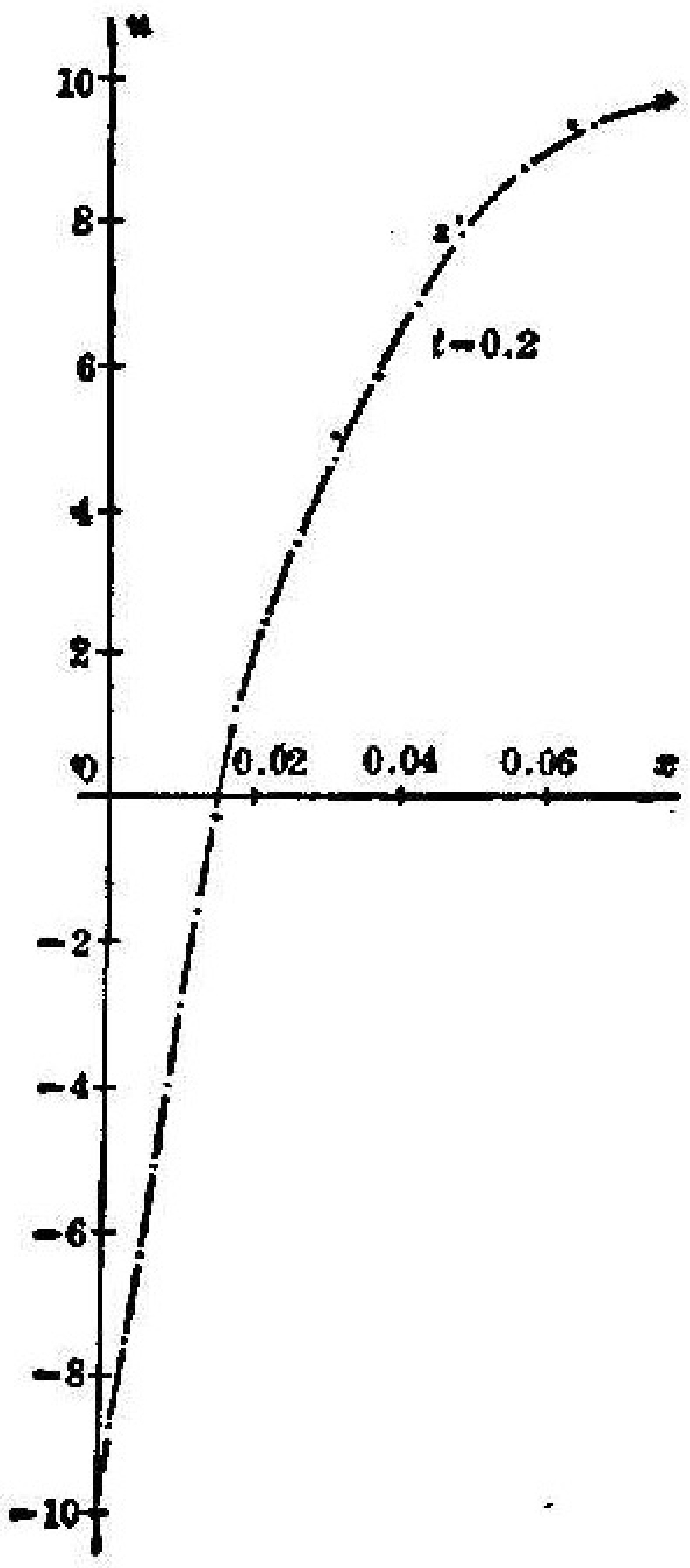

Fig. 7

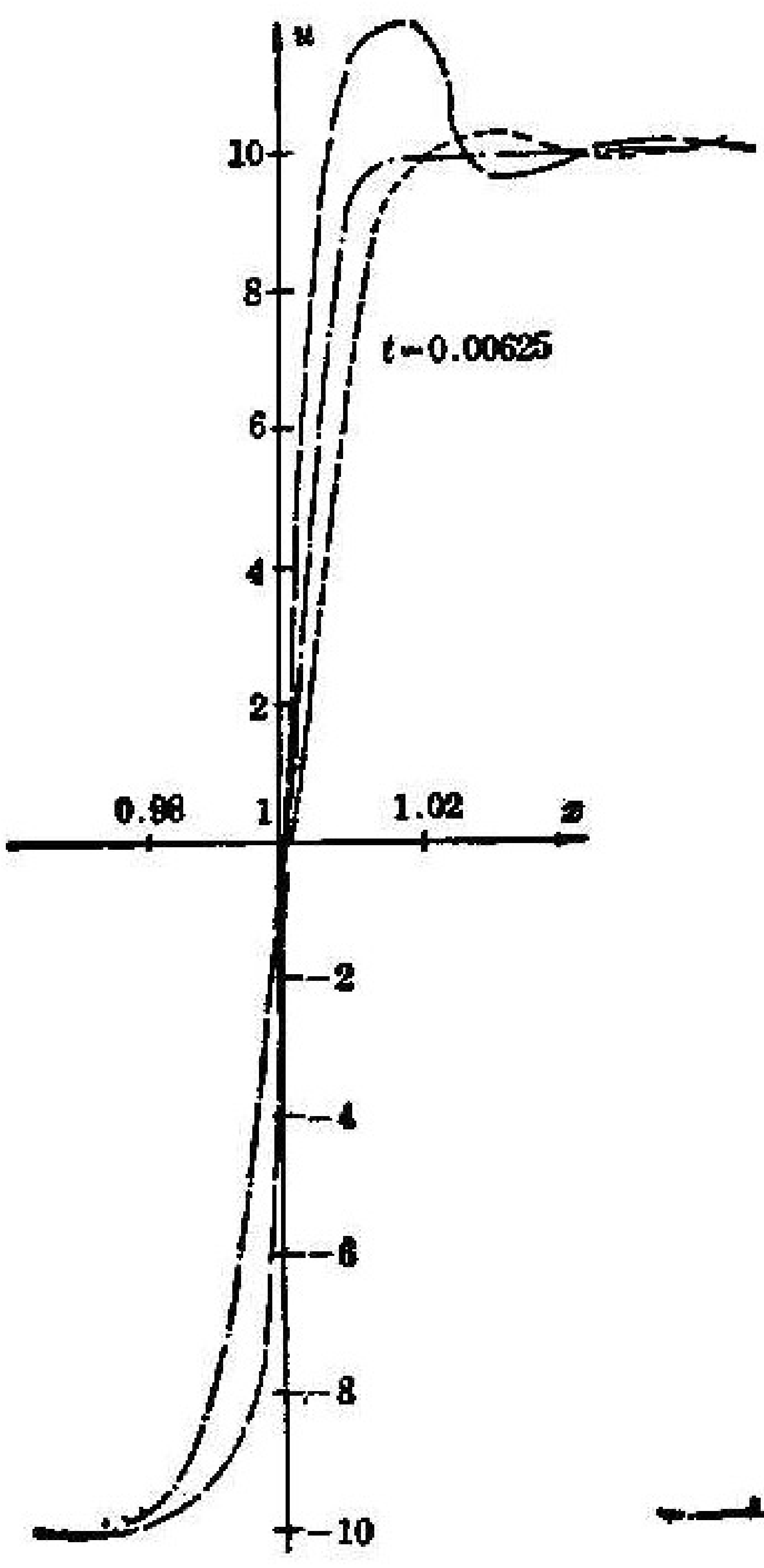

Fig. 8

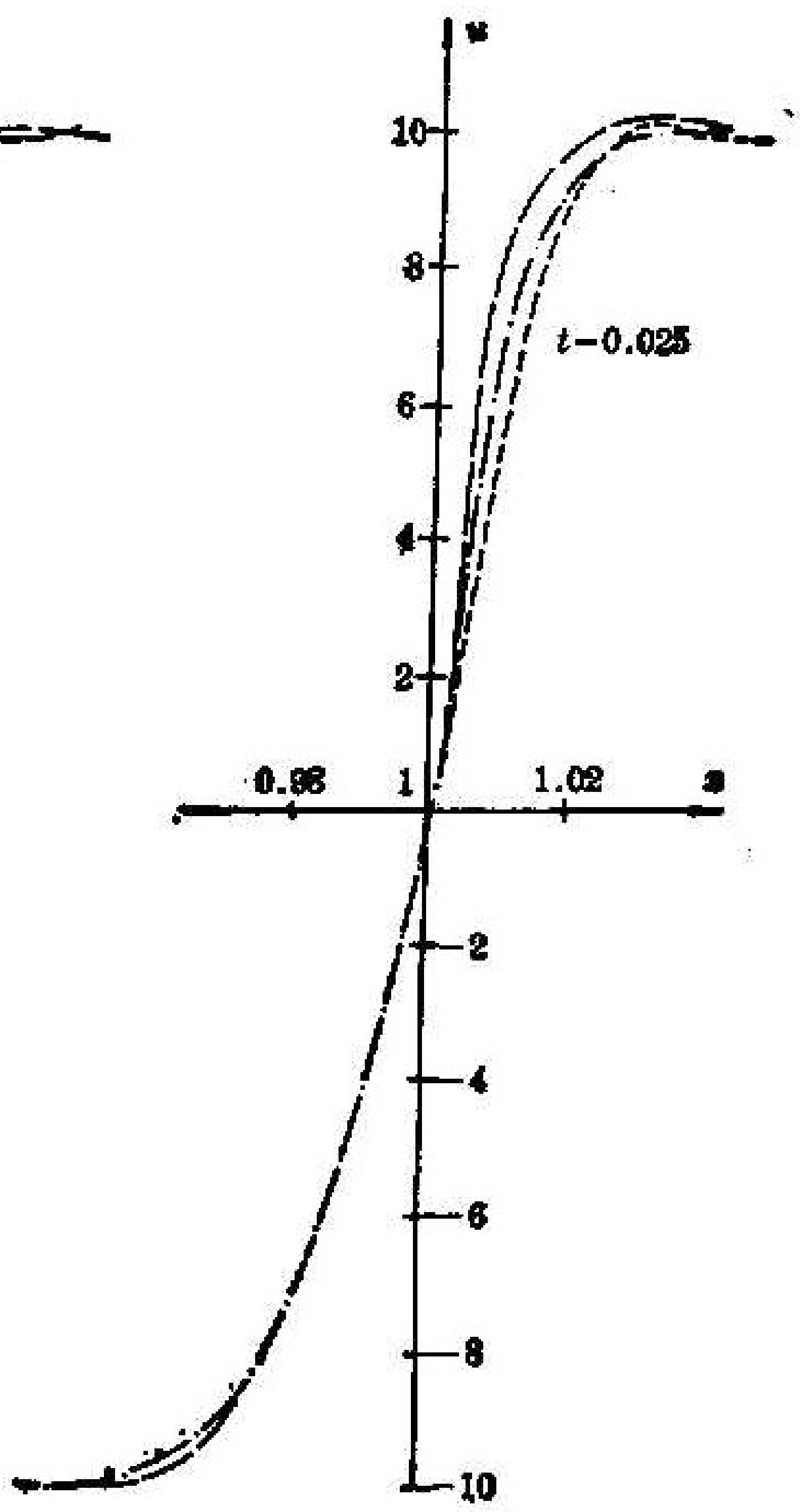

Fig. 9

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