NUMERICAL SOLUTION OF RADON'S PROBLEM IN A TWO DIMENSIONAL SPACE*

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§ 1

As in the Fourier transform of a function, we associate with a function $f(x)$ its Radon transform $g(\alpha, p)$, defined by the following integral of $f(x)$ over the hyperplane with unit normal α and distance p from the origin:

$$Rf=\int_{x\cdot\alpha=p} f(x)\,d\omega_x=g(\alpha, p) \tag{1}$$

for $p\in R^1$, x, $\alpha\in R^n$, $|\alpha|=1$. The Radon problem consists in solving equation (1) for $f(x)$ from $g(\alpha, p)$. This problem is of great importance in many applications, for instance, in the reconstruction of objects from X-ray pictures ([1], [2]).

In this paper we shall merely treat Radon's problem for $n=2$. Describing the unit normal α by its polar angle θ, we can rewrite (1) as

$$Rf=\int_{-\infty}^{\infty} f(\theta;\, p,\, r)\,dr=g(\theta, p), \tag{2}$$

$$f(\theta;\, p,\, r)=f(p\cos\theta+r\sin\theta,\ p\sin\theta-r\cos\theta),$$

or

$$4\pi\int_p^{\infty}\frac{\eta a(\eta)}{\sqrt{\eta^2-p^2}}\,d\eta=G(p),\quad G(p)=\int_0^{2\pi} g(\theta, p)\,d\theta,$$

where $a(\eta)$ is the average of f on the circle of radius η about the origin:

$$a(\eta)=\frac{1}{\omega\eta^{n-1}}\int_{|x|=\eta} f(x)\,ds_x.$$

The problem of determining the solution $a(\eta)$ (in particular, $f(\eta)$, if the function f has the property of circular symmetry[1], i.e. $f(x_1, x_2)=f(\eta)$, $x_1^2+x_2^2=\eta^2$) from the initial data $G(p)$ has been explored in [3] in greater detail.

Radon's problem (2) is not well-posed on the pair of spaces $(\bar{C}, L_2)$[4], where

$$L_2=L_2(H),$$

$$\bar{C}=\bar{C}(K_{\bar{T}})=\{f(x): f(x) \text{ is continuous and has compact support } K_\xi,\ 0<\xi\leqslant\bar{T}\},$$

$$H=\{(\alpha, p): p\in R^1,\ \alpha\in R^2,\ |\alpha|=1\}$$

is the unit cylinder in R^3 and K_ξ is the circle of radius ξ about the origin. This is because the range of Radon's integral operator R clearly does not coincide with L_2 and the inverse R^{-1} of the operator R is not continuous.

It should be pointed out that the reciprocity formula for $f(x)$ holds ([1], [2]):

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$$(2\pi i)^n f(z) = (\Delta_z)^{\frac{n-2}{2}} \int_{\Omega_x} d\omega_x \int_{p=-\infty}^{p=\infty} \frac{dg(x, p)}{p - z\cdot x}.$$

§ 2

Below, we shall use the finite-difference method to solve the Radon equation (2). In order to clarify the essentials of the method, in this section we study merely a semi-discrete scheme, where only the variable p is discretized:

$$p_i = i\tau,\ i = -n-1,\ -n,\ \cdots,\ n,\ n+1,\ \tau = \frac{\bar{T}}{n+1}$$

while the variable r is left continuous. The fully discrete case is discussed in section 3. Let f_τ be a vector-valued function:

$$f_\tau = f_\tau(\theta, r) = (f_{-n-1}(\theta, r), \cdots, f_{n+1}(\theta, r))$$

the components of which, i.e., $f_i(\theta, r)$, are defined on the segments:

$$\{(p_i, r):\ -\bar{T} \leqslant r \leqslant \bar{T}\} \quad (i = -n-1, \cdots, n+1).$$

Now let us consider the functional M_τ^α of the argument function f_τ, which may be taken to be an arbitrary continuous function with a continuous derivative:

$$M_\tau^\alpha[\theta; f_\tau, g_\tau] = \sum_{i=-n}^{n} \tau \left[\int_{-\bar{T}}^{\bar{T}} f_i(\theta, r)dr - g_i(\theta)\right]^2 + \alpha \sum_{i=-n}^{n} \tau \int_{-\bar{T}}^{\bar{T}} \left\{f_i^2(\theta, r) + \left[\frac{df_i}{dr}\right]^2\right\} dr$$

$$+\alpha \sum_{i=-n-1}^{n} \tau \int_{-\bar{T}}^{\bar{T}} \left[\frac{f_{i+1} - f_i}{\tau}\right]^2 dr,$$

where g_τ is a given vector:

$$g_\tau = g_\tau(\theta) = (g_{-n}(\theta), \cdots, g_n(\theta)).$$

Theorem 1. *For every g_τ and every positive parameter α, there exists a unique continuous function $f_\tau^\alpha(\theta, r)$ with a continuous derivative for which the functional $M_\tau^\alpha[\theta; f_\tau, g_\tau]$ attains its greatest lower bound:*

$$M_\tau^\alpha[\theta; f_\tau^\alpha, g_\tau] = \inf M_\tau^\alpha[\theta; f_\tau, g_\tau].$$

Proof. 1) The desired function $f_\tau^\alpha(\theta, r)$ should be determined by the integro-differential equation of Euler:

$$\alpha L[f_\tau] = \int_{-\bar{T}}^{\bar{T}} f_\tau(\theta, s)ds - g_\tau(\theta), \tag{3}$$

$$L[f_\tau] = \frac{d^2 f_\tau}{dr^2} + B f_\tau,$$

$$B = \begin{pmatrix} -\left(1+\frac{1}{\tau^2}\right) & \frac{1}{\tau^2} & & & \\ \frac{1}{\tau^2} & -\left(1+\frac{1}{\tau^2}\right) & \frac{1}{\tau^2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{\tau^2} & -\left(1+\frac{1}{\tau^2}\right) & \frac{1}{\tau^2} \\ & & & \frac{1}{\tau^2} & -\left(1+\frac{1}{\tau^2}\right) \end{pmatrix}$$

and the boundary conditions

$$f_\tau(\theta, -\overline{T})=0, \quad f_\tau(\theta, \overline{T})=0, \tag{4}$$

$$f_{n+1}(\theta, r)=f_n(\theta, r), \quad f_{-n-1}(\theta, r)=f_{-n}(\theta, r).$$

2) Under conditions (4) the associated homogeneous equation

$$\alpha L[f_\tau]=\int_{-\overline{T}}^{\overline{T}} f_\tau(\theta, s)\,ds$$

has only a trivial solution.

3) By means of Green's tensor $G(r, \zeta)$ for the differential operator $L[f_\tau]$ and the associated boundary condition ([5], p. 394), it is found that the desired solution $f_\tau^\alpha(\theta, r)$ is equivalent to solving the following integral equation of the second kind:

$$\alpha f_\tau(\theta, r)=\int_{-\overline{T}}^{\overline{T}} G(r, \zeta)\left[\int_{-\overline{T}}^{\overline{T}} f_\tau(\theta, s)\,ds-g_\tau(\theta)\right]d\zeta$$

and from 2) the last equation possesses a uniquely determined solution $f_\tau^\alpha(\theta, r)$.

Thus, Theorem 1 is proven.

As in [3], selecting $\alpha(\delta)=\delta^2$ we can now prove the following

Theorem 2. *Suppose that the function $f_T(x_1, x_2)\in \overline{C}(K_{\overline{T}})$ has a continuous derivative and satisfies equation* (2) *with right-hand side $g=g_T$:*

$$Rf_T=\int_{-\overline{T}}^{\overline{T}} f_T(p\cos\theta+r\sin\theta,\ p\sin\theta-r\cos\theta)\,dr=g_T(\theta, p).$$

Then, for every positive number ε, there exists $\delta(\varepsilon)$ and τ_0 such that for $\delta\leqslant\delta(\varepsilon)$ and $\tau\leqslant\tau_0$ the inequality

$$\sum_{i=-n}^{n} \tau[g_{T,i}^{(\tau)}(\theta)-g_i^{(\delta)}(\theta)]^2\leqslant\delta^2$$

implies

$$|f^{\alpha(\delta)}(\theta, p, r)-f_T^{(\tau)}(\theta, p, r)|<\varepsilon,$$

where $f_\tau^{\alpha(\delta)}(\theta, r)$ is the minimizer of functional $M_\tau^{\alpha(\delta)}[\theta; f_\tau, g_\tau^{(\delta)}]$:

$$f_\tau^{\alpha(\delta)}=f_\tau^{\alpha(\delta)}(\theta, r)=(f_{-n-1}^{\alpha(\delta)}(\theta, r), \cdots, f_{n+1}^{\alpha(\delta)}(\theta, r)),$$

$$f^{\alpha(\delta)}(\theta, p, r)=f_i^{\alpha(\delta)}(\theta, r)+\frac{f_{i+1}^{\alpha(\delta)}(\theta, r)-f_i^{\alpha(\delta)}(\theta, r)}{\tau}(p-p_i),$$

$$p\in[p_i, p_{i+1}], \quad i=-n-1, \cdots, n,$$

$$f_T^{(\tau)}=f_T^{(\tau)}(\theta, r)=(f_{T,-n-1}^{(\tau)}(\theta, r), \cdots, f_{T,n+1}^{(\tau)}(\theta, r)),$$

$$f_{T,i}^{(\tau)}=f_{T,i}^{(\tau)}(\theta, r)=f_T(p_i\cos\theta+r\sin\theta,\ p_i\sin\theta-r\cos\theta),$$

$$f_T^{(\tau)}(\theta, p, r)=f_{T,i}^{(\tau)}(\theta, r)+\frac{f_{T,i+1}^{(\tau)}(\theta, r)-f_{T,i}^{(\tau)}(\theta, r)}{\tau}(p-p_i),$$

$$p\in[p_i, p_{i+1}], \quad i=-n-1, \cdots, n,$$

$$g_T^{(\tau)}=g_T^{(\tau)}(\theta)=(g_{T,-n}^{(\tau)}(\theta), \cdots, g_{T,n}^{(\tau)}(\theta)), \quad g_{T,i}^{(\tau)}=g_T(\theta, p_i), \quad i=-n, \cdots, n,$$

$$g_\tau^{(\delta)}=g_\tau^{(\delta)}(\theta)=(g_{-n}^{(\delta)}(\theta), \cdots, g_n^{(\delta)}(\theta)).$$

Proof. Since the function $f_\tau^{\alpha(\delta)}(\theta, r)$ minimizes the functional $M_\tau^{\alpha(\delta)}[\theta; f_\tau, g_\tau^{(\delta)}]$,

$$M_\tau^{\alpha(\delta)}[\theta; f_\tau^{\alpha(\delta)}, g_\tau^{(\delta)}]\leqslant M_\tau^{\alpha(\delta)}[\theta; f_T^{(\tau)}, g_\tau^{(\delta)}].$$

Therefore

$$\sum_{i=-n}^{n}\tau\left[\int_{-\bar T}^{\bar T} f_i^{\alpha(\delta)}\,dr-g_i^{(\delta)}\right]^2+\alpha(\delta)\sum_{i=-n}^{n}\tau\int_{-\bar T}^{\bar T}\left[(f_i^{\alpha(\delta)})^2+\left(\frac{df_i^{\alpha(\delta)}}{dr}\right)^2\right]dr$$
$$+\sum_{i=-n-1}^{n}\tau\int_{-\bar T}^{\bar T}\left(\frac{f_{i+1}^{\alpha(\delta)}-f_i^{\alpha(\delta)}}{\tau}\right)^2 dr$$
$$\leqslant\sum_{i=-n}^{n}\tau\left[\int_{-\bar T}^{\bar T} f_{T,i}^{(\tau)}\,dr-g_i^{(\delta)}\right]^2+\alpha(\delta)\sum_{i=-n}^{n}\tau\int_{-\bar T}^{\bar T}\left[(f_{T,i}^{(\tau)})^2+\left(\frac{df_{T,i}^{(\tau)}}{dr}\right)^2\right]dr$$
$$+\alpha(\delta)\sum_{i=-n-1}^{n}\tau\int_{-\bar T}^{\bar T}\left(\frac{f_{T,i+1}^{(\tau)}-f_{T,i}^{(\tau)}}{\tau}\right)^2 dr$$
$$=\sum_{i=-n}^{n}\tau[g_{T,i}^{(\tau)}(\theta)-g_i^{(\delta)}]^2+\alpha(\delta)\sum_{i=-n-1}^{n}\tau\int_{-\bar T}^{\bar T}\left[f_T^2(\theta;p_i,r)\right.$$
$$\left.+\left(\frac{\partial f_T(\theta,p_i,r)}{\partial r}\right)^2+\left(\frac{\partial f_T(\theta,p_i,r)}{\partial r}\right)^2\right]dr+\eta_1$$
$$\leqslant\delta^2+\alpha(\delta)\int_{-\bar T}^{\bar T}dp\int_{-\bar T}^{\bar T}\left\{f_T^2+\left[\frac{\partial f_T}{\partial p}\right]^2+\left[\frac{\partial f_T}{\partial r}\right]^2\right\}dr+\eta_2$$
$$\leqslant\delta^2+\delta^2\int_{-\bar T}^{\bar T}dp\int_{-\bar T}^{\bar T}\left\{f_T^2+\left[\frac{\partial f_T}{\partial p}\right]^2+\left[\frac{\partial f_T}{\partial r}\right]^2\right\}dr+\delta^2=\delta^2 d,\quad \tau\leqslant\tau_0,$$
$$d=2+\int_{-\bar T}^{\bar T}dp\int_{-\bar T}^{\bar T}\left\{f_T^2(\theta;p,r)+\left[\frac{\partial f_T(\theta;p,r)}{\partial p}\right]^2+\left[\frac{\partial f_T(\theta;p,r)}{\partial r}\right]^2\right\}dr.$$

This, in turn, implies that

$$\sum_{i=-n}^{n}\tau\int_{-\bar T}^{\bar T}\left\{[f_i^{\alpha(\delta)}]^2+\left[\frac{df_i^{\alpha(\delta)}}{dr}\right]^2\right\}dr+\sum_{i=-n-1}^{n}\tau\int_{-\bar T}^{\bar T}\left[\frac{f_{i+1}^{\alpha(\delta)}-f_i^{\alpha(\delta)}}{\tau}\right]^2dr\leqslant d,$$
$$\sum_{i=-n}^{n}\tau\left[\int_{-\bar T}^{\bar T}f_i^{\alpha(\delta)}\,dr-g_i^{(\delta)}\right]^2\leqslant\delta^2 d.$$

Thus, the functions $f^{\alpha(\delta)}(\theta;p,r)$ and $f_T^{(\tau)}(\theta;p,r)$ belong to the compact subset M_τ of the space $C[S]$ (S: $[-T,T]\times[-T,T]$):

$$M_\tau=\left\{f(\theta;p,r):\iint_{-\bar T}^{\bar T}\left\{f^2+\left[\frac{\partial f}{\partial r}\right]^2+\left[\frac{\partial f}{\partial p}\right]^2\right\}dp\,dr\leqslant 12d\right\}$$

and hence

$$Rf^{\alpha(\delta)}(\theta;p,r)\in RM_\tau,\quad Rf_T^{(\tau)}(\theta;p,r)\in RM_\tau.$$

It follows from the continuity of R^{-1} on RM_τ ([4], p. 39) that for $\varepsilon>0$, there exists $\eta(\varepsilon)$ such that for $\|Rf^{\alpha(\delta)}-Rf_T^{(\tau)}\|<\eta(\varepsilon)$

$$|f^{\alpha(\delta)}(\theta;p,r)-f_T^{(\tau)}(\theta;p,r)|<\varepsilon.$$

Furthermore, since

$$\|Rf^{\alpha(\delta)}-Rf_T^{(\tau)}\|_{L_2}\leqslant 20(d+1)\delta^2.$$

We may choose

$$\delta(\varepsilon)=\frac{\eta(\varepsilon)}{\sqrt{20(d+1)}}.$$

Consequently, for $\delta\leqslant\delta(\varepsilon)$

$$\|Rf^{\alpha(\delta)}-Rf_T^{(\tau)}\|_{L_2}\leqslant\eta(\varepsilon)$$

and hence

$$|f^{\alpha(\delta)}(\theta;p,r)-f_T^{(\tau)}(\theta;p,r)|<\varepsilon.$$

This completes the proof of the theorem.

§ 3

In this section we study the fully discrete case:

$$p_i = i\tau, \quad i = -n-1, \cdots, 0, \cdots, n+1, \quad \tau = \frac{\overline{T}}{n+1},$$

$$r_j = jh, \quad j = -n, \cdots, 0, \cdots, n, \quad h = \frac{\overline{T}}{n}.$$

Let $f_{\tau h}$ be a vector-valued function:

$$f_{\tau h} = f_{\tau h}(\theta, j) = (f_{-n-1}(\theta, j), \cdots, f_{n+1}(\theta, j)),$$

the components of which, i.e. $f_i(\theta, j)$, are defined on the grid:

$$\{(p_i, r_j): j = -n, \cdots, n\} \quad i = -n-1, \cdots, n+1.$$

Now let us consider the functional $M^{\alpha}_{\tau h}$ of the argument function $f_{\tau h}$:

$$M^{\alpha}_{\tau h}[\theta; f_{\tau h}, g_{\tau}] = \sum_{i=-n}^{n} \tau \Big[\sum_{j=-n+1}^{n-1} h f_i(\theta, j) - g_i(\theta) \Big]^2$$

$$+ \alpha \sum_{i=-n}^{n} \tau \sum_{j=-n+1}^{n-1} h \left\{ f_i^2(\theta, j) + \left[\frac{f_{i+1}(\theta, j) - f_i(\theta, j)}{\tau} \right]^2 \right\}$$

$$+ \alpha \sum_{i=-n}^{n} \tau \sum_{j=-n}^{n-1} h \left[\frac{f_i(\theta, j+1) - f_i(\theta, j)}{h} \right]^2,$$

$$f_{\tau h}(\theta, -n) = 0, \quad f_{\tau h}(\theta, n) = 0,$$

where g_{τ} is a given vector:

$$g_{\tau} = g_{\tau}(\theta) = (g_{-n}(\theta), \cdots, g_n(\theta)).$$

Theorem 3. *For every g_{τ} and every positive parameter α there exists a unique function $f^{\alpha}_{\tau h}(\theta, j)$ for which the functional $M^{\alpha}_{\tau h}[\theta; f_{\tau h}, g_{\tau}]$ attains its greatest lower bound:*

$$M^{\alpha}_{\tau h}[\theta; f^{\alpha}_{\tau h}, g_{\tau}] = \inf M^{\alpha}_{\tau h}[\theta; f_{\tau h}, g_{\tau}].$$

Proof. 1) The desired function $f^{\alpha}_{\tau h}(\theta, j)$ should be determined by the Euler equation

$$\alpha L^h[f_{\tau h}] = \sum_{l=-n+1}^{n-1} h f_{\tau h}(\theta, l) - g_{\tau}(\theta), \tag{5}$$

$$L^h[f_{\tau h}] = \frac{f_{\tau h}(\theta, j+1) - 2f_{\tau h}(\theta, j) + f_{\tau h}(\theta, j-1)}{h^2} + Bf_{\tau h}(\theta, j)$$

and the boundary condition

$$f_{\tau h}(\theta, -n) = 0, \quad f_{\tau h}(\theta, n) = 0, \tag{6}$$

$$f_{-n-1}(\theta, j) = f_{-n}(\theta, j), \quad f_{n+1}(\theta, j) = f_n(\theta, j).$$

2) The homogeneous problem

$$\alpha L^h[f_{\tau h}] = \sum_{l=-n+1}^{n-1} h f_{\tau h}(\theta, l),$$

$$f_{\tau h}(\theta, -n) = 0, \quad f_{\tau h}(\theta, n) = 0$$

has only a trivial solution. Hence the inhomogeneous equation (5) under (6) possesses one and only one solution. This completes the proof of Theorem 3.

Theorem 4. *Suppose that the function $f_T(x_1, x_2) \in \overline{C}(K_{\overline{T}})$ has a continuous* *derivative and satisfies equation* (2) *with right-hand side* $g=g_T$:

$$Rf_T=\int_{-\bar{T}}^{\bar{T}} f_T(p\cos\theta+r\sin\theta,\ p\sin\theta-r\cos\theta)dr=g_T(\theta,p).$$

Then, for every positive number ε *there exist* $\delta(\varepsilon)$, τ_0 *and* h_0 *such that for* $\delta\leqslant\delta(\varepsilon)$, $\tau\leqslant\tau_0$ *and* $h\leqslant h_0$ *the inequality*

$$\sum_{i=-n}^{n}\tau[g_{T,i}^{(\tau)}(\theta)-g_i^{(\delta)}(\theta)]^2\leqslant\delta^2$$

implies

$$|f_{\tau h}^{\alpha(\delta)}(\theta;p,r)-f_T^{(\tau h)}(\theta;p,r)|<\varepsilon,$$

where $f_{\tau h}^{\alpha(\delta)}(\theta,j)$ *is the minimizer of functional* $M_{\tau h}^{\alpha(\delta)}[\theta;f_{\tau h},g_\tau^{(\delta)}]$:

$$f_{\tau h}^{\alpha(\delta)}=f_{\tau h}^{\alpha(\delta)}(\theta,j)=(f_{-n-1}^{\alpha(\delta)}(\theta,j),\ \cdots,\ f_{n+1}^{\alpha(\delta)}(\theta,j)),$$

$$f_{\tau h}^{\alpha(\delta)}(\theta;p,r)=f_i^{\alpha(\delta)}(\theta,j)+\frac{f_i^{\alpha(\delta)}(\theta,j+1)-f_i^{\alpha(\delta)}(\theta,j)}{h}(r-r_j)$$
$$+\frac{p-p_i}{\tau}\left\{\left[f_{i+1}^{\alpha(\delta)}(\theta,j)+\frac{f_{i+1}^{\alpha(\delta)}(\theta,j+1)-f_{i+1}^{\alpha(\delta)}(\theta,j)}{h}(r-r_j)\right]\right.$$
$$\left.-\left[f_i^{\alpha(\delta)}(\theta,j)+\frac{f_i^{\alpha(\delta)}(\theta,j+1)-f_i^{\alpha(\delta)}(\theta,j)}{h}(r-r_j)\right]\right\}$$

$$p\in[p_i,p_{i+1}],\ i=-n-1,\ \cdots,\ n,\ r\in[r_j,r_{j+1}],\ j=-n,\ \cdots,\ n-1,$$

$$f_{T,i}^{(\tau h)}(\theta,j)=f_T(p_i\cos\theta+r_j\sin\theta,\ p_i\sin\theta-r_j\cos\theta),$$
$$i=-n-1,\ \cdots,\ n+1,\quad j=-n,\ \cdots,\ n,$$

$$f_T^{(\tau h)}(\theta;p,r)=f_{T,i}^{(\tau h)}(\theta;j)+\frac{f_{T,i}^{(\tau h)}(\theta,j+1)-f_{T,i}^{(\tau h)}(\theta,j)}{h}(r-r_j)$$
$$+\frac{p-p_i}{\tau}\left\{\left[f_{T,i+1}^{(\tau h)}(\theta,j)+\frac{f_{T,i+1}^{(\tau h)}(\theta,j+1)-f_{T,i+1}^{(\tau h)}(\theta,j)}{h}(r-r_j)\right]\right.$$
$$\left.-\left[f_{T,i}^{(\tau h)}(\theta,j)+\frac{f_{T,i}^{(\tau h)}(\theta,j+1)-f_{T,i}^{(\tau h)}(\theta,j)}{h}(r-r_j)\right]\right\},$$

$$p\in[p_i,p_{i+1}],\ i=-n-1,\ \cdots,\ n,\ r\in[r_j,r_{j+1}],\ j=-n,\ \cdots,\ n-1.$$

The proof is analogous to that of Theorem 2.

References

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