THE PERTURBATION ANALYSIS OF THE PRODUCT OF SINGULAR VECTOR MATRICES UV^T *1)

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Abstract

Let A be an $n\times n$ nonsingular real matrix, which has singular value decomposition $A=U\Sigma V^T$. Assume A is perturbed to $\tilde{A}$ and $\tilde{A}$ has singular value decomposition $\tilde{A}=\tilde{U}\tilde{\Sigma}\tilde{V}^T$. It is proved that

$$\|\tilde{U}\tilde{V}^T-UV^T\|_F\leqslant\frac{2}{\sigma_n}\|\tilde{A}-A\|_F,$$

where σ_n is the minimum singular value of A; $\|\cdot\|_F$ denotes the Frobenius norm and n is the dimension of A.

This inequality is applicable to the computational error estimation of orthogonalization of a matrix, especially in the strapdown inertial navigation system.

§1. Introduction

For some engineering problems it is necessary to compute the coordinate transformation from one Cartesian coordinate system to another. For example, in a strapdown navigation system one of the key problems is to calculate the Direction Cosine Matrix (DCM), which transforms vectors from a body fixed coordinate system to a navigation reference system. The DCM is called strapdown matrix normally.

The strapdown matrix ought to be an orthogonal matrix. However because of the computational inaccuracies, e.g. the accumulation of rounding errors and the inaccuracy of the algorithm, it is impossible to get the exact solution from the computer. Part of the error between the exact DCM and the computed DCM can be corrected by using a suitable numerical method. The problem is how to correct the nonorthogonal error. The mathematical statement of this problem is: given a nonorthogonal matrix find an orthogonal matrix which is closest to the initial matrix in the sense of Euclidean norm of a matrix. The resultant matrix is the optimal orthogonalization matrix of the given matrix.

There exist iterative algorithms[1],[2] and a singular vector algorithm[3] for optimal orthogonalization of a matrix. The background of this paper is to estimate the computational error of the singular vector algorithm.

Let A be $n\times n$ real matrix, which has singular value decomposition $A=U\Sigma V^T$. It has been proved that the optimal orthogonalization matrix is $X=UV^T$. When A is nonsingular, the solution is unique[3].

The estimation of $\|\tilde{U}\tilde{V}^T-UV^T\|_F$ is given in this paper, where $\tilde{A}$ is the perturbed matrix of A, and $\tilde{A}$ has singular value decomposition $\tilde{A}=\tilde{U}\tilde{\Sigma}\tilde{V}^T$.

* Received April 22, 1985.

1) Projects supported by the Science Fund of the Chinese Academy of Sciences.

§ 2. The Perturbation of UV^T

Lemma 1[4]. *Let A and $\tilde{A}$ be $n\times n$ real matrices, and let their singular value decomposition be $A=U\Sigma V^T$ and $\tilde{A}=\tilde{U}\tilde{\Sigma}\tilde{V}^T$ respectively, where $\Sigma=\mathrm{diag}(\sigma_1, \cdots, \sigma_n)$, $\tilde{\Sigma}=\mathrm{diag}(\tilde{\sigma}_1, \cdots, \tilde{\sigma}_n)$, $\sigma_1\geqslant\cdots\geqslant\sigma_n$, $\tilde{\sigma}_1\geqslant\cdots\geqslant\tilde{\sigma}_n$. Then*

$$\|\tilde{\Sigma}-\Sigma\|_F\leqslant\|\tilde{A}-A\|_F. \tag{1}$$

Lemma 2[5]. *Let B and C be $n\times n$ normal matrices, $\Sigma=\mathrm{diag}(\sigma_1, \cdots, \sigma_n)$, $\sigma_1\geqslant\cdots\geqslant\sigma_n\geqslant 0$. Then*

$$\|\Sigma B-C\Sigma\|_F\geqslant\sigma_n\|B-C\|_F. \tag{2}$$

Theorem. *Let A be an $n\times n$ nonsingular real matrix, which has singular value decomposition $A=U\Sigma V^T$, when A is perturbed to $\tilde{A}$, which has singular value decomposition $\tilde{A}=\tilde{U}\tilde{\Sigma}\tilde{V}^T$. Then*

$$\|\tilde{U}\tilde{V}^T-UV^T\|_F\leqslant\frac{2}{\sigma_n}\|\tilde{A}-A\|_F, \tag{3}$$

where σ_n is the minimum singular value of A, $\|\cdot\|_F$ denotes the Frobenius norm and n is the dimension of A.

Proof. Since

$$\begin{aligned}\|\tilde{A}-A\|_F&=\|\tilde{U}\tilde{\Sigma}\tilde{V}^T-U\Sigma V^T\|_F=\|\tilde{U}\Sigma\tilde{V}^T-U\Sigma V^T+\tilde{U}(\tilde{\Sigma}-\Sigma)\tilde{V}^T\|_F\\&\geqslant\|\tilde{U}\Sigma\tilde{V}^T-U\Sigma V^T\|_F-\|\tilde{U}(\tilde{\Sigma}-\Sigma)\tilde{V}^T\|_F\end{aligned}$$

and U, V, $\tilde{U}$, $\tilde{V}$ are all orthogonal matrices, hence

$$\|\tilde{U}\Sigma\tilde{V}^T-U\Sigma V^T\|_F-\|\tilde{U}(\tilde{\Sigma}-\Sigma)\tilde{V}^T\|_F=\|U^T\tilde{U}\Sigma-\Sigma V^T\tilde{V}\|_F-\|\tilde{\Sigma}-\Sigma\|_F.$$

Using Lemma 1, 2, we have

$$\|U^T\tilde{U}\Sigma-\Sigma V^T\tilde{V}\|_F-\|\tilde{\Sigma}-\Sigma\|_F\geqslant\sigma_n\|U^T\tilde{U}-V^T\tilde{V}\|_F-\|\tilde{A}-A\|_F.$$

Also from the orthogonality of U and $\tilde{V}$,

$$\sigma_n\|U^T\tilde{U}-V^T\tilde{V}\|_F-\|\tilde{A}-A\|_F=\sigma_n\|\tilde{U}\tilde{V}^T-UV^T\|_F-\|\tilde{A}-A\|_F$$

thus

$$\|\tilde{A}-A\|_F\geqslant\sigma_n\|\tilde{U}\tilde{V}^T-UV^T\|_F-\|\tilde{A}-A\|_F$$

i.e.

$$\sigma_n\|\tilde{U}\tilde{V}^T-UV^T\|_F\leqslant 2\|\tilde{A}-A\|_F.$$

Since A is nonsingular and the minimum singular value is $\sigma_n>0$, it follows that

$$\|\tilde{U}\tilde{V}^T-UV^T\|_F\leqslant\frac{2}{\sigma_n}\|\tilde{A}-A\|_F.$$

§ 3. Numerical Examples

The theorem is illustrated by the next two examples. Computations are implemented on M-160 by means of LINPACK.

Example 1. Let

$$A=\begin{pmatrix} 0.40735173 & -0.80419803 & 0.11052590 \\ -0.88363382 & -0.77214510 & -0.54520913 \\ -0.90991876 & 0.75857107 & -0.86116686 \end{pmatrix}.$$

Assume A is perturbed to $\tilde{A}$,

$$\tilde{A}=\begin{pmatrix} 0.40735200 & -0.80419800 & 0.11052600 \\ -0.88363400 & -0.77214500 & -0.54520900 \\ -0.90991900 & 0.75857100 & -0.86116700 \end{pmatrix}.$$

Then the computing solution is as follows:

$$\sigma_3=0.227717;$$

$$\frac{2}{\sigma_3}\|\tilde{A}-A\|_F=41.679138\times 10^{-7};$$

$$\|\tilde{U}\tilde{V}^T-UV^T\|_F=1.759188\times 10^{-7}.$$

Thus the theorem is confirmed.

Example 2. Let

$$A=\begin{pmatrix} -1.172399 & -1.367204 & -1.047914 \\ 1.311614 & -0.874199 & -1.499384 \\ 0.644879 & -0.992129 & 0.607769 \end{pmatrix}.$$

Assume A is perturbed to $\tilde{A}$

$$\tilde{A}=\begin{pmatrix} -1.17240000 & -1.36720000 & -1.04791000 \\ 1.31161000 & -0.87419900 & -1.49938000 \\ 0.64487900 & -0.99212900 & 0.60776900 \end{pmatrix}.$$

Then the computing solution is as follows:

$$\sigma_3=1.23265;$$

$$\frac{2}{\sigma_3}\|\tilde{A}-A\|_F=13.081178\times 10^{-6};$$

$$\|\tilde{U}\tilde{V}^T-UV^T\|_F=2.56834\times 10^{-6}.$$

Also the theorem is confirmed.

§ 4. Conclusion

The theorem proved in this paper gives the perturbation analysis of the product of singular vector matrices UV^T. It estimates the computational error of an optimal orthogonal matrix, which is obtained by using singular value decomposition. This estimation is applicable to some engineering problems, especially to the calculation of Direction Cosine Matrix of a strapdown inertial navigation system.

Acknowledgment The author is grateful to Prof. Sun Ji-guang for his suggestions, which contributed immensely to the success of this work.

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