A NUMERICAL METHOD FOR A SYSTEM OF GENERALIZED NONLINEAR SCHRÖDINGER EQUATIONS*

CHANG QIAN-SHUN (常谦顺)
(*Institute of Applied Mathematics, Academia Sinica, Beijing, China*)

XU LIN-BAO (许林宝)
(*Institute of Applied Physics and Computational Mathematics, Beijing, China*)

§1. Introduction

In this paper, we consider the following initial-boundary value problem of the system of generalized nonlinear Schrödinger equations

$$\begin{cases} i\boldsymbol{u}-\dfrac{\partial}{\partial x}A(x)\dfrac{\partial \boldsymbol{u}}{\partial x}+\beta(x)q(|\boldsymbol{u}|^2)\boldsymbol{u}+F(x,t)\boldsymbol{u}=\boldsymbol{G}(x,t), \quad t>0,\ 0<x<1, & (1.1)\\ \boldsymbol{u}|_{x=0}=\boldsymbol{u}|_{x=1}=0, \quad t\geqslant 0, & (1.2)\\ \boldsymbol{u}|_{t=0}=\boldsymbol{u}_0(x), \quad 0<x<1, & (1.3)\end{cases}$$

where $\boldsymbol{u}(x,t)$ is an unknown complex functional vector, $A(x)=(a_m(x))$ is a real diagonal matrix, $\beta(x)$ and $q(s)$ are real functions, $\boldsymbol{u}_0(x)$ and $\boldsymbol{G}(x,t)$ are complex functional vectors. In [1] and [2] a class of stable and convergent finite difference schemes of (1.1) have been proved. In [3] the existence and uniqueness of the generalized solution for system (1.1) have been obtained. Now we consider a wide class of functions, which should satisfy one of the following conditions for the nonlinear terms $\beta(x)q(s)$:

(i) $K_\beta\geqslant\beta(x)\geqslant 0$; $Q(s)\geqslant 0$, $s\in[0,\infty)$, $Q(s)=\int_0^s q(z)dz$,

(ii) $|\beta(x)|\leqslant K_\beta$, $|q'(s)|\leqslant K_q$, $s\in[0,\infty)$,

(iii) $|\beta(x)|\leqslant K_\beta$, $q(|\boldsymbol{u}|^2)=|\boldsymbol{u}|^{2p}$, $0\leqslant p<2$.

We will construct a new finite difference scheme, which possesses two conservation quantities, and will prove that it is unconditionally stable and convergent. Furthermore, by using the Galerkin method, we will prove the existence and uniqueness of the generalized solution for problem (1.1)—(1.3), and the convergence of the iteration method in finding a solution of the finite difference scheme under the condition $k=O(h)$. The notations and conventions here are adopted as in [1].

§2. Convergence and Stability

The schemes in [1] and [2] have a common defect: they can not preserve the conservation of energy. Now we propose the following scheme to tackle this

* Received Septemebr 17, 1984.

problem:

$$\begin{cases} i(\phi_{m,j}^{n+1})_{\bar t}-\frac{1}{2}\{a_{m,j+1/2}[(\phi_{m,j}^{n+1})_x+(\phi_{m,j}^{n})_x]\}_{\bar x} \\ \quad +\frac{\beta_j}{2}\frac{Q(|\phi_j^{n+1}|^2)-Q(|\phi_j^n|^2)}{|\phi_j^{n+1}|^2-|\phi_j^n|^2}(\phi_{m,j}^{n+1}+\phi_{m,j}^{n}) \\ \quad +\frac{1}{2}\sum_{l=1}^{M} f_{m,l,j}^{n+1/2}(\phi_{m,j}^{n+1}+\phi_j^n)=G_{m,j}^{n+1/2}, \quad 1\le m\le M, 1\le j\le J-1, & (2.1) \\ \phi_{m,0}^n=\phi_{m,J}^n=0, \quad 1\le m\le M, & (2.2) \\ \phi_{m,j}^0=u_{0,m}(x_j), \quad 1\le m\le M, 1\le j\le J-1, & (2.3) \end{cases}$$

where $Q(s)=\int_0^s q(z)dz$. First we take a priori estimates for the finite difference solution.

Lemma 1. *Suppose that* $f_{m,l}(x,t)=f_{l,m}(x,t)$, $\|G_m(x,t)\|_{L_2}\le K_G$, $u_{0,m}\in L_2[0,1]$, $1\le m, l\le M$, *where* K_G *is a positive constant. Then there is an estimate for the solution of problem* (2.1)—(2.3):

$$\|\phi_m^n\|\le C_2,\ 0\le nk\le T,\ 1\le m\le M,$$

where C_2 *is a positive constant.*

Proof. Computing the inner product of both sides of (2.1) with $(\overline{\phi_{m,j}^{n+1}}+\overline{\phi_{m,j}^{n}})$, summing up for m from 1 to M, and taking the imaginary part in the resulting relation, we have

$$h\sum_{m=1}^{M}\sum_{j=1}^{J-1}(|\phi_{m,j}^{n+1}|^2)_{\bar t}=h\sum_{m=1}^{M}\sum_{j=1}^{J-1}I_m[G_{m,j}^{n+1/2}(\overline{\phi_{m,j}^{n+1}}+\overline{\phi_{m,j}^{n}})]. \tag{2.4}$$

By using the discrete Gronwall inequality, the conclusion of the lemma can be obtained.

Lemma 2 (Sobolev estimate [4]). *Suppose* $u\in L_q(R^n)$, $D^m u\in L_r(R^n)$, $1\le q, r\le\infty$. *Then for* $0\le j\le m$, $\frac{j}{m}\le\alpha\le 1$, *we have*

$$\|D^j u\|_{L_p}\le C\|D^m u\|_{L_r}^{\alpha}\|u\|_{L_q}^{1-\alpha},$$

where $\frac{1}{p}=\frac{j}{n}+\alpha\left(\frac{1}{r}-\frac{m}{n}\right)+(1-\alpha)\frac{1}{q}$, *and* C *is a positive constant.*

Lemma 3. *Suppose that the conditions of Lemma 1 are satisfied,* $0<a\le a_m(x)\le A$, $|f_{m,l}(x,t)|\le K_F$, $\left|\frac{\partial f_{m,l}(x,t)}{\partial t}\right|\le K_F$, $\left\|\frac{\partial G_m(x,t)}{\partial t}\right\|_{L_2}\le K_G$, $u_{0,m}(x)\in H_0^1[0,1]$, $1\le m, l\le M$, *where* a, A, K_F *and* K_G *are positive constants and* $q(s)\in C^1$, *and assume that one of the following conditions are satisfied:*

(i) $K_\beta\ge\beta(x)\ge 0$, $Q(s)\ge 0$, $s\in[0,\infty)$;

(ii) $|\beta(x)|\le K_\beta$, $|q'(s)|\le K_q$, $s\in[0,\infty)$;

(iii) $|\beta(x)|\le K_\beta$, $q(s)=s^p$, $0\le p<2$,

where p *is a real number,* K_β *and* K_q *are positive constants. Then for the solution of problem* (2.1)—(2.3), *there is an estimate*

$$\|(\phi_m^n)_x\|\le C_3,\ 0\le nk\le T,\ 1\le m\le M,$$

where C_3 *is a positive constant.*

Proof. Computing the inner product of both sides of (2.1) with $(\overline{\phi_{m,j}^{n+1}})_{\bar t}$, summing up for m from 1 to M, and taking the real part, we have

$$\frac{h}{2}\sum_{m=1}^{M}\sum_{j=1}^{J-1}[a_{m,j+1/2}|(\phi_{m,j}^{n+1})_x|^2]_{\bar t}+\frac{h}{2}\sum_{j=1}^{J-1}\beta_j[Q_j^{n+1}]_{\bar t}$$
$$+\frac{h}{2}\sum_{m=1}^{M}\sum_{l=1}^{M}\sum_{j=1}^{J-1}f_{m,l,j}^{n+1/2}[R_e(\phi_{l,j}^{n+1}\overline{\phi_{m,j}^{n+1}})]_{\bar t}$$
$$=h\sum_{m=1}^{M}\sum_{j=1}^{J-1}R_e[G_{m,j}^{n+1/2}(\overline{\phi_{m,j}^{n+1}})_{\bar t}]. \tag{2.5}$$

(2.5) implies

$$\frac{h}{2}\sum_{m=1}^{M}\sum_{j=0}^{J-1}a_{m,j+1/2}|(\phi_{m,j}^{n+1})_x|^2+\frac{h}{2}\sum_{j=1}^{J-1}\beta_jQ_j^{n+1}+\frac{h}{2}\sum_{m=1}^{M}\sum_{l=1}^{M}\sum_{j=1}^{J-1}f_{m,l,j}^{n+1/2}R_e(\phi_{l,j}^{n+1}\cdot\overline{\phi_{m,j}^{n+1}})$$
$$-h\sum_{m=1}^{M}\sum_{j=1}^{J-1}R_e(G_{m,j}^{n+1/2}\cdot\overline{\phi_{m,j}^{n+1}})$$
$$=\frac{h}{2}\sum_{m=1}^{M}\sum_{j=0}^{J-1}a_{m,j+1/2}|(\phi_{m,j}^{0})_x|^2+\frac{h}{2}\sum_{j=1}^{J-1}\beta_jQ_j^0+\frac{h}{2}\sum_{m=1}^{M}\sum_{l=1}^{M}\sum_{j=1}^{J-1}f_{m,l,j}^{1/2}\cdot R_e(\phi_{l,j}^0\cdot\overline{\phi_{m,j}^0})$$
$$+\frac{kh}{2}\sum_{\alpha=1}^{n}\sum_{m=1}^{M}\sum_{l=1}^{M}\sum_{j=1}^{J-1}(f_{m,l,j}^{\alpha+1/2})_t\cdot R_e(\phi_{l,j}^{\alpha}\cdot\overline{\phi_{m,j}^{\alpha}})-h\sum_{m=1}^{M}\sum_{j=1}^{J-1}R_e(G_{m,j}^{1/2}\cdot\overline{\phi_{m,j}^0})$$
$$-kh\sum_{\alpha=1}^{n}\sum_{m=1}^{M}\sum_{j=1}^{J-1}R_e[(G_{m,j}^{\alpha+1/2})_t\cdot\overline{\phi_{m,j}^{\alpha}}].$$

By the condition in the lemma, it follows that

$$\left|\frac{h}{2}\sum_{m=1}^{M}\sum_{j=0}^{J-1}a_{m,j+1/2}|(\phi_{m,j}^{n+1})_x|^2+\frac{h}{2}\sum_{j=1}^{J-1}\beta_jQ_j^{n+1}\right|\leqslant K_0, \tag{2.6}$$

where K_0 is a positive constant.

Since condition (i) holds, from (2.6) we get

$$\|(\phi_m^n)_x\|\leqslant C_3,\quad 1\leqslant m\leqslant M.$$

By condition (ii) and using the Taylor expression, we have

$$\left|\frac{h}{2}\sum_{j=1}^{J-1}\beta_jQ_j^{n+1}\right|\leqslant\frac{1}{2}K_\beta\Big[|q(0)|\cdot MC_2^2+\frac{M}{2}K_qh\sum_{j=1}^{J-1}\Big(\sum_{m=1}^{M}|\phi_{m,j}^{n+1}|^2\Big)^2\Big].$$

By the discrete Sobolev's inequality

$$\|\phi_m^{n+1}\|_\infty\leqslant K_1\|\phi_m^{n+1}\|+s\|(\phi_m^{n+1})_x\|,$$

where K_1 is a positive constant and $s>0$ is small enough, we obtain

$$\left|\frac{h}{2}\sum_{j=1}^{J-1}\beta_jQ_j^{n+1}\right|\leqslant\frac{1}{2}K_\beta\Big[|q(0)|MC_2^2+\frac{1}{2}K_qM\sum_{m=1}^{M}\Big(\|\phi_m^{n+1}\|_\infty^2\cdot\Big(h\sum_{j=1}^{J-1}|\phi_{m,j}^{n+1}|^2\Big)\Big)\Big]$$
$$\leqslant K_2+K_3s^2\sum_{m=1}^{M}\|(\phi_m^{n+1})_x\|,$$

where K_2 and K_3 are positive constants. Since

$$\left|\frac{h}{2}\sum_{m=1}^{M}\sum_{j=0}^{J-1}a_{m,j+1/2}|(\phi_{m,j}^{n+1})_x|^2\right|\geqslant\frac{a}{2}\sum_{m=1}^{M}\|(\phi_m^{n+1})_x\|^2,$$

choosing $s=\sqrt{\dfrac{a}{4K_3}}$, and from (2.6), we have

$$\frac{a}{4}\sum_{m=1}^{M}\|(\phi_m^{n+1})_x\|^2\leqslant K_0+K_2,$$

i.e.

$$\|(\phi_m^n)_x\|\leqslant C_3.$$

By condition (iii),

$$\left|\frac{h}{2}\sum_{j=1}^{J-1}\beta_j Q_j^{n+1}\right|-\left|\frac{h}{2(p+1)}\sum_{j=1}^{J-1}\beta_j|\phi_j^{n+1}|^{2(p+1)}\right|\leqslant\frac{K_\beta}{2}2^{Mp}\cdot h\sum_{m=1}^{M}\sum_{j=1}^{J-1}|\phi_{m,j}^{n+1}|^{2(p+1)}.$$

Let $p=2-\delta$; then $2\geqslant\delta>0$ and

$$\left|\frac{h}{2}\sum_{j=1}^{J-1}\beta_j Q_j^{n+1}\right|\leqslant 2^{Mp-1}\cdot K_\beta h\sum_{m=1}^{M}\sum_{j=1}^{J-1}|\phi_{m,j}^{n+1}|^{6-2\delta}. \tag{2.7}$$

Using Lemma 2 and the interpolation lemma[5], we have

$$\|\phi_m^{n+1}\|_{L_{6-2\delta}}\leqslant K_4\|(\phi_m^{n+1})_x\|^{\frac{1}{2}-\frac{1}{6-2\delta}}\cdot\|\phi_m^{n+1}\|^{\frac{1}{2}+\frac{1}{6-2\delta}},$$

where K_4 is a positive constant. Hence we get

$$h\sum_{j=1}^{J-1}|\phi_{m,j}^{n+1}|^{6-2\delta}\leqslant K_4^{6-2\delta}\cdot\|(\phi_m^{n+1})_x\|_{L_2}^{2-\delta}\cdot\|\phi_m^{n+1}\|_{L_2}^{4-\delta}.$$

By the inequality

$$ab\leqslant\frac{(\varepsilon a)^p}{p}+\frac{\left(\frac{1}{\varepsilon}b\right)^{p'}}{p'},\quad \frac{1}{p}+\frac{1}{p'}=1,\ a,\ b,\ p,\ p',\ \varepsilon>0,$$

we obtain the inequality

$$h\sum_{j=1}^{J-1}|\phi_{m,j}^{n+1}|^{6-2\delta}\leqslant\varepsilon\|(\phi_m^{n+1})_x\|_{L_2}^2+K_5\|\phi_m^{n+1}\|_{L_2}^{(4-\delta)\frac{2}{\delta}}\leqslant\varepsilon\|(\phi_m^{n+1})_x\|_{L_2}^2+K_6,$$

where K_5 and K_6 are positive constants dependent on ε. Substituting the above inequality into (2.7), we have

$$\left|\frac{h}{2}\sum_{j=1}^{J-1}\beta_j Q_j^{n+1}\right|\leqslant 2^{Mp-1}\cdot K_\beta\sum_{m=1}^{M}(\varepsilon\|(\phi_m^{n+1})_x\|_{L_2}^2+K_6).$$

By the above estimate and from (2.6), it follows that

$$\frac{a}{2}\sum_{m=1}^{M}\|(\phi_m^{n+1})_x\|^2\leqslant K_0+2^{Mp-1}\cdot K_\beta K_6 M+2^{Mp-1}\cdot K_\beta\varepsilon\sum_{m=1}^{M}\|(\phi_m^{n+1})_x\|^2.$$

Letting $\varepsilon=\frac{a}{K_\beta}\left(\frac{1}{2}\right)^{Mp+1}$, we obtain

$$\sum_{m=1}^{M}\|(\phi_m^{n+1})_x\|^2\leqslant\frac{4}{a}(K_0+2^{Mp-1}\cdot K_\beta K_6 M),$$

i.e.,

$$\|(\phi_m^n)_x\|\leqslant C_2.$$

Lemma 4. *Suppose that the conditions of Lemma 3 are satisfied. Then we have*

$$\|\phi_m^n\|_\infty\leqslant C_1,\ 0\leqslant nk\leqslant T,\ 1\leqslant m\leqslant M,$$

where C_1 is a positive constant.

Proof. It is obvious by Lemma 1, Lemma 3 and Sobolev's imbedding theorem.

Theorem 1. *Suppose that the conditions of Lemma 3 are satisfied, and assume that for the solution of problem (1.1)—(1.3), $u(x,t)\in C^{(4,3)}$ and $a_m(x)\in C^3$. Then the solution of the difference problem (2.1)—(2.3) converges to the solution of problem (1.1)—(1.3) in L_2 norm, and $\|u_h-u\|_{L_2}=O(k^2+h^2)$.*

Theorem 2. *Suppose that the conditions of Lemma 3 are satisfied. Then the solution of problem (2.1)—(2.3) is stable in L_2 norm for initial values.*

§ 3. Existence and Uniqueness

Definition. *The function vector* $\boldsymbol{u}(x, t)=(u_1(x, t), \cdots, u_m(x, t)) \in L^\infty(0, T; H_0^1)$, $\boldsymbol{u}_t(x, t) \in L^\infty(0, T; L_2)$, *is a generalized solution of the problem, if the following integrating identities are fulfilled:*

$$\begin{cases} i(u_{mt}, v_m)+(a_m u_{mx}, v_{mx})+(\beta q(|\boldsymbol{u}|^2)u_m, v_m)+\sum_{l=1}^{M}(f_{m,l}u_l, v_m)=(G_m, v_m), \\ \qquad \forall v_m \in H_0^1,\ t>0,\ 1\leqslant m\leqslant M, & (3.1) \\ (u_m(x, 0), v_m)=(u_{0,m}(x), v_m), \quad 1\leqslant m\leqslant M. & (3.2) \end{cases}$$

Let $\{W_j(x)\}\in H_0^1$ be a basis dense in H_0^1, where $W_j(x)$ are real functions. By using the Galerkin method, we construct the approximate solution of problem (1.1)—(1.3):

$$u_m^J(x, t)=\sum_{j=1}^{J} C_{mj}(t)W_j(x), \quad 1\leqslant m\leqslant M, \tag{3.3}$$

where the coefficients $C_{mj}(t)$ satisfy the following system

$$\begin{cases} i(u_{mt}^J, W_j)+(a_m u_{mx}^J, W_{jx})+(\beta q(|\boldsymbol{u}^J|^2)u_m^J, W_j)+\sum_{l=1}^{M}(f_{m,l}u_l^J, W_j)=(G_m, W_j), \\ \qquad j=1, 2, \cdots, J,\ 1\leqslant m\leqslant M, & (3.4) \\ (u_m^J(x, 0), W_j)=(u_{0,m}(x), W_j). & (3.5) \end{cases}$$

Lemma 5. *Suppose that the conditions of Lemma 3 are satisfied, Then for the solution* u_m^J *of problem* (3.4), (3.5), *there are estimates*

$$\|u_m^J\|_{L_2}\leqslant C_4, \quad \|u_{mx}^J\|_{L_2}\leqslant C_4, \quad \|u_m^J\|_\infty\leqslant C_4, \quad 1\leqslant m\leqslant M,$$

where C_4 *is a positive constant independent of* J.

Proof. Multiplying (3.4) by $\bar{C}_{mj}$, summing up for j and m, and taking the imaginary part, we have

$$\|u_m^J\|_{L_2}\leqslant C_4.$$

Multiplying (3.4) by $\overline{C'_{mj}(t)}$, summing up for j and m, and taking the real part, we get

$$\|u_{mx}^J\|_{L_2}\leqslant C_4.$$

From Sobolev's imbedding theorem, it follows that

$$\|u_m^J\|_\infty\leqslant C_4.$$

Lemma 6. *Suppose that the conditions of Lemma 3 are satisfied, and* $u_{0,m}(x) \in H^2$, $1\leqslant m\leqslant M$. *Then there is an estimate for the solution of problem* (3.4), (3.5)

$$\|u_{mt}^J\|_{L_2}\leqslant C_5, \quad 1\leqslant m\leqslant M,$$

where C_5 *is a positive constant.*

Proof. Differentiating (3.4) with respect to t, multiplying the resulting relation by $\overline{C'_{mj}(t)}$, and summing up for j and m, we obtain

$$i\sum_{m=1}^{J}(u_{mtt}^J, u_{mt}^J)+\sum_{m=1}^{M}(a_m u_{mxt}^J, u_{mxt}^J)+\sum_{m=1}^{M}(\beta q(|\boldsymbol{u}^J|^2) \text{[illegible]}$$

$$+\sum_{m=1}^{M}(\beta(q(|\boldsymbol{u}^J|^2))_t u_m^J, u_{mt}^J)+\sum_{m=1}^{M}\sum_{l=1}^{M}(f_{ml}u_{lt}^J, u_{mt}^J)+\sum_{m=1}^{M}\sum_{l=1}^{M}(f_{mlt}u_l^J, u_{mt}^J)$$

$$=\sum_{m=1}^{M}(G_{mt}, u_{mt}^J).$$

With condition (ii), $|q'(s)|\leqslant K_q$ holds. When conditions (i) and (iii) are fulfilled, letting

$$\max_{0<\xi<MC_1^2}|q'(\xi)|\leqslant K_q,$$

and using Lemma 5, we have

$$\left|\sum_{m=1}^{M}(\beta(q(|\boldsymbol{u}^J|^2))_t u_m^J, u_{mt}^J)\right|=\left|\sum_{m=1}^{M}\left(\beta q'\cdot\sum_{l=1}^{M}(u_{lt}^J\overline{u_l^J}+u_l^J\overline{u_{lt}^J})u_m^J, u_{mt}^J\right)\right|\leqslant K_7\cdot\sum_{m=1}^{M}\|u_{mt}^J\|_{L_2}^2,$$

where K_7 is a positive constant. Hence taking the imaginary part in (3.6), we get

$$\frac{1}{2}\frac{d}{dt}\left(\sum_{m=1}^{M}\|u_{mt}^J\|_{L_2}^2\right)\leqslant K_8\sum_{m=1}^{M}\|u_{mt}^J\|_{L_2}^2+K_9,$$

where K_8 and K_9 are positive constants. According to Gronwall's inequality, it follows that

$$\|u_{mt}^J\|_{L_2}\leqslant C_5,\quad 1\leqslant m\leqslant M.$$

Theorem 3. *Suppose that the conditions of Lemma 6 are satisfied. Then there exists the generalized solution of problem (1.1)—(1.3).*

Proof. By the uniform boundness of $\|u_m^J\|_{L_2}$, $\|u_{mx}^J\|_{L_2}$ and $\|u_{mt}^J\|_{L_2}$ for J, and the compact argument, we can choose the sequence $\{u_m^\nu\}$, such that

$$u_m^\nu(x,t)\xrightarrow{\text{strong}}u_m(x,t),\quad \text{in } L^\infty(0,T;L_2),$$

$$u_{mx}^\nu(x,t)\xrightarrow{\text{weak}}u_{mx}(x,t),\quad \text{in } L^\infty(0,T;H_0^1),$$

$$u_{mt}^\nu(x,t)\xrightarrow{\text{weak}}u_{mt}(x,t),\quad \text{in } L^\infty(0,T;L_2),$$

and

$$q(|\boldsymbol{u}^\nu|^2)\xrightarrow{\text{weak}}q(|\boldsymbol{u}|^2).$$

Let $J\to\infty$ in (3.4) for any fixed j. We have

$$i(u_{mt}, W_j)+(a_m u_{mx}, W_{jx})+(\beta q(|\boldsymbol{u}|^2)u_m, W_j)+\sum_{l=1}^{M}(f_{ml}u_l, W_j)$$

$$=(G_m, W_j),\quad 1\leqslant m\leqslant M.$$

Since $\{W_j(x)\}$ is dense in H_0^1, the above equalities hold for every $v\in H_0^1$. Hence the generalized solution for problem (1.1)—(1.3) exists.

By using the usual energy method, we can obtain

Theorem 4. *Suppose that the conditions of Lemma 6 are satisfied. Then the generalized solution of problem (1.1)—(1.3) is unique.*

§ 4. A Method for Solving Difference Equations

The system (2.1)—(2.3) of difference equations is a system of transcendental equations, which can be solved by means of iteration. In [1] we have proved that the iteration is convergent under the condition $k<\text{const}\cdot h^2$. In [2] the same result is also obtained under the condition $k<\text{const}\cdot h$. Now we write the formula of iteration for the system (2.1)—(2.3), and prove its convergence under conditions (i), (ii), (iii) and $k<\text{const}\cdot h$.

The formula of iteration for solving the system (2.1)—(2.3) is

$$\begin{cases} C_{mj}\phi_{m,j-1}^{n+1(s+1)}+B_{mj}\phi_{mj}^{n+1(s+1)}+D_{mj}\phi_{m,j+1}^{n+1(s+1)}=E_{mj}^{n+1(s)}, & (4.1)\\ \phi_{m,0}^{n+1(s+1)}=\phi_{m,J}^{n+1(s+1)}=0, \quad 1\leqslant m\leqslant M,\ j=1,2,\cdots,J-1, & (4.2)\end{cases}$$

where

$$C_{mj}=-\frac{k}{2h^2}a_{m,j-1/2},$$

$$D_{mj}=-\frac{k}{2h^2}a_{m,j+1/2},$$

$$B_{mj}=i-\frac{k}{2h^2}(a_{m,j-1/2}+a_{m,j+1/2}),$$

$$E_{m,j}^{n+1(s)}=kG_{mj}^{n+1/2}+i\phi_{mj}^{n}+\frac{k}{2h^2}[a_{m,j+1/2}\phi_{m,j+1}^{n}-(a_{m,j-1/2}+a_{m,j+1/2})\phi_{mj}^{n}+a_{m,j-1/2}\phi_{m,j-1}^{n}]$$
$$-\frac{h}{2}\Big[\sum_{l=1}^{m-1}f_{m,l,j}^{n+1/2}\phi_{lj}^{n+1(s+1)}+\sum_{l=m}^{M}f_{m,l,j}^{n+1/2}\phi_{lj}^{n+1(s)}+\sum_{l=1}^{M}f_{m,l,j}^{n+1/2}\phi_{lj}^{n}\Big]$$
$$-\frac{k\beta_j}{2}\,\frac{Q(|\phi_j^{n+1(s)}|^2)-Q(|\phi_j^n|^2)}{|\phi_j^{n+1(s)}|^2-|\phi_j^n|^2}(\phi_{mj}^{n+1(s)}+\phi_{mj}^{n}).$$

Theorem 5. *Suppose that the conditions of Lemma 3 are satisfied. Let the initial value of iteration $\phi_{mj}^{n+1(0)}=\phi_{mj}^{n}$, $1\leqslant m\leqslant M$, $1\leqslant j\leqslant J-1$, and let the time size $k<C_6h$, where C_6 is a positive constant. Then the iteration method (4.1), (4.2) is convergent.*

Proof. Let $\varepsilon_{mj}^{n+1(s)}=\phi_{mj}^{n+1}-\phi_{mj}^{n+1(s)}$. From (4.1), (4.2) we obtain

$$\begin{cases} C_{mj}\varepsilon_{m,j-1}^{n+1(s+1)}+B_{mj}\varepsilon_{mj}^{n+1(s+1)}+D_{mj}\varepsilon_{m,j+1}^{n+1(s+1)}=H_{mj}^{n+1(s)},\\ \varepsilon_{m0}^{n+1(s+1)}=\varepsilon_{mJ}^{n+1(s+1)}=0,\end{cases} \tag{4.3}$$

where

$$H_{mj}^{n+1(s)}=-\frac{h}{2}\Big[\sum_{l=1}^{m-1}f_{m,l,j}^{n+1/2}\varepsilon_{lj}^{n+1(s+1)}+\sum_{l=m}^{M}f_{m,l,j}^{n+1/2}\varepsilon_{lj}^{n+1(s)}$$
$$-\frac{k\beta_j}{2}\Big[\frac{Q(|\phi_j^{n+1}|^2)-Q(|\phi_j^n|^2)}{|\phi_j^{n+1}|^2-|\phi_j^n|^2}(\phi_{mj}^{n+1}+\phi_{mj}^{n})$$
$$-\frac{Q(|\phi_j^{n+1(s)}|^2)-Q(|\phi_j^n|^2)}{|\phi_j^{n+1(s)}|^2-|\phi_j^n|^2}(\phi_{mj}^{n+1(s)}+\phi_{mj}^{n})\Big].$$

When the condition $|B_{mj}|-|C_{mj}|-|D_{mj}|>0$ is fulfilled, from (4.3) it is not difficult to prove

$$\max_{1<j<J-1}|\varepsilon_{mj}^{n+1(s+1)}|\leqslant\max_{1<j<J-1}\frac{|H_{mj}^{n+1(s)}|}{|B_{mj}|-|C_{mj}|-|D_{mj}|},\quad 1\leqslant m\leqslant M. \tag{4.4}$$

Since $\phi_{mj}^{n+1(0)}=\phi_{mj}^{n}$, we have

$$|\varepsilon_{mj}^{n+1(s+1)}|\leqslant\max_{\substack{1<j<J-1\\1<m<M}}|\phi_{mj}^{n+1}-\phi_{mj}^{n}|.$$

In the following we will prove by induction

$$|\varepsilon_{mj}^{n+1(s)}|\leqslant\max_{\substack{1<j<J-1\\1<m<M}}|\phi_{mj}^{n+1}-\phi_{mj}^{n}|,\quad s=0,1,\cdots. \tag{4.5}$$

Suppose (4.5) holds for s. By Lemma 4 it follows that

$$|\phi_{mj}^{n+1(s)}|\leqslant 3C_1,\quad 1\leqslant m\leqslant M,\quad 1\leqslant j\leqslant J-1. \tag{4.6}$$

Letting

$$\max_{\xi\in[0,9MC_1^2]}(|q(\xi)|,\ q'(\xi)|)\leqslant K_{10},$$

K_{10}——a positive constant, and using the Taylor expression, (4.6) and Lemma 4, we get

$$\begin{aligned}|H_{mj}^{n+1(s)}|\leqslant&\frac{kK_F}{2}\sum_{l=1}^{m-1}|\varepsilon_{lj}^{n+1(s+1)}|+\frac{kK_F}{2}\sum_{l=m}^{M}|\varepsilon_{mi}^{n+1(s)}|\\&+\frac{kK_B}{2}\Big|\Big[q\Big(\frac{1}{2}|\phi_j^{n+1}|+\frac{1}{2}|\phi_j^n|^2\Big)\\&+\frac{1}{8}[q'(\xi_1)-q'(\xi_2)](|\phi_j^{n+1}|^2-|\phi_j^n|^2)\Big](\phi_{mj}^{n+1}+\phi_{mj}^n)\\&-\Big[q\Big(\frac{1}{2}|\phi_j^{n+1(s)}|^2+\frac{1}{2}|\phi_j^n|^2\Big)\\&+\frac{1}{8}[q'(\xi_3)-q'(\xi_4)](|\phi_j^{n+1(s)}|^2-|\phi_j^n|^2)\Big](\phi_{mj}^{n+1(s)}+\phi_{mj}^n)\Big|\\&\leqslant kK_{11}\Big[\sum_{l=1}^{M}|\varepsilon_{lj}^{n+1(s)}|^2+\sum_{l=1}^{m-1}|\varepsilon_{lj}^{n+1(s+1)}|^2\Big],\end{aligned}$$

where $\xi_i(i=1,2,3,4)$ and K_{11} are definite constants.

Choosing k such that

$$2\frac{k^2}{h^2}AC_7+(C_7k)^2\leqslant 1, \tag{4.7}$$

whore C_7 is a positive constant, which can be definite in the sequel, we get

$$|B_{mj}|-|C_{mj}|-|D_{mj}|=\sqrt{1+\Big(\frac{k}{2h^2}(a_{m,j-1/2}+a_{m,j+1/2})\Big)^2}-\frac{k}{2h^2}(a_{m,j-1/2}+a_{m,j+1/2})\geqslant C_7k.$$

Hence from (4.4),

$$\max_{1<j<J-1}|\varepsilon_{mj}^{n+1(s+1)}|\leqslant\frac{K_{11}}{C_7}\max_{1<j<J-1}\Big[\sum_{l=1}^{M}|\varepsilon_{lj}^{n+1(s)}|+\sum_{l=1}^{m-1}|\varepsilon_{lj}^{n+1(s+1)}|\Big]. \tag{4.8}$$

From (4.8), it follows that

$$\max_{1<j<J-1}|\varepsilon_{1j}^{n+1(s+1)}|\leqslant\frac{2MK_{11}}{C_7}\max_{\substack{1<j<J-1\\1<m<M}}|\varepsilon_{mj}^{n+1(s)}|.$$

Choosing C_7 such that

$$C_7>2MK_{11}, \tag{4.9}$$

we have

$$\max_{1<j<J-1}|\varepsilon_{1j}^{n+1(s+1)}|<\max_{\substack{1<j<J-1\\1<m<M}}|\varepsilon_{mj}^{n+1(s)}|.$$

From (4.8) and the above inequality we obtain

$$\max_{1<j<J-1}|\varepsilon_{2j}^{n+1(s+1)}|<\max_{\substack{1<j<J-1\\1<m<M}}|\varepsilon_{mj}^{n+1(s)}|$$

when (4.9) holds. Analogously, we have

$$\max_{\substack{1<j<J-1\\1<m<M}}|\varepsilon_{mj}^{n+1(s+1)}|<\max_{\substack{1<j<J-1\\1<m<M}}|\varepsilon_{mj}^{n+1(s)}|\leqslant\max_{\substack{1<j<J-1\\1<m<M}}|\phi_{mj}^{n+1}-\phi_{mj}^{n}|$$

when (4.9) and (4.7) holds. In the meantime, we can obtain

$$\max_{\substack{1<j<J-1\\1<m<M}}|\varepsilon_{mj}^{n+1(s+1)}|\leqslant\frac{2MK_{11}}{C_7}\max_{\substack{1<j<J-1\\1<m<M}}|\varepsilon_{mj}^{n+1(s)}|.$$

Hence choosing C_7 and k, such that

$$C_7>2MK_{11},\quad k<\frac{h}{\sqrt{C_7^2h^2+2AC_7}}\equiv C_8h,$$

we have got the convergence of the iteration method.

§ 5. Some Remarks

(1) It is known that the difference scheme preserving the conservation of energy is essential for computational efficiency; especially, for the soliton computation, by using this scheme we can see the nature of soliton, more clearly.

(2) The treatment of nonlinear term. For the factor in the nonlinear term Q, taking the Taylor expression, it is easy to obtain

$$\hat{Q}=q\left(\frac{1}{2}|\phi_j^{n+1(s)}|^2+\frac{1}{2}|\phi_j^n|^2\right)+O(|\phi_j^{n+1(s)}|^2-|\phi_j^n|^2).$$

For a given error ε, as $|\phi_j^{n+1(s)}|^2-|\phi_j^n|^2\geqslant\varepsilon$. Let

$$Q=\frac{Q(|\phi_j^{n+1(s)}|^2)-Q(|\phi_j^n|^2)}{|\phi_j^{n+1(s)}|^2-|\phi_j^n|^2};$$

as $|\phi_j^{n+1(s)}|^2-|\phi_j^n|<\varepsilon$, let

$$Q\approx q\left(\frac{1}{2}|\phi_j^{n+1(s)}|^2+\frac{1}{2}|\phi_j^n|^2\right).$$

We expect that the total error will be small for this treatment.

We are grateful to Professor Guo Bo-ling for his valuable suggestion and help.

References

[1] 常谦顺,广义非线性 Schrödinger 方程组的守恒差分格式, *Scientia Sinica*, 3 (1983), 202—214.

[2] Zhu Yoa-Lan, Implicit Difference Schemes for the Generalized Non-Linear Schrödinger System, *Journal of Computational Mathematics*, 1: 2(1983), 116—129.

[3] Guo Bo-ling (郭柏灵), The Global Solution for Some Systems of Nonlinear Schrödinger Equations, Proc. 1980 Beijing DD-Symposium, 1227—1246.

[4] A. Friedman, Partial Differential Equations, Holt, New York, 1969.

[5] A. Menikoff, The Existence of Unbounded Solutions of the KdV Equation, *Comm. Pure and Appl. Math.*, 25 (1972), 407—432.