THE EXACT ESTIMATION OF THE HERMITE-FEJÉR INTERPOLATION*1)

SUN XIE-HUA (孙燮华)
(*Hangzhou University, Hangzhou, China*)

Abstract

The exact pointwise estimation of the Hermite-Fejér interpolation process based on the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ $(-1<\alpha,\ \beta<0)$ is given. The method employed is useful for other extended H-F interpolations also.

§ 1. Introduction and Result

Let $f\in C[-1, 1]$ and (with $x_k=x_{kn}^{(\alpha,\beta)}$, $k=1, 2, \cdots, n$)

$$-1<x_{nn}^{(\alpha,\beta)}<\cdots<x_{2n}^{(\alpha,\beta)}<x_{1n}^{(\alpha,\beta)}<1$$

be the roots of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ defined by

$$P_n^{(\alpha,\beta)}(x)=(1-x)^{-\alpha}(1+x)^{-\beta}\frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}[(1-x)^{\alpha+n}(1+x)^{\beta+n}],\quad \alpha,\ \beta>-1.$$

Define the Hermite-Fejér interpolation by

$$H_n^{(\alpha,\beta)}(f, x)=\sum_{k=1}^{n}f(x_k)v_k(x)l_k^2(x),\tag{1.1}$$

where

$$l_k(x)=l_{kn}^{(\alpha,\beta)}(x)=P_n^{(\alpha,\beta)}(x)/[P_n'^{(\alpha,\beta)}(x_k)(x-x_k)],$$

$$v_k(x)=v_{kn}^{(\alpha,\beta)}(x)=\{1-x[\alpha-\beta+(\alpha+\beta+2)x_k]+(\alpha-\beta)x_k+(\alpha+\beta+1)x_k^2\}/(1-x_k^2),$$

$$k=1, \cdots, n.$$

Denote by $\omega(t)$ the given modulus of continuity and $H_\omega=\{f;\ \omega(f, t)\leq\omega(t)\}$, where $\omega(f, t)$ is the modulus of continuity of f. In what follows, c, c_1, $c_2>0$ or the sign "O" will always denote different constants that are independent of f, n and x but dependent on α and β. The sign "$A\sim B$" means that there exist constants c_1 and c_2 such that

$$c_1A\leq B\leq c_2A.$$

In recent years there has been a great amoumt of research concerning the degree of approximation by interpolation process (1.1). P. Vertesi[1] proved that

$$|H_n^{(\alpha,\beta)}(f, x)-f(x)|=O(1)\sum_{i=1}^{n}\left[\omega\left(f, \frac{i\sqrt{1-x^2}}{n}\right)+\omega\left(f, \frac{i^2|x|}{n^2}\right)\right]i^{2r-1},$$

where $r=\max\left(\alpha,\ \beta,\ -\frac{1}{2}\right)$. Many authors (see [2]) investigated the special cases $H_n^{(-\frac{1}{2},\,-\frac{1}{2})}(f, x)$ and $H_n^{(0,0)}(f, x)$. Here we cite the following works only:

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$$|H_n^{(-\frac{1}{2},-\frac{1}{2})}(f,x)-f(x)|=O\Big(\frac{1}{n}\Big)\sum_{k=1}^{n}\omega\Big(f,\ \frac{\sqrt{1-x^2}}{k}+\frac{1}{k^2}\Big)^{[3]},$$

$$|H_n^{(-\frac{1}{2},-\frac{1}{2})}(f,x)-f(x)|\leqslant C\Big\{\omega(f,\ |\theta-\theta_{k_0}|^2|x|+|\theta-\theta_{k_0}|\sqrt{1-x^2})$$
$$+C\ \frac{T_n^2(x)}{n}\int_{\frac{1}{n}}^{1}\omega(f,\ t^2|x|+t\sqrt{1-x^2})t^{-2}\,dt\Big\}^{[4]},$$

where $\theta=\arccos x$ and $\theta_{k_0}=(2k_0-1)\pi/(2n)$ satisfy the inequality $|\theta-\theta_{k_0}|\leqslant\pi/(2n)$,

$$|H_n^{(-\frac{1}{2},-\frac{1}{2})}(f,x)-f(x)|=O\Big(\frac{1}{n}\Big)\sum_{k=1}^{n}\omega\Big(f,\ \frac{\sqrt{1-x^2}|T_n(x)|}{k}+\frac{|x||T_n(x)|}{k^2}\Big)^{[5]}.$$

The improvement of the degree of approximation naturally gives rises to the question of whether the above results are exact? This is a problem raised by Prof. Shen Xie-chang[2]. The purpose of the paper is to answer the above question and to give the exact pointwise degree of $H_n^{(\alpha,\beta)}(f,x)$ $(-1<\alpha,\ \beta\leqslant0)$. Our method is useful for finding the exact degree of other H-F type interpolations.

Our main result is the following

Theorem. *For every* $f\in C[-1,1]$ *and* $-1<\alpha,\ \beta\leqslant0$, *we have*

$$\sup_{f\in H_\omega}|H_n^{(\alpha,\beta)}(f,x)-f(x)|\sim\begin{cases}L_\alpha(x), & if\quad 0\leqslant x\leqslant1,\\ L_\beta(x), & if\quad -1\leqslant x\leqslant0,\end{cases}\tag{1.2}$$

where

$$L_t(x)=\begin{cases}\omega(|x-x_j|)v_j(x)+K_t(x), & if\quad -1<t\leqslant-\dfrac{1}{2},\\[2ex] \omega(|x-x_j|)v_j(x)+\omega(1)(P_n^{(\alpha,\beta)}(x))^2+I_t(x)+t\cdot J_t(x), & if\ -\dfrac{1}{2}<t\leqslant0,\end{cases}$$

and

$$K_t(x)=n(P_n^{(\alpha,\beta)}(x))^2\sum_{i=1}^{n}\omega\Big(\frac{i}{n}\sqrt{1-x^2}+\frac{i^2}{n^2}\Big)\Big(\sqrt{1-x^2}+\frac{i}{n}\Big)^{1+2t}\cdot i^{-2},\quad -1<t\leqslant-\frac{1}{2},$$

$$I_t(x)=n(1-x^2)(P_n^{(\alpha,\beta)}(x))^2\sum_{i=1}^{n}\omega\Big(\frac{i}{n}\sqrt{1-x^2}+\frac{i^2}{n^2}\Big)\Big(\sqrt{1-x^2}+\frac{i}{n}\Big)^{-1+2t}\cdot i^{-2},$$
$$-\frac{1}{2}<t\leqslant0,$$

$$J_t(x)=n^{-2t}(P_n^{(\alpha,\beta)}(x))^2\sum_{i=1}^{n}\omega\Big(\frac{i}{n}\sqrt{1-x^2}+\frac{i^2}{n^2}\Big)\cdot i^{-1+2t},\quad -\frac{1}{2}<t\leqslant0,$$

x_j *satisfies* $|x-x_j|=\min\limits_{1\leqslant k\leqslant n}|x-x_k|$.

§ 2. Lemmas

In order to prove the above Theorem we need some known results and the following lemmas. Let $x=\cos\theta$, $x_k=\cos\theta_k$,

$$|P_n'^{(\alpha,\beta)}(x_k)|\sim k^{-\alpha-\frac{3}{2}}n^{\alpha+2},\quad 0<\theta_k\leqslant\frac{\pi}{2},\tag{2.1}$$

$$\theta_k\sim n^{-1}\{k\pi+O(1)\},\tag{2.2}$$

$$|P_n^{(\alpha,\beta)}(x)|=\begin{cases}O(\theta^{-\alpha-\frac{1}{2}}n^{-\frac{1}{2}}), & \text{if } \frac{c}{n}\leqslant\theta\leqslant\frac{\pi}{2},\\ O(n^\alpha), & \text{if } 0\leqslant\theta\leqslant\frac{c}{n},\end{cases} \tag{2.3}$$

$$P_n^{(\alpha,\beta)}(x)=(-1)^n P_n^{(\beta,\alpha)}(-x), \tag{2.4}$$

$$P_n^{\prime(\alpha,\beta)}(x)=\frac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{2.5}$$

From (2.2) we have

$$\sqrt{1-x_k^2}=\sin\theta_k\sim\begin{cases}k/n, & 0<\theta_k\leqslant\frac{\pi}{2},\\ (n+1-k)/n, & \frac{\pi}{2}\leqslant\theta_k<\pi.\end{cases} \tag{2.6}$$

Thus (2.1) and (2.6) yield

$$|P_n^{\prime(\alpha,\beta)}(\cos\theta_k)|\sim n^{\frac{1}{2}}/\sin^{\alpha+\frac{3}{2}}\theta_k,\quad 0<\theta_k\leqslant\frac{\pi}{2}. \tag{2.7}$$

Let x_j be the zero of $P_n^{(\alpha,\beta)}(x)$ which is nearest to x, $i=|j-k|$.

Lemma 2.1. *The following estimate is valid:*

$$|x-x_k|\sim\left(\frac{i}{n}\sqrt{1-x^2}+\frac{i^2}{n^2}\right),\quad k\neq j. \tag{2.8}$$

For the proof see [7].

Lemma 2.2. *For α, $\beta>-1$, we have*

$$|l_j^{(\alpha,\beta)}(x)|\sim 1. \tag{2.9}$$

Proof. Applying the mean value theorem of differential, we obtain

$$|l_j^{(\alpha,\beta)}(x)|=\left|\frac{P_n^{(\alpha,\beta)}(x)-P_n^{(\alpha,\beta)}(x_j)}{P_n^{\prime(\alpha,\beta)}(x_j)(x-x_j)}\right|=\left|\frac{P_n^{\prime(\alpha,\beta)}(\xi_j)}{P_n^{\prime(\alpha,\beta)}(x_j)}\right|$$

with ξ_j strictly between x and x_j. Denote by z_k $(k=1,\cdots,n-2)$ the zeros of $P_n^{\prime\prime(\alpha,\beta)}(x)$. If $4\leqslant j\leqslant n-3$, then $z_{j+1}<\xi_j<z_{j-3}$. By (2.5) and [8, Lemma 3.4] it follows that

$$\left|\frac{P_n^{\prime(\alpha,\beta)}(\xi_j)}{P_n^{\prime(\alpha,\beta)}(x_j)}\right|\sim\left|\frac{nP_{n-1}^{(\alpha+1,\beta+1)}(\xi_j)}{P_n^{\prime(\alpha,\beta)}(x_j)}\right|=O(1).$$

If $1\leqslant j\leqslant 3$ or $n-2\leqslant j\leqslant n$, we can verify immediately that the above holds.

To complete the proof we need a result of P. Erdös and P. Turán[9] (also see [10]): For an arbitrary matrix X,

$$l_k(x)+l_{k+1}(x)\geqslant 1,\quad \text{if } x\in[x_{k+1},x_k],$$

where $l_k(x)$ is the fundamental polynomial of Lagrange interpolation corresponding to matrix X. Thus, if $x\in[x_n,x_1]$, it is easy to see that

$$|l_j^{(\alpha,\beta)}(x)|\geqslant c.$$

If $x\in(x_1,1]$ or $x\in[-1,x_n)$, we consider the fundamental polynomial corresponding to the zeros of $(1-x^2)P_n^{(\alpha,\beta)}(x)$. Similarly we can prove that the above holds still. Thus Lemma 2.1 is established.

Lemma 2.3. *The following relations are valid:*

$$J_t(x)\geqslant c\omega(1)(P_n^{(\alpha,\beta)}(x))^2, \tag{2.10}$$

$$K_t(x)\geqslant c\omega(1)(P_n^{(\alpha,\beta)}(x))^2, \tag{2.11}$$

$$I_t(x)\geqslant cJ_t(x), \quad -1+\varepsilon\leqslant x\leqslant 1-\varepsilon,\ 0<\varepsilon<1. \tag{2.12}$$

The proof is obvious.

§ 3. The Proof of the Theorem

Since $v_k(x)\geqslant 0$ $(\forall x\in[-1, 1])$, we have

$$\sup_{f\in H_\omega}|H_n^{(\alpha,\beta)}(f, x)-f(x)|=\sum_{k=1}^n\omega(|x-x_k|)v_k(x)l_k^2(x)$$

$$=\omega(|x-x_j|)v_j(x)l_j^2(x)$$

$$+\frac{1}{2}(\alpha+\beta+2)\sum_{k\neq j}\omega(|x-x_k|)\frac{(P_n^{(\alpha,\beta)}(x))^2}{(1-x_k^2)(P_n'^{(\alpha,\beta)}(x_k))^2}$$

$$+\frac{1}{2}(\alpha+\beta+2)\sum_{k\neq j}\omega(|x-x_k|)\frac{(1-x^2)(P_n^{(\alpha,\beta)}(x))^2}{(1-x_k^2)(P_n'^{(\alpha,\beta)}(x_k))^2(x-x_k)^2}$$

$$-\frac{1}{2}(\alpha+\beta)\sum_{k\neq j}\omega(|x-x_k|)\frac{(P_n^{(\alpha,\beta)}(x))^2}{(P_n'^{(\alpha,\beta)}(x_k))^2(x-x_k)^2}$$

$$+(\beta-\alpha)\sum_{k\neq j}\omega(|x-x_k|)\frac{(x-x_k)(P_n^{(\alpha,\beta)}(x))^2}{(1-x_k^2)(P_n'^{(\alpha,\beta)}(x_k))^2(x-x_k)^2}$$

$$:=I_j(x)+\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4:=I_j(x)+\Sigma,$$

where all terms except Σ_4 are nonnegative. Due to (2.4), without loss of generality, we may assume $x\geqslant 0$. Let $m=m(\alpha, \beta)$ be an integer such that if $1\leqslant k\leqslant m$, then $0<\theta_k^{(\alpha,\beta)}\leqslant\frac{\pi}{2}$; if $m<k\leqslant n$, then $\frac{\pi}{2}<\theta_k^{(\alpha,\beta)}<\pi$. Now we shall distinguish and discuss different cases.

Case 3.1. $\alpha>-\frac{1}{2}$.

3.1.1. (a) **First assume** $\frac{1}{2}\leqslant x\leqslant 1$. **We write**

$$\Sigma_2=\frac{1}{2}(\alpha+\beta+2)\left\{\sum_{k=1}^{j-1}+\sum_{k=j+1}^{m}+\sum_{k=m+1}^{[\frac{3}{4}n]}+\sum_{k=[\frac{3}{4}n]+1}^{n}\right\}$$

$$:=\frac{1}{2}(\alpha+\beta+2)\{\Sigma_{21}+\Sigma_{22}+\Sigma_{23}+\Sigma_{24}\}. \tag{3.1}$$

If $j=1$, then $\Sigma_{21}=0$. By (2.7), (2.8) and the fact

$$\sin\theta_k\sim\sin\frac{1}{2}(\theta+\theta_k),\quad 0\leqslant x\leqslant 1,\ j+1\leqslant k\leqslant\left[\frac{3}{4}n\right], \tag{3.2}$$

it follows that

$$\Sigma_{22}\sim\frac{(1-x^2)(P_n^{(\alpha,\beta)}(x))^2}{n}\sum_{k=j+1}^{m}\frac{\omega(|x-x_k|)\sin^{1+2\alpha}\theta_k}{|x-x_k|^{1-2\alpha}\left|\sin\frac{1}{2}(\theta-\theta_k)\right|^{1+2\alpha}\sin^{1+2\alpha}\frac{1}{2}(\theta+\theta_k)}\sim I(x). \tag{3.3}$$

Obviously,

$$\Sigma_{21}+\Sigma_{23}\leqslant cI(x). \tag{3.4}$$

Since

$$|x-x_k|\sim 1,\quad \frac{1}{2}\leqslant x\leqslant 1,\ m+1\leqslant k\leqslant n, \tag{3.5}$$

we have

$$\Sigma_{24}\sim\frac{(1-x^2)(P_n^{(\alpha,\beta)}(x))^2}{n}\sum_{k=m+1}^{n}\omega(1)\sin^{1+2\beta}\theta_k\sim(1-x^2)(P_n^{(\alpha,\beta)}(x))^2\omega(1)\leqslant cI(x). \tag{3.6}$$

From (3.1), (3.3), (3.4) and (3.6) we obtain

$$\Sigma_2\sim\Sigma_{22}\sim I(x),\quad \frac{1}{2}\leqslant x\leqslant 1. \tag{3.7}$$

(b) If $0\leqslant x<\frac{1}{2}$, it is clear that

$$\sin\theta_k\sim 1 \text{ and } \sin\frac{1}{2}(\theta+\theta_k)\sim 1,\quad 0\leqslant x<\frac{1}{2},\ j+1\leqslant k\leqslant\left[\frac{3}{4}n\right]. \tag{3.8}$$

From (2.7), (2.8) and (3.7)

$$(\Sigma_{22}+\Sigma_{23})\sim\frac{(P_n^{(\alpha,\beta)}(x))^2}{n}\left\{\sum_{k=j+1}^{m}\frac{\omega\left(\frac{i}{n}\right)\sin^{1+2\alpha}\theta_k}{\sin^2\frac{1}{2}(\theta-\theta_k)}+\sum_{k=m+1}^{\left[\frac{3}{4}n\right]}\frac{\omega\left(\frac{i}{n}\right)\sin^{1+2\beta}\theta_k}{\sin^2\frac{1}{2}(\theta-\theta_k)}\right\}$$

$$\sim n(P_n^{(\alpha,\beta)}(x))^2\sum_{i=1}^{n}\omega\left(\frac{i}{n}\right)\frac{1}{i^2}\sim I(x). \tag{3.9}$$

It is easy to verify that

$$\Sigma_{21}+\Sigma_{24}\leqslant cI(x). \tag{3.10}$$

Summarizing (3.9), (3.10) and noting (3.7) we have

$$\Sigma_2\sim I(x),\quad 0\leqslant x\leqslant 1. \tag{3.11}$$

3.1.2. Now we turn to the estimation of Σ_3. (a) If $\frac{1}{2}\leqslant x\leqslant 1$, write

$$\Sigma_3=-\frac{1}{2}(\alpha+\beta)\left\{\sum_{k=1}^{j-1}+\sum_{k=j+1}^{m}+\sum_{k=m+1}^{\left[\frac{3}{4}n\right]}+\sum_{k=\left[\frac{3}{4}n\right]+1}^{n}\right\}$$

$$:=-\frac{1}{2}(\alpha+\beta)\{\Sigma_{31}+\Sigma_{32}+\Sigma_{33}+\Sigma_{34}\}. \tag{3.12}$$

Since

$$\sin\theta_k\leqslant\sin\theta+2\left|\sin\frac{1}{2}(\theta-\theta_k)\right|,$$

we have

$$\Sigma_{31}+\Sigma_{32}\leqslant c|\alpha+\beta|\frac{(P_n^{(\alpha,\beta)}(x))^2}{n}\left\{\sum_{\substack{k=1\\k\neq j}}^{m}\frac{\omega(|x-x_k|)\sin^2\theta}{|x-x_k|^{1-2\alpha}\left|\sin\frac{1}{2}(\theta-\theta_k)\right|^{1+2\alpha}}\right.$$

$$\left.+\sum_{\substack{k=1\\k\neq j}}^{m}\frac{\omega(|x-x_k|)\sin^2\frac{1}{2}(\theta-\theta_k)}{|x-x_k|^{1-2\alpha}\left|\sin\frac{1}{2}(\theta-\theta_k)\right|^{1+2\alpha}}\right\}\leqslant c|\alpha+\beta|\{I(x)+J(x)\}. \tag{3.13}$$

From (3.5) and (2.10) it follows that

$$\Sigma_{33}+\Sigma_{34}\leqslant c|\alpha+\beta|(P_n^{(\alpha,\beta)}(x))^2\omega(1)\leqslant c|\alpha+\beta|J(x). \tag{3.14}$$

Using (3.2) and the fact

$$\sin\theta_k\sim\left\{\sin\theta+\sin\frac{1}{2}(\theta_k-\theta)\right\},\quad \frac{1}{2}\leqslant x\leqslant 1,\ j+1\leqslant k\leqslant\left[\frac{3}{8}n\right], \tag{3.15}$$

we get

$$\Sigma_3\geqslant c|\alpha+\beta|\frac{(P_n^{(\alpha,\beta)}(x))^2}{n}\sum_{k=j+1}^{[\frac{3}{8}n]}\frac{\omega(|x-x_k|)\sin^{1+2\alpha}\theta_k}{\sin^2\frac{1}{2}(\theta-\theta_k)}$$

$$\geqslant c|\alpha+\beta|\frac{(P_n^{(\alpha,\beta)}(x))^2}{n}\sum_{k=j+1}^{[\frac{3}{8}n]}\frac{\omega(|x-x_k|)}{\left|\sin\frac{1}{2}(\theta-\theta_k)\right|^{1-2\alpha}}\geqslant c|\alpha+\beta|J(x). \tag{3.16}$$

Similarly,

$$\Sigma_3\geqslant c|\alpha+\beta|\frac{(P_n^{(\alpha,\beta)}(x))^2}{n}\sum_{k=j+1}^{[\frac{3}{8}n]}\frac{\omega(|x-x_k|)\sin^2\theta_k}{|x-x_k|^{1-2\alpha}\left|\sin\frac{1}{2}(\theta-\theta_k)\right|^{1+2\alpha}}$$

$$\geqslant c|\alpha+\beta|\sin^2\theta\frac{(P_n^{(\alpha,\beta)}(x))^2}{n}\sum_{k=j+1}^{[\frac{3}{8}n]}\frac{\omega(|x-x_k|)}{|x-x_k|^{1-2\alpha}\left|\sin\frac{1}{2}(\theta-\theta_k)\right|^{1+2\alpha}}$$

$$\geqslant c|\alpha+\beta|I(x). \tag{3.17}$$

Combining (3.13)—(3.14) and (3.16)—(3.17), we obtain

$$\Sigma_3\sim|\alpha+\beta|\{I(x)+J(x)\},\quad \frac{1}{2}\leqslant x\leqslant 1. \tag{3.18}$$

(b) If $0\leqslant x\leqslant\frac{1}{2}$, the case is analogous to 3.1.1.(b). Then we obtain

$$\Sigma_3\sim|\alpha+\beta|\{I(x)+J(x)\},\quad 0\leqslant x\leqslant 1. \tag{3.19}$$

3.1.3. Consider Σ_1. It is easy to verify that

$$\Sigma_1\leqslant c\frac{(P_n^{(\alpha,\beta)}(x))^2}{n}\left\{\sum_{k=1}^{m}\omega(1)\sin^{1+2\alpha}\theta_k+\sum_{k=m+1}^{n}\omega(1)\sin^{1+2\beta}\theta_k\right\}$$

$$\leqslant c\omega(1)(P_n^{(\alpha,\beta)}(x))^2. \tag{3.20}$$

3.1.4. For Σ_4, using a similar method, we can obtain

$$\Sigma_4\leqslant c|\alpha+\beta|\{I(x)+J(x)\},\quad 0\leqslant x\leqslant 1. \tag{3.21}$$

Collecting (3.11), (3.19), (3.20) and (3.21), we get

$$\Sigma:=\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4\leqslant c\{\omega(1)(P_n^{(\alpha,\beta)}(x))^2+I(x)+|\alpha+\beta|J(x)\}. \tag{3.22}$$

Case 3.2. $-1<\alpha\leqslant-\frac{1}{2}$.

3.2.1.(a) If $\frac{1}{2}\leqslant x\leqslant 1$, then by (2.8) and (3.2)

$$\Sigma_{32}\sim\frac{(P_n^{(\alpha,\beta)}(x))^2}{n}\sum_{k=j+1}^{m}\omega(|x-x_k|)\frac{\sin^{3+2\alpha}\theta_k}{|x-x_k|^{3+2\alpha}|x-x_k|^{-1-2\alpha}}\sim K(x).$$

Obviously,

$$\Sigma_{31}+\Sigma_{33}+\Sigma_{34}\leqslant cK(x).$$

Then, we get

$$\Sigma_3\sim K(x),\quad \frac{1}{2}\leqslant x\leqslant 1. \tag{3.23}$$

(b) If $0\le x\le\frac{1}{2}$, it is easy to verify that the above is valid. Then, we have

$$\sum\nolimits_3\sim K(x),\quad 0\le x\le 1. \tag{3.24}$$

3.2.2. For $\sum_1$, we have by (2.11)

$$\sum\nolimits_1\le c\omega(1)(P_n^{(\alpha,\beta)}(x))^2\le cK(x),\quad 0\le x\le 1. \tag{3.25}$$

3.2.3. Now turn to estimation of $\sum_4$. Write

$$\sum\nolimits_4=(\beta-\alpha)\left\{\sum_{k=1}^{j-1}+\sum_{k=j+1}^{m}+\sum_{k=m+1}^{[\frac{3}{4}n]}+\sum_{k=[\frac{3}{4}n]+1}^{n}\right\}:=\sum\nolimits_{41}+\sum\nolimits_{42}+\sum\nolimits_{43}+\sum\nolimits_{44}.$$

We have

$$\sum\nolimits_{42}+\sum\nolimits_{43}\le c\frac{(P_n^{(\alpha,\beta)}(x))^2}{n}\sum_{k=j+1}^{[\frac{3}{4}n]}\frac{\omega(|x-x_k|)}{|x-x_k|^{-1-2\alpha}\left|\sin\frac{1}{2}(\theta-\theta_k)\right|^{3+2\alpha}}\le cK(x)$$

and

$$\sum\nolimits_{44}\le c\omega(1)(P_n^{(\alpha,\beta)}(x))^2\le cK(x).$$

For $\sum_{41}$, write again

$$\sum\nolimits_{41}=\sum_{k=1}^{[\frac{1}{2}j]}+\sum_{k=[\frac{1}{2}j]+1}^{j-1}:=\sum\nolimits_{41}'+\sum\nolimits_{41}''.$$

First estimate $\sum_{41}''$; here $j\sim k$, $\theta\sim\theta_k$. So

$$\sum\nolimits_{41}''\le c\frac{(P_n^{(\alpha,\beta)}(x))^2}{n}\sum_{k=[\frac{1}{2}j]+1}^{j-1}\frac{\omega(|x-x_k|)}{|x-x_k|^{-1-2\alpha}\left|\sin\frac{1}{2}(\theta-\theta_k)\right|^{2+2\alpha}\sin\theta_k}$$

$$\le cn(P_n^{(\alpha,\beta)}(x))^2\sum_{i=1}^{n}\frac{\omega\left(\frac{i}{n}\sqrt{1-x^2}+\frac{i^2}{n^2}\right)}{ij\left(\sqrt{1-x^2}+\frac{i}{n}\right)^{-1-2\alpha}}\le cK(x).$$

For $\sum_{41}'$, here

$$\frac{1}{2}j\le i=j-k\le j,$$

i.e. $i\sim j$; then we have

$$|x-x_k|\sim\left(\frac{j}{n}\sin\theta+\frac{j^2}{n^2}\right)\sim\sin^2\theta$$

and

$$\sum\nolimits_{41}'\le c\frac{(P_n^{(\alpha,\beta)}(x))^2\omega(\sin^2\theta)}{n\sin^2\theta}\sum_{k=1}^{[\frac{1}{2}j]}\frac{1}{\sin^{-1-2\alpha}\theta_k}$$

$$\le c\sin^{2\alpha}\theta(P_n^{(\alpha,\beta)}(x))^2\omega(\sin^2\theta).$$

From (3.24)

$$K(x)\ge c\sum\nolimits_3\ge c\sum_{k=1}^{[\frac{1}{2}j]}\omega(\sin^2\theta)\frac{\sin^{3+2\alpha}\theta_k(P_n^{(\alpha,\beta)}(x))^2}{n\sin^4\theta}$$

$$\ge c\omega(\sin^2\theta)\frac{(P_n^{(\alpha,\beta)}(x))^2}{n^{2\alpha+4}\sin^4\theta}\sum_{k=1}^{[\frac{1}{2}j]}k^{2\alpha+3}\ge c\sin^{2\alpha}\theta(P_n^{(\alpha,\beta)}(x))^2\omega(\sin^2\theta).$$

Collecting these estimates, we get

$$\Sigma_4 \leqslant cK(x), \quad 0 \leqslant x \leqslant 1. \tag{3.26}$$

Since

$$1-x^2 \leqslant 2(1-x_k^2+|x-x_k|),$$

it follows from (3.24) and the proof of (3.26) that

$$\Sigma_2 \leqslant c\left\{\Sigma_3+\sum_{k\neq j}\omega(|x-x_k|)\frac{(P_n^{(\alpha,\beta)}(x))^2}{(1-x_k^2)(P_n'^{(\alpha,\beta)}(x_k))^2|x-x_k|}\right\} \leqslant cK(x). \tag{3.27}$$

Combining (3.24)—(3.27), we get

$$\Sigma \leqslant cK(x), \quad 0 \leqslant x \leqslant 1. \tag{3.28}$$

Now we will prove that the reverse inequality holds.

If $-1<\alpha<0$, since $v_k(x) \geqslant c>0$ ($\forall x \in [0, 1]$, $k=1, \cdots, n$), we have by (3.19), (3.24) and (2.10)

$$\Sigma \geqslant c\{\omega(1)(P_n^{(\alpha,\beta)}(x))^2+I(x)+|\alpha|J(x)\}, \quad -\frac{1}{2}<\alpha<0 \tag{3.29}$$

and

$$\Sigma \geqslant cK(x), \quad -1<\alpha \leqslant -\frac{1}{2} \tag{3.30}$$

respectively.

Now assume $\alpha=0$. To complete the proof we are going to prove that

$$\Sigma \sim \{\omega(1)(P_n^{(\alpha,\beta)}(x))^2+I(x)\}, \quad \alpha=0. \tag{3.31}$$

Indeed, for $\frac{1}{2} \leqslant x \leqslant 1$,

$$v_k^{(0,\beta)}(x)=\{(1+\beta)(1-x_k^2)-\beta(1-x)+(\beta+2)x_k(1-x)\}/(1-x_k^2)$$
$$\sim\left\{(1-x_k^2)+\frac{(1-x^2)}{1-x_k^2}\right\}, \quad j+1 \leqslant k \leqslant \left[\frac{3}{8}n\right].$$

So

$$\Sigma \geqslant \sum_{k=j+1}^{[\frac{3}{8}n]}\omega(|x-x_k|)\frac{(1-x^2)(P_n^{(\alpha,\beta)}(x))^2}{(1-x_k^2)(P_n'^{(\alpha,\beta)}(x_k))^2(x-x_k)^2} \geqslant cI(x) \tag{3.32}$$

and

$$\Sigma \geqslant \sum_{k=j+1}^{[\frac{3}{8}n]}\omega(|x-x_k|)\frac{(1-x_k^2)(P_n^{(\alpha,\beta)}(x))^2}{(P_n'^{(\alpha,\beta)}(x_k))^2(x-x_k)^2} \geqslant c\omega(1)(P_n^{(\alpha,\beta)}(x))^2. \tag{3.33}$$

(3.32) and (3.33) yield

$$\Sigma \geqslant c\{\omega(1)(P_n^{(\alpha,\beta)}(x))^2+I(x)\}. \tag{3.34}$$

On the other hand, since

$$v_k^{(0,\beta)}(x) \leqslant c\left\{\frac{1-x^2}{1-x_k^2}+\frac{1-x_k}{1+x_k}\right\}, \quad \forall x \in [-1, 1], \ k=1, \cdots, n,$$

we have by (3.11) and a similar method that

$$\Sigma \leqslant c\left\{\Sigma_3+\sum_{\substack{k=1\\k\neq j}}^{n}\omega(|x-x_k|)\frac{(1-x_k)(P_n^{(\alpha,\beta)}(x))^2}{(1+x_k)(P_n'^{(\alpha,\beta)}(x_k))^2(x-x_k)^2}\right\}$$
$$\leqslant c\left\{I(x)+\sum_{\substack{k=1\\k\neq j}}^{n}\right\} \leqslant c\{I(x)+\omega(1)(P_n^{(\alpha,\beta)}(x))^2\}. \tag{3.35}$$

Combining (3.34) and (3.35), it follows that (3.29) is valid, if $\frac{1}{2} \leqslant x \leqslant 1$.

If $0\leqslant x\leqslant\frac{1}{2}$, by (3.22), (2.10) and (2.12) it is clear that

$$\sum\leqslant c\{\omega(1)(P_n^{(\alpha,\beta)}(x))^2+I(x)\},\quad 0\leqslant x\leqslant\frac{1}{2},$$

and the reverse inequality follows at once on using the fact

$$v_k^{(0,\beta)}(x)\geqslant 0,\quad \left[\frac{n}{8}\right]\leqslant k\leqslant j-1.$$

This proves that (3.29) holds for $0\leqslant x\leqslant 1$.

Finally, summarizing (3.22), (3.28)—(3.31) and Lemma 2.2, we establish the Theorem.

§ 4. Some Remarks

4.1. If $\alpha<0$, we have

$$v_j^{(\alpha,\beta)}(x)\sim 1,\quad 0\leqslant x\leqslant 1$$

and

$$I_j(x)\sim\begin{cases}\omega\left(\dfrac{(1-x^2)^{\frac{\alpha}{2}+\frac{3}{4}}|P_n^{(\alpha,\beta)}(x)|}{\sqrt{n}}\right), & \dfrac{1}{2}\theta_1^{(\alpha,\beta)}\leqslant\theta\leqslant\dfrac{\pi}{2}\\ \omega\left(\dfrac{1}{n^2}\right), & 0\leqslant\theta\leqslant\dfrac{1}{2}\theta_1^{(\alpha,\beta)}.\end{cases}$$

If $\alpha=0$, then

$$v_j^{(0,\beta)}(x)\leqslant c,$$

$$I_j(x)\sim\omega\left(\frac{(1-x^2)^{\frac{3}{4}}|P_n^{(0,\beta)}(x)|}{\sqrt{n}}\right)v_j(x),\quad \frac{1}{2}\theta_1\leqslant\theta\leqslant\frac{\pi}{2},$$

$$I_j(x)\sim c\omega\left(\frac{1}{n^2}\right)v_1(x),\quad 0\leqslant\theta\leqslant\frac{1}{2}\theta_1.$$

Applying these estimates and Lemma 2.3, we can obtain more explicit expressions than (1.2).

4.2. When $\alpha\geqslant 0$ or $\beta\geqslant 0$, the interpolation process (1.1) does not converge to f in general. If $\alpha=0$, using the Theorem, it follows that

$$\sup_{f\in H_\omega}\|H_n^{(0,\beta)}(f,x)-f(x)\|_{[0,1]}\sim 1.$$

4.3. If $\alpha\geqslant 0$ or $\beta\geqslant 0$, from the proof of the Theorem and Lemma 2.3 it follows for $x\in[-1+\varepsilon,1-\varepsilon]\ (\varepsilon>0)$ that

$$|H_n^{(\alpha,\beta)}(f,x)-f(x)|\leqslant cI(x)\leqslant cn(P_n^{(\alpha,\beta)}(x))^2\sum_{i=1}^n\omega\left(\frac{i}{n}\right)\frac{1}{i^2}$$

$$\leqslant(P_n^{(\alpha,\beta)}(x))^2\sum_{k=1}^n\omega\left(\frac{1}{k}\right).$$

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