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TWO-DIMENSIONAL REPRODUCING KERNEL AND SURFACE INTERPOLATION*

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Abstract

One-dimensional polynomial interpolation does not guarantee the convergency and the stability during numerical computation. For two (or multi)-dimensional interpolation, difficulties are much more raising. There are many fundamental problems, which are left open.

In this paper, we begin with the discussion of reproducing kernel in two variables. With its help we deduce a two-dimensional interpolation formula. According to this formula, the process of interpolation will converge uniformly, whenever the knot system is thickened in finitely. We have also proven that the error function will decrease monotonically in the sense of Соболев norm when the number of knot points is increased.

In our formula, knot points may be chosen arbitrarily without any request of regularity about their arrangement. We also do not impose any restriction on the number of knot points. For the case of multi-dimensional interpolation, these features may be important and essential.

§1. The *W* Space and Reproducing Kernel

1. Let

$$W=\left\{\{u|u\in C([a, b]\times[c, d]);\frac{\partial u}{\partial \xi},\ \frac{\partial u}{\partial \eta},\ \frac{\partial^2 u}{\partial \xi\partial \eta}\in L^2([a, b]\times[c, d])\right\}.$$

We define the inner product

$$(u, v)=\int_c^d d\eta\int_a^b\left(uv+\frac{\partial u}{\partial \xi}\cdot\frac{\partial v}{\partial \xi}+\frac{\partial u}{\partial \eta}\cdot\frac{\partial v}{\partial \eta}+\frac{\partial^2 u}{\partial \xi\partial \eta}\cdot\frac{\partial^2 v}{\partial \xi\partial \eta}\right)d\xi$$

and the norm

$$\|\cdot\|=(\cdot, \cdot)^{\frac{1}{2}}.$$

2. Definition. *Suppose* $K_{xy}(\cdot)\in W$, $(x\in[a, b], y\in[c, d])$. *If for any* $u\in W$ *we have*

$$u(x, y)=(u(\cdot), K_{xy}(\cdot)),$$

then we call $K_{xy}(\cdot)$ *the reproducing kernel in* W *space.*

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3. Definition.

$$\begin{cases} R_M(M') \overset{\text{def.}}{=\!=} R_{xy}(\xi, \eta) = R_x(\xi)\cdot R_y(\eta), \quad x, \xi \in [a, b], y, \eta \in [c, d], \\ R_x(\xi) = \dfrac{1}{2\,\text{sh}(b-a)}[\text{ch}(\xi+x-a-b)+\text{ch}(|\xi-x|+a-b)], \\ R_y(\eta) = \dfrac{1}{2\,\text{sh}(d-c)}[\text{ch}(\eta+y-c-d)+\text{ch}(|\eta-y|+c-d)]. \end{cases} \tag{1}$$

Evidently $R_{xy} \in W$, $R_{xy}(\xi, \eta) = R_{\xi\eta}(x, y)$. It is easy to verify that $R_{xy}(\xi, \eta)$ is a reproducing kernel in W space.

Proof. For any $u \in W$,

$$\begin{aligned}(u(\cdot), R_{xy}(\cdot)) &= \int_c^d d\eta \int_a^b \{u(\xi, \eta) R_{xy}(\xi, \eta) + u'_\xi(\xi, \eta) R'_{xy}(\xi, \eta)_\xi \\ &\quad + u'_\eta(\xi, \eta) R'_{xy}(\xi, \eta)_\eta + u''_{\xi\eta}(\xi, \eta) R''_{xy}(\xi, \eta)_{\xi\eta}\} d\xi \\ &= \int_c^d d\eta \Big\{ R_y(\eta) \int_a^b (uR_x(\xi) + u'_\xi R'_x(\xi)_\xi) d\xi \\ &\quad + R'_y(\eta)_\eta \int_a^b \Big(u'_\eta R_x(\xi) + \frac{\partial}{\partial \xi}(u'_\eta) R'_x(\xi)_\xi\Big) d\xi \Big\}. \end{aligned} \tag{2}$$

Using the reproducing property of one-dimensional kernel $R_x(\xi)$[1], we have

$$\int_a^b (uR_x(\xi) + u'_\xi R'_x(\xi)_\xi) d\xi = u(x, \eta), \tag{3}$$

$$\int_a^b \Big(u'_\eta R_x(\xi) + \frac{\partial}{\partial \xi}(u'_\eta) R'_x(\xi)_\xi\Big) d\xi = u'_\eta(x, \eta). \tag{4}$$

So,

$$(u(\cdot), R_{xy}(\cdot)) = \int_c^d \{u(x, \eta) R_y(\eta) + u'_\eta(x, \eta) R'_y(\eta)_\eta\} d\eta. \tag{5}$$

Again, by the reproducing property of $R_y(\eta)$, we get

$$(u(\cdot), R_{xy}(\cdot)) = u(x, y).$$

It is also very easy to verify that $R_{xy}(\xi, \eta)$ holds many properties that $R_x(\xi)$ possesses. For instance, $R_{xy}(\xi, \eta)$ satisfies Lipschitz condition and is positively bounded both above and below[1].

§ 2. Formula of Interpolation

Let $E = [a, b] \times [c, d]$, and let $\{M_i\}_1^n$ be n distinct knots of interpolation on E with $M_i = (x_i, y_i)$. Denote $R_M(M') = R_{xy}(\xi, \eta)$, $M' = (\xi, \eta)$.

Put

$$\phi_k(M) = R_{M_k}(M), \quad k = 1, 2, \cdots, n. \tag{6}$$

By the reproducing property of $R_{M_k}(M)$, it follows that

$$(u(M), \phi_k(M)) = u(M_k), \quad u \in W, k = 1, 2, \cdots, n. \tag{7}$$

Evidently $\{\phi_k(M)\}_1^n$ forms a linearly independent system in W. Using Gram-Schmidt process, we can get the orthonormal system $\{\overline{\phi}_k(M)\}_1^n$:

$$\overline{\phi}_k(M) = \sum_{i=1}^{k} \beta_{ki} \phi_i(M), \quad k = 1, 2, \cdots, n. \tag{8}$$

In order to describe the degree of thickness of the knot system, we introduce a measure called the degree of sparseness of $\{M_i\}_1^n$ on E.

Definition. *We call*

$$\rho=\max_{M\in E}\inf_{1\le i\le n} d(M, M_i)$$

the degree of sparseness of knot system $\{M_i\}_1^n$ on E, where $d(M, M_i)$ denotes the distance between M and M_i.

Theorem 1. *Suppose $u\in W$, then the linear operator*

$$H_n(u; M)=\sum_{j=1}^n(u, \overline{\phi}_j)\overline{\phi}_j(M)=\sum_{j=1}^n\Big(\sum_{i=1}^j\beta_{ji}u(M_i)\Big)\overline{\phi}_j(M) \tag{9}$$

satisfies the condition of interpolation:

$$H_n(u; M_k)=u(M_k)\equiv u_k,\quad k=1, 2, \cdots, n. \tag{10}$$

Proof. Let $X_n^\phi=\text{span}(\{\overline{\phi}_j\}_1^n)$ and let PX_n^ϕ be the projective operator to X_n^ϕ. Then

$$(PX_n^\phi u)(M)=\sum_{j=1}^n(u, \overline{\phi}_j)\overline{\phi}_j(M), \tag{11}$$

and $H_n(u; M)$ is exactly the operator of projection $(PX_n^\phi u)(M)$. Recalling that $\phi_j\in X_n^\phi$ $(j=1, 2, \cdots, n)$, we have

$$H_n(u; M_k)=(H_n(u; \cdot), \phi_k(\cdot))=(PX_n^\phi u, \phi_k)$$
$$=(u, PX_n^\phi\phi_k)=(u, \phi_k)=u_k,\quad k=1, 2, \cdots, n.$$

Theorem 1 has been proven.

§ 3. Convergence Property of Interpolation Process

We shall prove, that $H_n(u; M)$ converges to $u(M)$ uniformly when $\rho\to 0$.

Theorem 2. *Let $u\in W$ and let $\{M_i\}_1^n$ be the knot system of interpolation. If $\rho\to 0$, then*

1° $H_n(u; M)\xrightarrow{\text{uniformly}} u(M)$,

2° $\|H_n(u; M)-u(M)\|\ge\|H_{n+1}(u; M)-u(M)\|$,

where the knot system corresponding to $H_{n+1}(u; M)$ comes from $\{M_i\}_1^n$ by adding one new point M_{n+1}.

Proof.

$$|u(M)-H_n(u; M)|=|(u(\cdot), R_M(\cdot))-\sum_{i=1}^n(u(\cdot), \overline{\phi}_i(\cdot))\overline{\phi}_i(M)|$$
$$=|(u(\cdot), R_M(\cdot)-\sum_{i=1}^n\overline{\phi}_i(\cdot)\overline{\phi}_i(M))|\le\|u\|\,\|R_M(\cdot)-\sum_{i=1}^n\overline{\phi}_i(\cdot)\overline{\phi}_i(M)\|$$
$$=\|u\|\,\|R_M(\cdot)-\sum_{i=1}^n(R_M(\cdot), \overline{\phi}_i(\cdot))\overline{\phi}_i(\cdot)\|. \tag{12}$$

Evidently, the distance between $R_M(\cdot)$ and its projection on $\text{span}(\{\phi_i\}_1^n)$ must be no more than the distance between $R_M(\cdot)$ and its projection on any one of the $\phi_i(\cdot)$'s. Hence

$$\|R_M(\cdot)-\sum_{i=1}^n(R_M(\cdot), \overline{\phi}_i(\cdot))\overline{\phi}_i(\cdot)\|$$
$$\le\|R_M(\cdot)-(R_M(\cdot), \overline{\phi}_i(\cdot))\overline{\phi}_i(\cdot)\|\ \text{for all } 1\le i\le n. \tag{13}$$

Let

$$\min_{1\le i\le n} d(M, M_i)=d(M, M_k), M_k=(x_k, y_k). \tag{14}$$

In order to prove the uniform convergency of interpolation operator $H_n(u, M)$, we have only to verify that, for any $\varepsilon>0$, there exists a $\delta>0$ such that

$$I=\|R_M(\cdot)-(R_M(\cdot), \overline{\phi}_{i_*}(\cdot))\overline{\phi}_{i_*}(\cdot)\|<\varepsilon, \quad \text{when } \rho<\delta, \tag{15}$$

i.e.,

$$I=\|R_M(\cdot)-\overline{\phi}_{i_*}(M)\overline{\phi}_{i_*}(\cdot)\|<\varepsilon. \tag{16}$$

Since the solution of least square problem is unique, the left side of inequality (13) is invariant with respect to the permutation of $\phi_1, \phi_2, \cdots, \phi_n$. For the sake of simplicity in estimation of (15), we may suppose

$$\phi_{i_*}(M)=\phi_1(M).$$

Thus

$$\begin{aligned}
&\overline{\phi}_{i_*}(M)=R_{M^*}(M)/\|R_{M^*}(\cdot)\|, \quad M^*=(x_{i_*}, y_{i_*})\equiv(x^*, y^*),\\
&I=\|R_M(\cdot)-R_{M^*}(M)R_{M^*}(\cdot)/\|R_{M^*}(\cdot)\|^2\|\\
&\quad\leqslant\|R_M(\cdot)-R_{M^*}(\cdot)\|+\|(1-R_{M^*}(M)/\|R_{M^*}(\cdot)\|^2)R_{M^*}(\cdot)\|,
\end{aligned} \tag{17}$$

where

$$\|R_{M^*}(\cdot)\|^2=(R_{M^*}(\cdot), R_{M^*}(\cdot))=R_{M^*}(M^*).$$

Therefore by (17)

$$I\leqslant\|R_M(\cdot)-R_{M^*}(\cdot)\|+\frac{|R_{M^*}(M^*)-R_{M^*}(M)|}{R_{M^*}(M^*)}\|R_{M^*}(\cdot)\|. \tag{18}$$

As mentioned in the end of Section 1, $R_{M^*}(M)$ satisfies Lipschitz condition with respect to M, and $R_{M^*}(M)$ positively both bounded above and bounded below, so

$$0<\|R_{M^*}(\cdot)\|/R_{M^*}(M^*)\leqslant\text{const}.$$

By the definition of M^*, $M^*\to M$, when $\rho\to0$. Hence there exists a $\delta_0>0$ such that the second term in (18) is less than $\varepsilon/3$ when ever $\rho<\delta_0$.

It remains to prove

$$\|R_M(\cdot)-R_{M^*}(\cdot)\|<\frac{2}{3}\varepsilon. \tag{19}$$

To this end, we write

$$\begin{aligned}
\|R_M(\cdot)-R_{M^*}(\cdot)\|^2=&\iint_E\{(R_M(M')-R_{M^*}(M'))^2+(R_M(M')-R_{M^*}(M'))_\xi'^2\\
&+(R_M(M')-R_{M^*}(M'))_\eta'^2+(R_M(M')-R_{M^*}(M'))_{\xi\eta}''^2\}dM'\\
=&I_1+I_2+I_3+I_4,
\end{aligned} \tag{20}$$

where

$$I_1=\iint_E(R_M(M')-R_{M^*}(M'))^2\,dM',$$

$$I_2=\iint_E(R_M(M')-R_{M^*}(M'))_\xi'^2\,dM',$$

$$I_3=\iint_E(R_M(M')-R_{M^*}(M'))_\eta'^2\,dM',$$

$$I_4=\iint_E(R_M(M')-R_{M^*}(M'))_{\xi\eta}''^2\,dM'.$$

Remembering that $R_M(M')$ satisfies Lipschitz condition, we have immediately

$I_1 < \varepsilon^2/9$, provided n is large enough so that the degree of sparseness of knot system $\rho < \delta_1$.

In order to estimate I_2, we denote

$$f_M(M') \equiv f_{xy}(\xi, \eta) \overset{\text{def.}}{=\!=} R'_x(\xi)_\xi \cdot R_y(\eta).$$

$f_M(M')$ is defined almost everywhere on E, and $|f_M(M')| \leqslant \text{const}$, $\xi \neq x$. This implies $(f_M(M') - f_{M^*}(M'))^2 \leqslant \text{const}$ with $\xi \neq x$ and $\xi \neq x^*$. On the other hand, there exists a $\delta' > 0$ such that

$$\iint_e \{f_M(M') - f_{M^*}(M')\}^2 dM' < \varepsilon^2/18 \tag{21}$$

when the measure of $e \subset E$ $m(e) < \delta'$. Following, we take s as the union of two stripes in E: $S = S(x^*, R_1) \cup S(x, R_2)$, where $S(x^*, R_1) = \{M'(\xi, \eta) \in E: |x^* - \xi| < R_1\}$, $S(x, R_2) = \{M'(\xi, \eta) \in E: |x - \xi| < R_2\}$. If we take R_1, R_2 to be sufficiently small so that $m(S) < \delta'$. Then

$$\iint_S \{f_M(M') - f_{M^*}(M')\}^2 dM' \leqslant \iint_{E\backslash S} \{f_M(M') - f_{M^*}(M')\}^2 dM' + \varepsilon^2/18. \tag{22}$$

On $E \backslash S$

$$\left|\frac{\partial}{\partial \xi} f_M(M')\right| \leqslant \text{const}, \quad \left|\frac{\partial}{\partial \eta} f_M(M')\right| \leqslant \text{const}.$$

Hence

$$|f_M(M') - f_{M^*}(M')| = |f_{M'}(M) - f_{M'}(M^*)| \leqslant \text{const} \cdot d(M, M^*). \tag{23}$$

By (23), it follows that there exists a positive number $\delta_2(\varepsilon)$ independent of M, such that when $\rho < \delta_2$, the first part in the right side of (22)

$$\iint_{E\backslash S} \{f_M(M') - f_{M^*}(M')\}^2 dM' < \varepsilon^2/18. \tag{24}$$

Combining (22) with (24), we have $I_2 < \varepsilon^2/9$, when $\rho < \delta_2$.

Similarly, it can be proven that $I_3 < \varepsilon^2/9$, $I_4 < \varepsilon^2/9$, when $\rho < \delta_3$, $\rho < \delta_4$ respectively. Taking $\delta = \min\{\delta_0, \delta_1, \delta_2, \delta_3, \delta_4\}$, then (15) holds, and the first part of the theorem has been proven.

Considering the fact that the interpolation functions $H_n(u; M)$ and $H_{n+1}(u; M)$ are exactly the projective operators on $\text{span}(\{\phi_i\}_1^n)$ and $\text{span}(\{\phi_i\}_1^{n+1})$ respectively, the second part of the theorem, monotonically decreasing property of the error, is self-evident. Theorem 2 has been completely proven.

Reference

[1] Zhang Mian, Cui Ming-gen, Deng Zhong-xing, A new uniformly convergent iterative method by interpolation, where error decreases monotonically, *J. Computational Mathematics*, 3: 4 (1985), 365—372