AN ANALYSIS OF PENALTY-NONCONFORMING FINITE ELEMENT METHOD FOR STOKES EQUATIONS*

HAN HOU-DE (韩厚德)
(Peking University, Beijing, China)

Abstract

In this paper, the penalty-nonconforming finite element method for Stokes equations is considered. An abstract error estimate is given. For Crouzeix-Raviart nonconforming triangular elements, in particular, the analysis shows that the "reduced integration" technique is not necessary in the integration of the penalty term on each element. It means that a loss of precision is avoided in this penalty method.

§1. Introduction

We consider the numerical analysis of a class of finite element method for Stokesian flow problems of the type

$$\begin{cases} -\mu\Delta \boldsymbol{u}+\nabla p=\boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u}=0 & \text{in } \Omega, \\ \boldsymbol{u}=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where μ is the viscosity, $\boldsymbol{u}=(u_1, \cdots, u_n)$ is the velocity field, p is the pressure, $\boldsymbol{f}$ is the body force density, and Ω is an open bounded domain in $\mathbb{R}^n$. $\partial\Omega$ is the boundary of Ω satisfying the Lipschitz condition.

As usual, let $H^m(\Omega)$, $H_0^m(\Omega)$ denote the Sobolev spaces with norm $\|\cdot\|_{m,\Omega}$, and $V=(H_0^1(\Omega))^n$, $M=\{q\in L^2(\Omega), \int_\Omega q\,dx=0\}$. Then the boundary value problem (1.1) is equivalent to the following variational problem:

Find $(\boldsymbol{u}, p)\in V\times M$, such that

$$\begin{cases} a(\boldsymbol{u}, \boldsymbol{v})+b(\boldsymbol{v}, p)=\langle \boldsymbol{f}, \boldsymbol{v}\rangle, & \forall \boldsymbol{v}\in V, \\ b(\boldsymbol{u}, q)=0, & \forall q\in M, \end{cases} \tag{1.2}$$

where

$$a(\boldsymbol{u}, \boldsymbol{v})=\mu\sum_{i,j=1}^{n}\int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}dx,$$

$$b(\boldsymbol{v}, q)=-\int_\Omega q(\operatorname{div}\boldsymbol{v})dx=-(\operatorname{div}\boldsymbol{v}, q),$$

$$\langle \boldsymbol{f}, \boldsymbol{v}\rangle=\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,dx.$$

A direct finite-element approximation of problem (1.2) leads to the so-called

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mixed finite element methods using conforming and nonconforming finite elements which had been studied extensively, see [1]—[5]. An alternative formulation of (1.2) is provided by the exterior penalties where (1.2) is replaced by a family of perturbations consisting of unconstrained problems depending on a penalty parameter $\varepsilon>0$.

Let ε be an arbitrary positive number. Then a penalty approximation of the variational problem (1.2) consists of finding $(\boldsymbol{u}_\varepsilon, p_\varepsilon)\in V\times M$, such that

$$\begin{cases} a(\boldsymbol{u}_\varepsilon, \boldsymbol{v})+b(\boldsymbol{v}, p_\varepsilon)=\langle \boldsymbol{f}, \boldsymbol{v}\rangle, & \forall \boldsymbol{v}\in V, \\ b(\boldsymbol{u}_\varepsilon, q)-\varepsilon(p_\varepsilon, q)=0, & \forall q\in M. \end{cases} \tag{1.3}$$

For any $\boldsymbol{v}\in V$, we have

$$\int_\Omega \operatorname{div} \boldsymbol{v}\, dx=0;$$

then we can eliminate the pressure p_ε from the last equation and get

$$p_\varepsilon=-\frac{1}{\varepsilon}\operatorname{div}\boldsymbol{u}_\varepsilon, \quad \text{in } \Omega. \tag{1.4}$$

Finally, we obtain the penalty approximation of the variational problem (1.2) containing only unknown functions $\boldsymbol{u}_\varepsilon$:

$$a(\boldsymbol{u}_\varepsilon, \boldsymbol{v})+\varepsilon^{-1}(\operatorname{div}\boldsymbol{u}_\varepsilon, \operatorname{div}\boldsymbol{v})=\langle \boldsymbol{f}, \boldsymbol{v}\rangle, \quad \forall \boldsymbol{v}\in V. \tag{1.5}$$

The variational problem (1.5) and (1.4) is equivalent to problem (1.3). The significant advantage in the penalty variational problem (1.5) is that the pressure does not appear explicitly in the variational formulation; hence the corresponding finite element schemes can be constructed to have fewer unknowns than the standard mixed methods.

Finite element methods based on (1.5) have been proposed by several authors[5—9], who on the basis of numerical experiments, have determined that it is necessary to use reduced integration of the penalty terms in formulation (1.5) in order to obtain physically reasonable results. These reduced-integration-penalty schemes also have been studied mathematically by several authors. In particular, we refer to the work of Oden, Kikuchi and Song[10].

In this paper, nonconforming finite elements are applied to penalty finite element methods for Stokes equations. Moreover, an abstract error estimate is given. For nonconforming triangular elements, in particular, the reduced integration technique is not necessary. It means that the integration of the penalty term on each element is required to integrate exactly.

§ 2. Nonconforming Finite Element Approximation

First of all, we recall the basic convergence theorem for penalty problem (1.5).

Theorem 2.1. *Given $\varepsilon>0$, let $\boldsymbol{u}_\varepsilon\in V$ be the solution of (1.5) and let p_ε be the function given by (1.4). Then $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ converges strongly to solution $(\boldsymbol{u}, p)$ of (1.2) in $V\times M$ as $\varepsilon\to 0$. Moreover, the following estimates hold*

$$\|\boldsymbol{u}-\boldsymbol{u}_\varepsilon\|_V+|p-p_\varepsilon|_M\le C\varepsilon,$$

where C is a constant independent of ε.

The proof can be found in [10].

We discuss the nonconforming finite element approximation to problem (1.3). Let V_h, M_h be two finite dimensional spaces, and $M_h\subset M$; in general, V_h is not a subspace of V. Suppose $V_h\subset(L^2(\Omega))^n$. We extend the definitions of $a(\boldsymbol{u}, \boldsymbol{v})$ and $b(\boldsymbol{v}, q)$ to $(V_h\cup V)\times(V_h\cup V)$ and $(V_h\cup V)\times M$ respectively. Let $a_h(\boldsymbol{u}, \boldsymbol{v})$ and $b_h(\boldsymbol{v}, q)$ denote those extensions, and

$$a_h(\boldsymbol{u}, \boldsymbol{v})=a(\boldsymbol{u}, \boldsymbol{v}), \quad \forall \boldsymbol{u}, \boldsymbol{v}\in V,$$
$$b_h(\boldsymbol{v}, q)=b(\boldsymbol{v}, q), \quad \forall \boldsymbol{v}\in V, q\in M.$$

Furthermore, suppose that

(1) there are three constants $\alpha>0$, $A>0$, $B>0$ such that

$$\begin{cases} a_h(\boldsymbol{v}_h, \boldsymbol{v}_h)\geqslant\alpha\|\boldsymbol{v}_h\|_h^2, & \forall \boldsymbol{v}_h\in V_h, \\ |a_h(\boldsymbol{u}_h, \boldsymbol{v}_h)|\leqslant A\|\boldsymbol{u}_h\|_h\|\boldsymbol{v}_h\|_h, & \forall \boldsymbol{u}_h, \boldsymbol{v}_h\in V_h, \\ |b_h(\boldsymbol{v}_h, q_h)|\leqslant B\|\boldsymbol{v}_h\|_h\|q_h\|_M, & \forall \boldsymbol{v}_h\in V_h, q_h\in M_h, \end{cases} \tag{2.1}$$

where $\|\cdot\|_h$ denotes the norm of space V_h;

(2) there is an operator ρ_h: $V_h\to M_h$ satisfying

$$-b_h(\boldsymbol{v}_h, q_h)=(\rho_h\boldsymbol{v}_h, q_h), \quad \forall q_h\in M_h; \tag{2.2}$$

(3) there is a constant C independent of h, such that

$$\|\boldsymbol{v}\|_{0,\Omega}\leqslant C\|\boldsymbol{v}_h\|_h, \quad \forall \boldsymbol{v}_h\in V_h; \tag{2.3}$$

(4) there exists a constant $\beta_h>0$, such that

$$\sup_{\boldsymbol{v}_h\in V_h}\frac{b_h(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_h}\geqslant\beta_h\|q_h\|_M, \quad \forall q_h\in M_h. \tag{2.4}$$

The nonconforming finite element approximation of (1.3) is the solution to the following problem:

Find $(\boldsymbol{u}_h^\varepsilon, p_h^\varepsilon)\in V_h\times M_h$, such that

$$\begin{cases} a_h(\boldsymbol{u}_h^\varepsilon, \boldsymbol{v}_h)+b_h(\boldsymbol{v}_h, p_h^\varepsilon)=\langle \boldsymbol{f}, \boldsymbol{v}_h\rangle, & \forall \boldsymbol{v}_h\in V_h, \\ b_h(\boldsymbol{u}_h^\varepsilon, q_h)-\varepsilon(p_h^\varepsilon, q_h)=0, & \forall q_h\in M_h. \end{cases} \tag{2.5}$$

To begin with, we prove the following lemmas.

Lemma 2.1. *Suppose that the hypotheses (1)—(4) hold. Then problem (2.5) has a unique solution $(\boldsymbol{u}_h^\varepsilon, p_h^\varepsilon)$, and the following estimates hold*

$$\begin{cases} \|\boldsymbol{u}_h^\varepsilon\|_h\leqslant C\|\boldsymbol{f}\|_{0,\Omega}, \\ \|p_h^\varepsilon\|_M\leqslant\dfrac{C}{\beta_h}\|\boldsymbol{f}\|_{0,\Omega}, \end{cases} \tag{2.6}$$

where C is a constant dependent only on α, A and B.

Proof. By condition (2.2), problem (2.5) can be rewritten as follows:

$$\begin{cases} a_h(\boldsymbol{u}_h^\varepsilon, \boldsymbol{v}_h)+\dfrac{1}{\varepsilon}(\rho_h\boldsymbol{v}_h, \rho_h\boldsymbol{u}_h^\varepsilon)=\langle \boldsymbol{f}, \boldsymbol{v}_h\rangle, & \forall \boldsymbol{v}_h\in V_h, \\ p_h^\varepsilon=-\dfrac{1}{\varepsilon}\rho_h\boldsymbol{u}_h^\varepsilon. \end{cases} \tag{2.5*}$$

Obviously, problem (2.5)* is equivalent to problem (2.5). It is straightforward to see that problem (2.5)* has a unique solution $(\boldsymbol{u}_h^\varepsilon, p_h^\varepsilon)$ by the Lax-Milgram theorem[11]. Setting $v_h=u_h^\varepsilon$ in (2.5)*, we obtain

$$\|u_h^\varepsilon\|_h^2\leqslant\frac{1}{\alpha}\Big\{a_h(u_h^\varepsilon,\ u_h^\varepsilon)+\frac{1}{\varepsilon}(\rho_h u_h^\varepsilon,\ \rho_h u_h^\varepsilon)\Big\}=\frac{1}{\alpha}\langle f,\ u_h^\varepsilon\rangle\leqslant\frac{1}{\alpha}\|f\|_{0,\Omega}\|u_h^\varepsilon\|_{0,\Omega}.$$

From condition (2.3), we have

$$\|u_h^\varepsilon\|_h\leqslant C\|f\|_{0,\Omega}.$$

On the other hand, by inequality (2.4) we have

$$\beta_h\|p_h^\varepsilon\|_M\leqslant\sup_{v_h\in V_h}\frac{b_h(v_h,\ p_h^\varepsilon)}{\|v_h\|_h}=\sup_{v_h\in V_h}\frac{-a_h(u_h^\varepsilon,\ v_h)+\langle f,\ v_h\rangle}{\|v_h\|_h}$$
$$\leqslant A\|u_h^\varepsilon\|_h+C\|f\|_{0,\Omega}\leqslant C\|f\|_{0,\Omega}.$$

Finally, we get

$$\|p_h^\varepsilon\|_M\leqslant\frac{C}{\beta_h}\|f\|_{0,\Omega}.$$

Furthermore, we will prove that when $\varepsilon\to0$, $(u_h^\varepsilon,\ p_h^\varepsilon)$ converges to $(u_h,\ p_h)$ satisfying

$$\begin{cases}a_h(u_h,\ v_h)+b_h(v_h,\ p_h)=\langle f,\ v_h\rangle, & \forall v_h\in V_h,\\ b_h(u_h,\ q_h)=0, & \forall q_h\in M_h.\end{cases}\tag{2.7}$$

We have

Theorem 2.2. *Assume that hypotheses* (1)—(4) *hold. Then* $(u_h^\varepsilon,\ p_h^\varepsilon)$ *converges to* $(u_h,\ p_h)$ *when* $\varepsilon\to0$ *and the following estimates hold*

$$\|u_h^\varepsilon-u_h\|_h\leqslant C\beta_h^{-2}\varepsilon\|f\|_{0,\Omega},\tag{2.8}$$
$$\|p_h^\varepsilon-p_h\|_M\leqslant C\beta_h^{-3}\varepsilon\|f\|_{0,\Omega},\tag{2.9}$$

where C *is a constant dependent only on* α, A *and* B.

Proof. From (2.5) and (2.7), we obtain

$$a_h(u_h^\varepsilon-u_h,\ v_h)=b_h(v_h,\ p_h-p_h^\varepsilon),\quad\forall v_h\in V_h.\tag{2.10}$$

By inequality (2.4), we have

$$\|p_h^\varepsilon-p_h\|_M\leqslant\frac{A}{\beta_h}\|u_h^\varepsilon-u_h\|_h.\tag{2.11}$$

Taking $v_h=u_h^\varepsilon-u_h$ in (2.10), we get

$$\|u_h^\varepsilon-u_h\|_h^2\leqslant\frac{1}{\alpha}a_h(u_h^\varepsilon-u_h,\ u_h^\varepsilon-u_h)=\frac{1}{\alpha}b_h(u_h^\varepsilon-u_h,\ p_h-p_h^\varepsilon)$$
$$=\frac{\varepsilon}{\alpha}(p_h^\varepsilon,\ p_h-p_h^\varepsilon)\leqslant\frac{\varepsilon}{\alpha}\|p_h^\varepsilon\|_M\|p_h-p_h^\varepsilon\|_M\leqslant C\|f\|_{0,\Omega}\frac{\varepsilon}{\beta_h^2}\|u_h^\varepsilon-u_h\|_h;$$

the last inequality is from (2.6) and (2.11). Hence inequalities (2.8) and (2.9) follow immediately.

For the error estimates of $\|u-u_h\|_h$ and $\|p-p_h\|_M$, we have

Theorem 2.3. *There exists a constant* C *independent of* h, *such that*

$$\|u-u_h\|_h+\beta_h\|p-p_h\|_M\leqslant C\Big(1+\frac{1}{\beta_h}\Big)\{\inf_{v_h\in V_h}\|u-v_h\|_h+\inf_{q_h\in M_h}\|p-q_h\|_M+\|E_h\|_{V_h'}\},\tag{2.12}$$

where $(u,\ p)$ *is the solution to* (1.2), *and*

$$E_h(u,\ p;\ v_h)=a_h(u,\ v_h)+b_h(v_h,\ p)-\langle f,\ v_h\rangle,\tag{2.13}$$

$$\|E_h\|_{V_h'}=\sup_{v_h\in V_h}\frac{|E_h(u,p;v_h)|}{\|v_h\|_h}. \tag{2.14}$$

Proof. By equality (2.13), we know that, for arbitrary $v_h\in V_h$, (u, p) satisfies

$$\begin{cases} a_h(u, v_h)+b_h(v_h, p)=\langle f, v_h\rangle+E_h(u, p; v_h), & \forall v_h\in V_h, \\ b_h(u, q_h)=0, & \forall q_h\in M_h. \end{cases} \tag{2.15}$$

From (2.7) and (2.15), we get

$$a_h(u-u_h, v_h)=b_h(v_h, p_h-p)+E_h(u, p; v_h), \quad \forall v_h\in V_h. \tag{2.16}$$

For arbitrary $v_h\in V_h$ and $q_h\in M_h$, we have

$$\begin{aligned} a_h(v_h-u_h, v_h-u_h)&=a_h(v_h-u, v_h-u_h)+a_h(u-u_h, v_h-u_h)\\ &=a_h(v_h-u, v_h-u_h)+b_h(v_h-u_h, p_h-p)+E_h(u, p; v_h-u_h)\\ &=a_h(v_h-u, v_h-u_h)+b_h(v_h-u, p_h-p)+b_h(u-u_h, q_h-p)\\ &\quad+E_h(u, p; v_h-u_h). \end{aligned}$$

Moreover, we obtain

$$\begin{aligned} \|v_h-u_h\|_h^2\leqslant\frac{1}{\alpha}\{&A\|u-v_h\|_h\|v_h-u_h\|_h+B\|u-v_h\|_h\|p-p_h\|_M\\ &+B\|u-u_h\|_h\|p-q_h\|_M+E_h\|_{V_h'}\|v_h-u_h\|_h\}. \end{aligned}$$

Then there is a constant C_1 dependent only on α, A and B, such that

$$\begin{aligned} \|u-u_h\|_h^2\leqslant C_1\{&\|u-v_h\|_h^2+\|u-v_h\|_h\|p-p_h\|_M\\ &+\|u-u_h\|_h\|p-q_h\|_M+\|E_h\|_{V_h'}^2\}, \quad \forall v_h\in V_h,\ q_h\in M_h. \end{aligned} \tag{2.17}$$

On the other hand, inequality (2.4) yields

$$\begin{aligned} \beta_h\|q_h-p_h\|_M&\leqslant\sup_{v_h\in V_h}\frac{b_h(v_h, q_h-p_h)}{\|v_h\|_h}\leqslant\sup_{v_h\in V_h}\frac{b_h(v_h, p-p_h)-b_h(v_h, p-q_h)}{\|v_h\|_h}\\ &\leqslant A\|u-u_h\|_h+\|E_h\|_{V_h'}+B\|p-q_h\|_M, \quad \forall q_h\in M_h. \end{aligned}$$

Furthermore, we have

$$\beta_h\|p-p_h\|_M\leqslant A\|u-u_h\|_h+\|E_h\|_{V_h'}+(B+\beta_h)\|p-q_h\|_M, \quad \forall q_h\in M_h. \tag{2.18}$$

After combination of estimates (2.17) and (2.18), the error estimate (2.12) follows immediately.

Finally, an abstract error estimate for penalty-nonconforming finite element approximation is given by Theorems 2.2 and 2.3.

Theorem 2.4. *Suppose that hypotheses* (1)—(4) *hold and* $(u_h^\varepsilon, p_h^\varepsilon)$ *is the solution to problem* (2.5)*; *then we have the following error estimate*

$$\begin{aligned} \|u_h^\varepsilon-u\|_h+\beta_h\|p_h^\varepsilon-p\|_M\leqslant\left(1+\frac{1}{\beta_h}\right)\Big\{&\inf_{v_h\in V_h}\|u-v_h\|_h\\ &+\inf_{q_h\in M_h}\|p-q_h\|_M+\|E_h\|_{V_h'}+\frac{\varepsilon}{\beta_h}\|f\|_{0,\Omega}\Big\}. \end{aligned} \tag{2.19}$$

§ 3. Nonconforming Triangular Elements

In this section we shall confine ourselves to the case $n=2$. Moreover, suppose Ω is an open convex polygon. $\bar{\Omega}$ is divided into some triangles $\{K\}$. Let $\mathscr{T}_h$

denote this triangulation satisying

(1) $\bar{\Omega} = \sum_{K \in \mathscr{T}_h} K$.

(2) For each distinct K_1 and $K_2 \in \mathscr{T}_h$, either $K_1 \cap K_2$ is empty or K_1 and K_2 have a common vertex or K_1 and K_2 have a common side.

(3) Let

$$h_K = \operatorname{diam}(K),$$
$$\rho_K = \sup\{\operatorname{diam}(S);\ S \text{ is a circle contained in } K\},$$
$$h = \max_{K \in \mathscr{T}_h}\{h_K\},$$
$$\rho = \min_{K \in \mathscr{T}_h}\{\rho_K\}.$$

Suppose $h/\rho \leqslant \sigma$, where $\sigma > 0$ is a constant.

(I) Linear elements for the velocity field

Let N_0 denote the set of the midpoints of the sides of $\mathscr{T}_h$ on the boundary of Ω and N_1 denote the set of the midpoints of the sides of $\mathscr{T}_h$ in the interior of Ω. Suppose

$$\mathring{S}_h = \{v \mid v|_K \in P_1(K),\ \forall K \in \mathscr{T}_h,\ v \text{ is continuous on } N_1 \text{ and } v(b) = 0,\ \forall b \in N_0\},$$

$$M_h = \{q \mid q|_K \in P_0(K),\ \int_\Omega q\, dx = 0\},$$

$$V_h = \mathring{S}_h \times \mathring{S}_h,$$

where $P_k(K)$ denote the space of all polynomials of degree $\leqslant k$ on domain K. Obviously, M_h is a subspace of M, but V_h is not a subspace of V. Therefore we need to define the approximate bilinear forms:

$$a_h(\boldsymbol{u}, \boldsymbol{v}) \equiv \mu \sum_{K \in \mathscr{T}_K} \sum_{i,j=1}^{n} \int_K \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j}\, dx, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V_h \cup V,$$

$$b_h(\boldsymbol{v}, p) \equiv -\sum_{K \in \mathscr{T}_K} \int_K (\operatorname{div} \boldsymbol{v}) p\, dx, \quad \forall \boldsymbol{v} \in V_h \cup V,\ p \in M.$$

Let $\|\boldsymbol{v}\|_h = (a_h(\boldsymbol{v}, \boldsymbol{v}))^{1/2}$, $\forall \boldsymbol{v} \in V_h$. It is straightforward to see that hypothesis (2.1) holds. In this case, operator ρ_h is given by

$$\rho_h \boldsymbol{v}_h = -\operatorname{div} \boldsymbol{v}_h \text{ on } K, \quad \forall K \in \mathscr{T}_h,\ \boldsymbol{v}_h \in V_h. \tag{3.1}$$

Obviously, $\rho_h \boldsymbol{v}_h \in M_h$ and

$$(\rho_h \boldsymbol{v}_h, q_h) = -b_h(\boldsymbol{v}_h, q_h), \quad \forall \boldsymbol{v}_h \in V_h,\ q_h \in M_h. \tag{3.2}$$

Hence, the penalty-nonconforming finite element approximation (2.5)* is reduced to

$$\begin{cases} a_h(\boldsymbol{u}_h^\varepsilon, \boldsymbol{v}_h) + \dfrac{1}{\varepsilon} \sum\limits_{K \in \mathscr{T}_h} (\operatorname{div} \boldsymbol{u}_h^\varepsilon, \operatorname{div} \boldsymbol{v}_h)_K = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle, & \forall \boldsymbol{v}_h \in V_h, \\ p_h^\varepsilon = -\dfrac{1}{\varepsilon} \rho_h \boldsymbol{u}_h^\varepsilon, \end{cases} \tag{3.3}$$

where $(\operatorname{div} \boldsymbol{u}_h^\varepsilon, \operatorname{div} \boldsymbol{v}_h)_K = \int_K \operatorname{div} \boldsymbol{u}_h^\varepsilon \cdot \operatorname{div} \boldsymbol{v}_h\, dx$.

It is clear that the reduced integration technique is not necessary for this case. For spaces V_h and M_h, we have

Lemma 3.1. *There exists a constant C independent of h, such that*

$$\sup_{v_h\in V_h}\frac{|E_h(u,p;v_h)|}{\|v_h\|_h}\leqslant Ch(|u|_{2,\Omega}+|p|_{1,\Omega}),$$

where (u, p) is the solution to problem (1.2) and $u\in(H^2(\Omega))^2$, $p\in H^1(\Omega)$.

Proof. By Green's formulation, we obtain

$$\begin{aligned}E_h(u,p;v_h)&=\mu\sum_{K\in\mathcal{T}_h}\int_{\partial K}\frac{\partial u}{\partial\nu}\cdot v_h\,dS-\sum_{K\in\mathcal{T}_h}\int_{\partial K}(v_h\cdot\nu)p\,dS\\&=\mu\sum_{K\in\mathcal{T}_h}\Big\{\int_{\partial K}\frac{\partial u_1}{\partial x_1}\nu_1v_h^1\,dS+\int_{\partial K}\frac{\partial u_1}{\partial x_2}\nu_2v_h^1\,dS+\int_{\partial K}\frac{\partial u_2}{\partial x_1}\nu_1v_h^2\,dS\\&\quad+\int_{\partial K}\frac{\partial u_2}{\partial x_2}\nu_2v_h^2\,dS\Big\}-\sum_{K\in\mathcal{T}_h}\Big\{\int_{\partial K}p\nu_1v_h^1\,dS+\int_{\partial K}p\nu_2v_h^2\,dS\Big\},\end{aligned}$$

where $u=(u_1, u_2)$, $v_h=(v_h^1, v_h^2)$, $\nu=(\nu_1,\nu_2)$ denotes the unit outer normal vector on ∂K. Let

$$T_i(\varphi, v_h)=\sum_{K\in\mathcal{T}_h}\int_{\partial K}\varphi\nu_iv_h\,dS,\quad\forall\varphi\in H^1(\Omega),\ v_h\in\mathring{S}_h.$$

By the result about Crouzeix-Raviart triangular elements in [12], we know that there exists a constant C independent of h, such that

$$|T_i(\varphi, v_h)|\leqslant Ch|\varphi|_{1,\Omega}\|v_h\|_h,\quad\forall\varphi\in H^1(\Omega),\ v_h\in\mathring{S}_h.$$

Therefore, we obtain

$$\begin{aligned}|E_h(u,p;v_h)|&\leqslant Ch\Big\{|p|_{1,\Omega}+\Big|\frac{\partial u_1}{\partial x_1}\Big|_{1,\Omega}+\Big|\frac{\partial u_1}{\partial x_2}\Big|_{1,\Omega}+\Big|\frac{\partial u_2}{\partial x_1}\Big|_{1,\Omega}+\Big|\frac{\partial u_2}{\partial x_2}\Big|_{1,\Omega}\Big\}\|v_h\|\\&\leqslant Ch\{|p|_{1,\Omega}+|u|_{2,\Omega}\}\|v_h\|_h.\end{aligned}$$

The proof is completed.

For $v_h\in\mathring{S}_h$, the discrete imbedding theorem holds. Then there is a constant C independent of h, such that

$$\|v_h\|_{0,\Omega}\leqslant C\|v_h\|_h,\quad\forall v_h\in\mathring{S}_h.\tag{3.4}$$

The proof can be found in [13] with hardly any change. Hence we have

Lemma 3.2. *There is a constant C independent of h, such that*

$$\|v_h\|_{0,\Omega}\leqslant C\|v_h\|_h,\quad\forall v_h\in V_h.$$

Futhermore, we have

Lemma 3.3. *There exists a constant β independent of h, such that*

$$\sup_{v_h\in V_h}\frac{b_h(v_h,q_h)}{\|v_h\|_h}\geqslant\beta\|q_h\|_M,\quad\forall q_h\in M_h.$$

Lemma 3.4. *For given $u\in V\cap(H^2(\Omega))^2$ and $p\in M\cap H^1(\Omega)$, we have*

$$\inf_{v_h\in V_h}\|u-v_h\|_h\leqslant Ch|u|_{2,\Omega},$$

$$\inf_{q_h\in M_h}\|p-q_h\|_{0,\Omega}\leqslant Ch|p|_{1,\Omega}.$$

The result in Lemma 3.3 has been shown in [2] and used implicitly in [1]. The proof of Lemma 3.4 can be found in [11].

An application of Theorem 2.4 yields the following error estimate:

Theorem 3.1. *Suppose that the solution (u, p) of problem (1.2) satisfies*

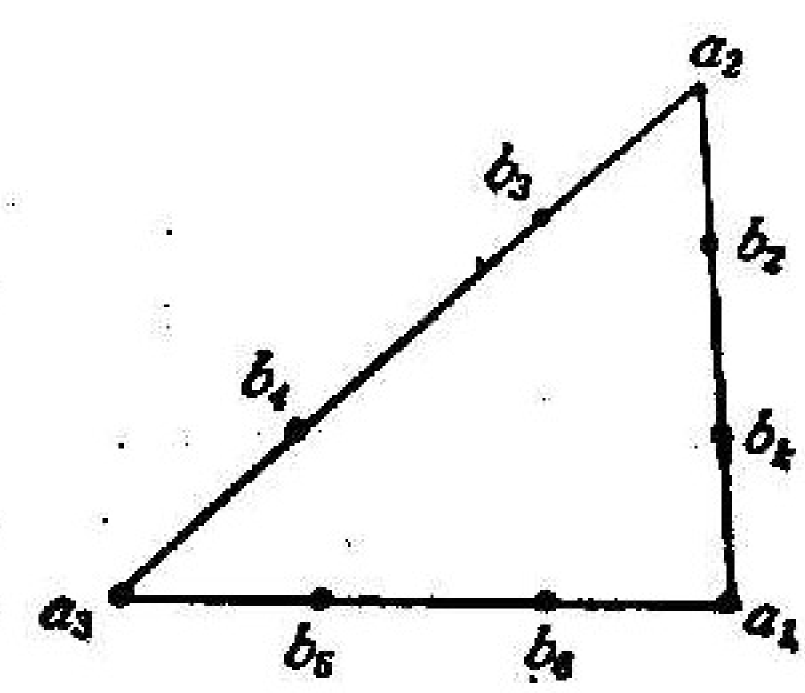
$u\in(H^2(\Omega))^2$, $p\in H^1(\Omega)$; *then the following inequality holds*

$$\|u-u_h^\varepsilon\|_h+\|p-p_h^\varepsilon\|_{0,\Omega}\leqslant C\{h(|u|_{2,\Omega}+|p|_{1,\Omega})+\varepsilon\|f\|_{0,\Omega}\}, \tag{3.5}$$

where C is a constant dependent only on α, A, B and β. $(u_h^\varepsilon, p_h^\varepsilon)$ is the solution to (3.3).

(II) Quadratic elements for velocity field

For each triangle $K\in\mathcal{T}_h$, let a_1, a_2, a_3 be the vertices of K and b_1, b_2, b_3, b_4, b_5, b_6 be the six Gauss-Legendre points of its sides. We use $(\lambda_1, \lambda_2, \lambda_3)$ as the barycentric coordinates of a point x of K. On triangle K, there exists a "neutral function" (unique up to a multiplicative constant)



$$\phi_{0,K}(x)=2-3(\lambda_1^2+\lambda_2^2+\lambda_3^2)$$

which vanishes on the six Gaussian nodes b_1, b_2, $\cdots$, b_6 and belongs to $P_2(x)$.

Let

$\mathring{W}_h=\{v_h|v_h|_K\in P_2(K)$, v_h is continuous on the two Gaussian nodes of each interside of $\mathcal{T}_h$; and $v_h(b_i)=0$, when $b_i\in\partial\Omega\}$.

Suppose

$\mathring{X}_h=\{v_h|v_h|_K\in P_2(K)$, v_h is continuous on Ω; and $v_h|_{\partial\Omega}=0\}$,

$\Phi_h=\{\phi_h|\phi_h|_K=\alpha_K\phi_{0,K}(x),\ \alpha_K\in\mathbb{R}^1\}$.

By proposition 1 in [14], we know $\mathring{W}_h=\mathring{X}_h\oplus\Phi_h$. Assume

$$\begin{cases} M_h=\left\{q_h\middle|q_h|_K\in P_1(K),\int_\Omega q_h\,dx=0\right\},\\ V_h=\mathring{W}_h\times\mathring{W}_h \text{ with norm } \|v_h\|_h=\sqrt{a_h(v_h, v_h)}. \end{cases} \tag{3.6}$$

Similarly, in this case operator ρ_h is given by

$$\rho_h v_h=-\operatorname{div} v_h \text{ on } K,\quad \forall K\in\mathcal{T}_h,\ v_h\in V_h, \tag{3.1*}$$

where $\rho_h v_h\in M_h$, and the penalty nonconforming finite element approximation $(u_h^\varepsilon, p_h^\varepsilon)$ is given by

$$\begin{cases} a_h(u_h^\varepsilon, v_h)+\dfrac{1}{\varepsilon}\sum\limits_{K\in\mathcal{T}_h}(\operatorname{div} u_h^\varepsilon, \operatorname{div} v_h)=\langle f, v_h\rangle, & \forall v_h\in V_h,\\ p_h^\varepsilon=-\dfrac{1}{\varepsilon}\rho_h u_h^\varepsilon. \end{cases} \tag{3.3*}$$

In problem (3.3)*, the integrations of penalty terms on each triangle are required to integrate exactly. For the spaces V_h and M_h defined by (3.6), similarly, we have

Lemma 3.1*. *There is a constant C independent of h, such that*

$$\sup_{v_h\in V_h}\frac{|E_h(u, p; v_h)|}{\|v_h\|_h}\leqslant Ch^2(|u|_{3,\Omega}+|p|_{2,\Omega}),$$

where (u, p) is the solution to problem (1.2) *and* $u\in(H^3(\Omega))^2$, $p\in H^2(\Omega)$.

Lemma 3.2*. *There is a constant C independent of h, such that*

$$\|v_h\|_{0,\Omega}\leqslant C\|v_h\|_h,\quad \forall v_h\in V_h.$$

Lemma 3.3*. *There exists a constant β independent of h, such that*

$$\sup_{v_h\in V_h}\frac{|b_h(v_h, q_h)|}{\|v_h\|_h}\geqslant\beta\|q_h\|_M,\quad \forall q_h\in M_h.$$

Lemma 3.4*. *For given* $u\in V\cap(H^3(\Omega))^2$ *and* $p\in M\cap H^2(\Omega)$, *we have*

$$\inf_{v_h\in V_h}\|u-v_h\|_h\leq Ch^2|u|_{3,\Omega},$$

$$\inf_{q_h\in M_h}\|p-q_h\|_M\leq Ch^2|p|_{2,\Omega},$$

where C is a constant independent of h.

A sketch of the proof of Lemma 3.2* was given in [14]. The proofs of the other three lemmas are basically the same. Finally, we obtain

Theorem 3.2. *Suppose that* $u\in(H^3(\Omega))^2$, $p\in H^2(\Omega)$ *and* (u, p) *is the solution to* (1.2); *then the following inequality holds:*

$$\|u-u_h^\varepsilon\|_h+\|p-p_h^\varepsilon\|_M\leq C\{h^2[|u|_{3,\Omega}+|p|_{2,\Omega}]+\varepsilon\|f\|_{0,\Omega}\},$$

where C is a constant independent of h and ε.

Conclusion. The nonconforming finite element method for Stokes equations has special advantages: by use of nonconforming elements for the standard mixed method, the optimal error estimate or quasi-optimal error estimate can be obtained [1], [3], [14]; by use of nonconforming Crouzeix–Raviart triangular elements for the penalty variational problem (1.5), the reduced integration technique is not necessary. It means that a loss of precision is avoided in this penalty method.

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