INFINITE ELEMENT APPROXIMATION TO AXIAL SYMMETRIC STOKES FLOW*

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We considered in [1] the finite element approximation to axial symmetric Stokes flow in a bounded domain. The problem for the flow passing an obstacle in an unbounded domain is also frequently encountered. In this paper, we are going to give approximate solutions for this problem by an approach stated in [2]. An iterative method[3-5] is used to calculate the combined stiffness matrix.

§ 1. The Reduction to a System of Finite Algebraic Equations

Let us consider a rigid body in a 3-dimensional space, around which there is incompressible viscous fluid with steady velocity u. We assume that the flow at infinity is homogeneous with a velocity u_∞, and the Reynolds number is so small that the assumption of Stokes flow is acceptable. We can always replace u with $u-u_\infty$; therefore it is no harm to deem $u_\infty=0$. Now we give the classical formulation of the axial symmetric Stokes flow. Let $x=(x_1, x_2)\in R^2$, $R^2_+=\{x\in R^2;\ x_1>0\}$, and introduce in R^2 the polar coordinates (r, θ). Suppose there is a broken line Γ with end points at the x_2-axis and Ω is the exterior of Γ in R^2_+ (Fig. 1). Consider the following problem: to find $u(x)=(u_1(x), u_2(x))$, $p(x)$, satisfying

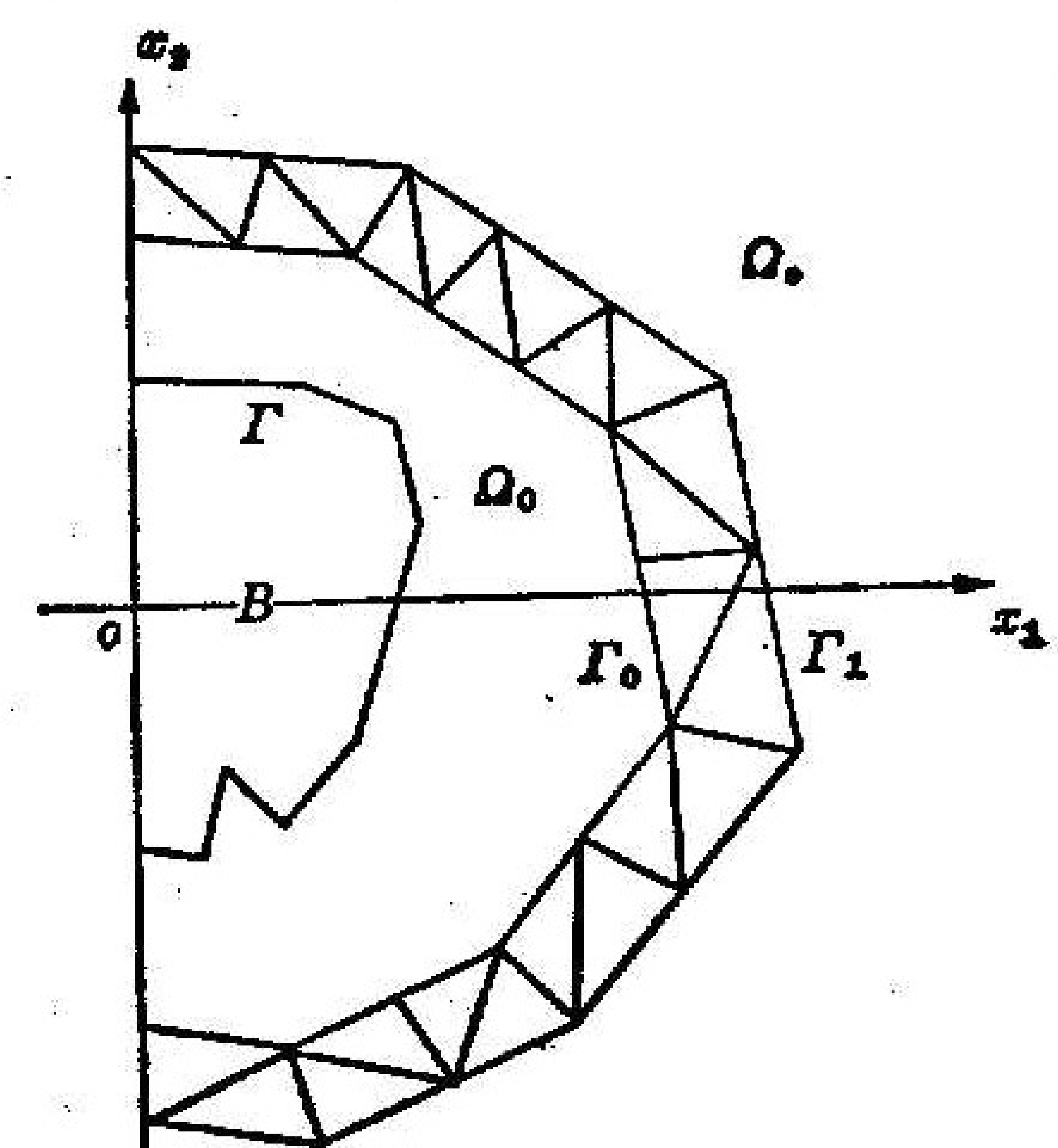

Fig. 1

$$\nu(-\nabla(x_1\nabla u_1)/x_1+u_1/x_1^2)+\partial p/\partial x_1=0, \quad x\in\Omega,$$

$$-\nu\nabla(x_1\nabla u_2)/x_1+\partial p/\partial x_2=0, \quad x\in\Omega,$$

$$\frac{\partial}{\partial x_1}(x_1u_1)+\frac{\partial}{\partial x_2}(x_1u_2)=0, \quad x\in\Omega,$$

$$u=u_*(x), \quad x\in\Gamma,$$

$$u_1=0, \quad x\in\partial\Omega\cap\{x_1=0\},$$

$$u=0,\ p=0, \quad |x|=\infty;$$

where ν is a positive constant and $u_*(x)$ is a known function.

We define some weighted Sobolev spaces for the above problem. There is no harm in assuming $|x|\geqslant\delta>0$ for every point x in Ω. The following semi-norm and

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norm

$$|f|_{m,\beta,\Omega}=\Big(\sum_{|\alpha|=m}\int_\Omega x_1|x|^{2(m-\beta)}|D^\alpha f|^2\,dx\Big)^{1/2},$$

$$\|f\|_{m,\beta,\Omega}=\Big(\sum_{i=0}^{m}|f|^2_{i,\beta,\Omega}\Big)^{1/2}$$

are defined and the corresponding Hilbert spaces are denoted by $Z^{m,\beta}(\Omega)$. We also define the norms as

$$|f|_{1,\beta,*,\Omega}=(|f|^2_{1,\beta,\Omega}+\|f/x_1\|^2_{0,\beta-1,\Omega})^{1/2},$$

$$\|f\|_{1,\beta,*,\Omega}=(|f|^2_{1,\beta,*,\Omega}+\|f\|^2_{0,\beta,\Omega})^{1/2}.$$

The corresponding Hilbert spaces are denoted by $Z^{1,\beta}_*(\Omega)$, and $Z^{2,\beta}_+(\Omega)$ is a set such that $f\in Z^{2,\beta}_+(\Omega)$ if and only if $f\in Z^{1,\beta}_*(\Omega)$ and $\|D^{(0,2)}f/x_1\|_{0,\beta-2,\Omega}$ is finite. The above definitions are equivalent to that in [1] when Ω is bounded.

Let $H(\Omega)=Z^{1,1}_*(\Omega)\times Z^{1,1}(\Omega)$, $H_0(\Omega)=\{u\in H(\Omega);\ u|_{\partial\Omega\setminus(x_1=0)}=0\}$. Consider the bilinear form

$$a(u,v)_\Omega=\nu\int_\Omega x_1(\nabla u_1\cdot\nabla v_1+\nabla u_2\cdot\nabla v_2+u_1v_1/x_1^2)dx,\quad u,v\in H(\Omega),\tag{1}$$

defined in $H(\Omega)\times H(\Omega)$, and the bilinear form

$$b(v,p)_\Omega=-\int_\Omega p\Big\{\frac{\partial}{\partial x_1}(x_1v_1)+\frac{\partial}{\partial x_2}(x_1v_2)\Big\}dx,\quad v\in H(\Omega),\ p\in Z^{0,0}(\Omega),\tag{2}$$

defined in $H(\Omega)\times Z^{0,0}(\Omega)$. The definitions for bilinear forms with respect to other domains are similar. Let $H(\Gamma)$ be the trace space of $H(\Omega)$ on Γ; then the weak formulation for the original problem is: to find $(u,p)\in H(\Omega)\times Z^{0,0}(\Omega)$, such that

$$a(u,v)_\Omega+b(v,p)_\Omega=0,\quad \forall v\in H_0(\Omega),\tag{3}$$

$$b(u,q)_\Omega=0,\quad \forall q\in Z^{0,0}(\Omega),\tag{4}$$

$$u|_\Gamma=u_*,\tag{5}$$

where $u_*\in H(\Gamma)$. The solution of this problem exists and is unique.

Let us consider the infinite element approximation to problem (3)—(5). We construct a broken line Γ_0: $r=r_0(\theta)$, $|\theta|<\frac{\pi}{2}$, which divides Ω into Ω_* and Ω_0, where Ω_0 lies between Γ and Γ_0 and Ω_* is the exterior of Γ_0. We assume that Γ_0 is star-shaped with respect to the point 0, i.e. each ray from the point 0 intersects Γ_0 at most at one point. Especially, it may happen that $\Gamma_0=\Gamma$; then Ω_0 is empty.

Taking a constant $\xi>1$, we construct similar curves $\Gamma_1,\Gamma_2,\cdots,\Gamma_k,\cdots$ of Γ_0 with 0 as the center and $\xi,\xi^2,\cdots,\xi^k,\cdots$ as constants of proportionality. Let

$$\Omega_k=\Big\{(r,\theta);\ \xi^{k-1}r_0(\theta)<r<\xi^k r_0(\theta),\ |\theta|<\frac{\pi}{2}\Big\},$$

$$\Omega_{*,k}=\Big\{(r,\theta);\ r_0(\theta)<r<\xi^k r_0(\theta),\ |\theta|<\frac{\pi}{2}\Big\}.$$

Domain Ω is triangulated in such a way that $\Omega_0,\Omega_1,\Omega_2,\cdots$ consist exactly of finite triangular elements, and the triangulation of $\Omega_1,\Omega_2,\cdots,\Omega_k,\cdots$ is geometrically similar. In each element, second order interpolation is used for u and p is constant, just as in [1]. For definiteness, we assume that each subdomain Ω_k is divided into some quadrilaterals by the rays from point 0; then each quadrilateral is further divided into two triangles.

To get the infinite element approximation to (3)—(5), we eliminate variable p in equation (3). Let

$$V(\Omega)=\{u\in H(\Omega);\ b(u,\ q)_\Omega=0,\quad \forall q\in Z^{0,0}(\Omega)\},$$
$$V_0(\Omega)=V(\Omega)\cap H_0(\Omega).$$

Then the solution u of (3)—(5) satisfies: $u\in V(\Omega)$,

$$a(u,\ v)=0,\quad \forall v\in V_0(\Omega), \tag{6}$$

and (5) holds.

Corresponding to the above triangulation, let the subspaces of $H(\Omega)$, $Z^{0,0}(\Omega)$ be $H_h(\Omega)$ and $P_h(\Omega)$ respectively. Let

$$V_h(\Omega)=\{u\in H_h(\Omega);\ b(u,\ q)_\Omega=0,\quad \forall q\in P_h(\Omega)\},$$
$$V_{0h}(\Omega)=V_h(\Omega)\cap H_0(\Omega).$$

Then the infinite element approximation of (6) is: $u_h\in V_h(\Omega)$, and

$$a(u_h,\ v)=0,\quad \forall v\in V_{0h}(\Omega), \tag{7}$$
$$u_h|_\Gamma=u_*. \tag{8}$$

Of course, u_* is piecewise quadratic here.

To meet the need of the following discussion, we should extend formulation (7), (8). Let Ω_h be any subdomain of Ω which consists exactly of some elements, finite or infinte, in accordance with the above triangulation. We may define spaces $H(\Omega_h)$, $H_0(\Omega_h)$, $Z^{0,0}(\Omega_h)$ and their subspaces $H_h(\Omega_h)$, $H_{0h}(\Omega_h)$, $P_h(\Omega_h)$ and so on in a same way. And

$$V_h(\Omega_h)=\{u\in H_h(\Omega_h);\ b(u,\ q)_{\Omega_h}=0,\quad \forall q\in P_h(\Omega_h)\},$$
$$V_{0h}(\Omega_h)=V_h(\Omega_h)\cap H_{0h}(\Omega_h).$$

We denote by $W(\Omega_h)$ the following problem: to find $u_h\in V_h(\Omega_h)$, such that

$$a(u_h,\ v)=0,\quad \forall v\in V_{0h}(\Omega_h),$$
$$u_h|_{\partial\Omega_h\setminus(x_1=0)}=u_*.$$

It is obvious that (7), (8) is just problem $W(\Omega)$.

The solutions of problems $W(\Omega)$, $W(\Omega_*)$ exist and are unique, because we have

Lemma 1. *$a(u,\ v)_{\Omega_h}$ is symmetric and positive definite on $H_0(\Omega_h)$.*

Proof. By (1),

$$a(u,\ u)_{\Omega_h}=\nu(|u_1|^2_{1,1,*,\Omega_h}+|u_2|^2_{1,1,\Omega_h}).$$

Using an inequality of Poincaré–Friedrichs type[6] we obtain

$$a(u,\ u)_{\Omega_h}\geqslant C^{-1}\|u\|^2_{H(\Omega_h)}.$$

Here and after C is always a certain constant not necessarily the same. Q.E.D.

Now we consider the solution of problem $W(\Omega)$. The values $(u_1,\ u_2)$ on the nodes of Γ_k are arranged to be a column vector z_k in an anti-clockwise direction. Since $u_1=0$ at the x_2-axis, these two values are excluded. Therefore if there are N nodes on Γ_k, then z_k is a $2N-2$ dimensional vector. z_1 is uniquely determined by z_0

due to the existence and uniqueness of the solution of problem $W(\Omega_*)$. Hence there is a real matrix X such that

$$z_1 = X z_0;$$

by similarity

$$z_k = X z_{k-1}, \quad k=1, 2, \cdots.$$

Therefore

$$z_k = X^k z_0. \tag{9}$$

As a result of (9), to solve problem $W(\Omega)$, it suffices to calculate only the values of u_h on the nodes of Ω_0.

Now we define the combined stiffness matrix K_s. Let u_h be the solution of $W(\Omega_*)$ with a boundary value z_0. Because the dependence of u_h on z_0 is linear, $a(u_h, u_h)_{\Omega_*}$ can be expressed as a quadratic form $z_0^T K_s z_0$. By (1), K_s is a symmetric positive definite matrix.

Regarding the condition $b(u_h, q)=0$, $\forall q \in P_h(\Omega_0)$, as a restriction in problem $W(\Omega)$, and introducing a Lagrangian multiplier, we get a new formulation of problem $W(\Omega)$ as: to find $(u_h, p_h) \in H_h(\Omega_0) \times P_h(\Omega_0)$, such that

$$a(u_h, v)_{\Omega_0} + z_0^T K_s \bar{z}_0 + b(v, p_h)_{\Omega_0} = 0, \quad \forall v \in H_h(\Omega_0),\ v|_\Gamma = 0, \tag{10}$$

$$b(u_h, q)_{\Omega_0} = 0, \quad \forall q \in P_h(\Omega_0), \tag{11}$$

$$u_h|_\Gamma = u_*,$$

where $\bar{z}_0$ is the value of v on the nodes of Γ_0.

We will prove in § 3 that (u_h, p_h) is an approximation of (u, p) in some sense. Now we only prove the following

Theorem 1. *The solution of problem* (10), (11), (8) *exists and is unique in the sense that p_h may differ within a constant.*

Proof. It suffices to prove that the corresponding homogeneous problem only possesses null solution if an additional restriction $\int_{\Omega_0} x_1 p_h\, dx = \int_\Omega x_1 q\, dx = 0$ is assumed. Let u_h, p_h be the solution of such a homogeneous problem. Taking $q = p_h$ in (11), we obtain

$$b(u_h, p_h) = 0.$$

Then taking $v = u_h$ in (10), we obtain

$$a(u_h, u_h)_{\Omega_0} + z_0^T K_s z_0 = 0.$$

K_s is positive definite, so $z_0 = 0$; hence $u_h = 0$. Afterwards, we take such a v that it vanishes at every node except one middle point of a side s of an element. Then by (2) and Green's formula we know that p_h is the same in the neighboring elements of s. But s is arbitrary, so p_h is a constant, and we obtain $p_h = 0$ from the restriction. Q.E.D.

Therefore, to get an approximate solution of problem (3)—(5), the important matter is to obtain matrices X and K_s.

§ 2. The Calculation of K_s and X

We consider problem $W(\Omega_1)$, if

$$\int_{\partial\Omega_1} a_1 u_* \cdot n\, dx = 0, \tag{12}$$

where n is a unit exterior normal vector, then we can take $u^{(0)} \in V_\lambda(\Omega_1)$ such that $u^{(0)}|_{\partial\Omega_1 \setminus (x_1=0)} = u_*$. Let $u - u^{(0)}$ be a new unknown. By the Lax-Milgram theorem, the existence and uniqueness of the solution of problem $W(\Omega_1)$ is proved.

Let us write down condition (12) again in the form of vectors. We drop the last component of z_k; the obtained $2N-3$ dimensional vector is denoted by y_k. Then equation (12) can be understood as a formula where z_1 is uniquely determined by z_0 and y_1, i.e.

$$z_1 = B_1 y_1 + B_1' z_0, \tag{13}$$

where

$$B_1 = \begin{pmatrix} I \\ * \cdots * \end{pmatrix}, \quad B_1' = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ * \end{pmatrix} (* \cdots *),$$

I is the unit matrix.

Let u_h be the solution of problem $W(\Omega_1)$. Then the dependence of u_h on z_0, z_1 is linear. Using (13) we may write

$$a(u_h, u_h)_{\Omega_1} = (z_0^T, y_1^T) \begin{pmatrix} K_1 & -A_1^T \\ -A_1 & K_1' \end{pmatrix} \begin{pmatrix} z_0 \\ y_1 \end{pmatrix},$$

where K_1, K_1' are symmetric positive definite matrices. By similarity, let u_h be the solution of problem $W(\Omega_k)$ for $k=1, 2, \cdots$; then

$$a(u_h, u_h)_{\Omega_k} = \xi^{k-1} (z_{k-1}^T, y_k^T) \begin{pmatrix} K_1 & -A_1^T \\ -A_1 & K_1' \end{pmatrix} \begin{pmatrix} z_{k-1} \\ y_k \end{pmatrix}. \tag{14}$$

It is necessary to restrict the boundary condition for domain $\Omega_{*,k}$ as

$$\int_{\partial\Omega_{*,k}} a_1 u_* \cdot n\, dx = 0.$$

Corresponding to condition (13) we have

$$z_k = B_k y_k + B_k' z_0. \tag{15}$$

Let u_h be the solution of problem $W(\Omega_{*,k})$. Then we have

$$a(u_h, u_h)_{\Omega_{*,k}} = (z_0^T, y_k^T) \begin{pmatrix} K_k & -A_k^T \\ -A_k & K_k' \end{pmatrix} \begin{pmatrix} z_0 \\ y_k \end{pmatrix}. \tag{16}$$

K_k is an approximation of K_* as k is large enough. To prove this, we prove

Lemma 2. *Each eigenvalue λ of matrix X satisfies inequality $|\lambda| < \xi^{-1/2}$.*

Proof. Let λ be an eigenvalue, and g the corresponding eigenvector. Taking g as boundary value z_0 and solving problem $W(\Omega_*)$, we obtain solution u_h. By (9)

$$z_k = X^k z_0 = \lambda^k g.$$

By similarity

$$a(u_h, u_h)_{\Omega_*} = \sum_{k=1}^{\infty} a(u_h, u_h)_{\Omega_k} = \sum_{k=1}^{\infty} \xi^{k-1} \lambda^{2(k-1)} a(u_h, u_h)_{\Omega_1}.$$

But $a(u_h, u_h)_{\Omega_*} < +\infty$, hence $|\lambda| < \xi^{-1/2}$. Q.E.D.

Theorem 2.

$$\lim_{k\to+\infty} K_k = K_*.$$

Proof. For the time being, we denote by u the solution of problem $W(\Omega_*)$ with boundary value z_0, by $u^{(k)}$ the solution of problem $W(\Omega_{*,k})$ with boundary value z_0 and $y_k=0$. It is easy to see from (15) that

$$|z_k| \leqslant C\xi^{-2k}. \tag{17}$$

We consider problem $W(\Omega_*\backslash\overline{\Omega}_{*,k})$ with boundary value z_k. Let $w^{(k)}$ be the solution. By similarity

$$a(w^{(k)}, w^{(k)})_{\Omega_*\backslash\overline{\Omega}_{*,k}} = \xi^k z_k^T K_* z_k.$$

By (17)

$$a(w^{(k)}, w^{(k)})_{\Omega_*\backslash\overline{\Omega}_{*,k}} \leqslant C\xi^{-3k}.$$

Let

$$\widetilde{w}^{(k)} = \begin{cases} u^{(k)}, & \Omega_{*,k}, \\ w^{(k)}, & \Omega_*\backslash\overline{\Omega}_{*,k}. \end{cases}$$

Owing to the minimum property of quadratic functional,

$$a(u, u)_{\Omega_*} \leqslant a(\widetilde{w}^{(k)}, \widetilde{w}^{(k)})_{\Omega_*} \leqslant a(u^{(k)}, u^{(k)})_{\Omega_{*,k}} + C\xi^{-3k}. \tag{18}$$

On the other hand, let

$$\overline{w}^{(k)} = \begin{cases} u, & \Omega_{*,k-1}, \\ w_1^{(k)}, & \Omega_k, \\ w^{(k)}, & \Omega_*\backslash\overline{\Omega}_{*,k}, \end{cases}$$

where $w_1^{(k)}$ is a solution of problem $W(\Omega_k)$ with a suitable boundary value such that $\overline{w}^{(k)} \in H_h(\Omega_*)$. Also owing to the minimum property of quadratic functional,

$$\begin{aligned} a(u^{(k)}, u^{(k)})_{\Omega_{*,k}} &\leqslant a(\overline{w}^{(k)}, \overline{w}^{(k)})_{\Omega_{*,k}} \\ &= a(u, u)_{\Omega_{*,k-1}} + a(w_1^{(k)}, w_1^{(k)})_{\Omega_k} \\ &= a(u, u)_{\Omega_*} - a(u, u)_{\Omega_*\backslash\overline{\Omega}_{*,k-1}} + a(w_1^{(k)}, w_1^{(k)})_{\Omega_k}. \end{aligned} \tag{19}$$

By Lemma 2,

$$|X^k z_0| \leqslant C\eta^k, \quad \eta < \xi^{-\frac{1}{2}};$$

hence

$$a(u, u)_{\Omega_*\backslash\overline{\Omega}_{*,k-1}} \leqslant C\xi^k\eta^{2k} \to 0, \quad k\to\infty.$$

By the same reason

$$a(w_1^{(k)}, w_1^{(k)})_{\Omega_k} \to 0, \quad k\to\infty.$$

By (18), (19) we obtain

$$\lim_{k\to\infty} a(u^{(k)}, u^{(k)})_{\Omega_{*,k}} = a(u, u)_{\Omega_*}. \tag{20}$$

We rewrite it in matrix form, and the conclusion of this theorem follows. Q.E.D.

Using Theorem 2 we have already got an iterative scheme for calculating K_*. In fact, if K_k, K_k', A_k are known, then

$$\begin{aligned} &(z_0^T, y_{2k}^T)\begin{pmatrix} K_{2k} & -A_{2k}^T \\ -A_{2k} & K_{2k}' \end{pmatrix}\begin{pmatrix} z_0 \\ y_{2k} \end{pmatrix} \\ &= \min_{y_k}\left\{(z_0^T, y_k^T)\begin{pmatrix} K_k & -A_k^T \\ -A_k & K_k' \end{pmatrix}\begin{pmatrix} z_0 \\ y_k \end{pmatrix} + \xi^k(z_k^T, y_{2k}^T)\begin{pmatrix} K_k & -A_k^T \\ -A_k & K_k' \end{pmatrix}\begin{pmatrix} z_k \\ y_{2k} \end{pmatrix}\right\}, \end{aligned}$$

where z_k satisfies (15). After some calculation we obtain

$$K_{2k}=K_k-2C_k^TA_k+C_k^TK_k'C_k+\xi^k(B_k'^T+C_k^TB_k^T)K_k(B_k'+B_kC_k),$$
$$K_{2k}'=D_k^TK_k'D_k+\xi^k\{D_k^TB_k^TK_kB_kD_k-2A_kB_kD_k+K_k'\},$$
$$A_{2k}=D_k^TA_k-D_k^TK_k'C_k+\xi^k(A_k-D_k^TB_k^TK_k)(B_k'+B_kC_k),$$

where

$$C_k=(\xi^kB_k^TK_kB_k+K_k')^{-1}(A_k-\xi^kB_k^TK_kB_k'),$$
$$D_k=\xi^k(\xi^kB_k^TK_kB_k+K_k')^{-1}B_k^TA_k^T.$$

The rate of convergence of the above scheme is very fast, but this scheme is not stable. We give a stable iterative scheme[5] as follows:

Lemma 3. *Let u_h be the solution of problem $W(\Omega_*)$. Then the problem: to find $w^{(k)}\in V_h(\Omega_{*,k})$, such that $w^{(k)}|_{\Gamma_*}=u_h|_{\Gamma_*}$, and*

$$a(w^{(k)},v)=0,\quad \forall v\in V_h(\Omega_{*,k}),\ v|_{\Gamma_*}=0 \tag{21}$$

has a unique solution; moreover

$$a(u_h,u_h)_{\Omega_*}\geqslant a(w^{(k+1)},w^{(k+1)})_{\Omega_*,k+1}\geqslant a(w^{(k)},w^{(k)})_{\Omega_*,k}, \tag{22}$$
$$\lim_{k\to\infty}a(w^{(k)},w^{(k)})_{\Omega_*,k}=a(u_h,u_h)_{\Omega_*}. \tag{23}$$

Proof. Thanks to the Lax-Milgram theorem, the solution of problem (21) exists. And owing to the minimum property of quadratic functional,

$$a(w^{(k)},w^{(k)})_{\Omega_*,k}\leqslant a(w^{(k+1)},w^{(k+1)})_{\Omega_*,k}$$
$$\leqslant a(w^{(k+1)},w^{(k+1)})_{\Omega_*,k+1}\leqslant a(u_h,u_h)_{\Omega_*,k+1}\leqslant a(u_h,u_h)_{\Omega_*}.$$

We take a natural number l. As $k\geqslant l$,

$$a(w^{(k)},w^{(k)})_{\Omega_*,l}\leqslant a(w^{(k)},w^{(k)})_{\Omega_*,k}.$$

Hence $a(w^{(k)},w^{(k)})_{\Omega_*,l}$ are bounded uniformly. But $V_h(\Omega_{*,l})$ is a finite dimensional space. We can take a converging subseries, and then take a diagonal subseries with respect to l, still denoted by $\{w^{(k)}\}$. Let w be its limit; then

$$a(w,w)_{\Omega_*,l}\leqslant a(u_h,u_h)_{\Omega_*}.$$

Let $l\to\infty$; we obtain

$$a(w,w)_{\Omega_*}\leqslant a(u_h,u_h)_{\Omega_*}.$$

But u_h is the unique minimum point; hence $w=u_h$. Because the limit is unique, the original series $\{w^{(k)}\}$ converges to u_h, i.e.

$$\lim_{k\to\infty}a(w^{(k)},w^{(k)})_{\Omega_*,l}=a(u_h,u_h)_{\Omega_*,l}.$$

Then (23) follows. Q.E.D.

Theorem 3. *Let $K_z^{(0)}$ be a $2N-2$ order symmetric non-negative matrix; then the problem: to find $v^{(k)}\in V_h(\Omega_{*,k})$, such that*

$$a(v^{(k)},v)_{\Omega_*,k}+\xi^kz_k^TK_z^{(0)}\bar{z}_k=0,\quad \forall v\in V_h(\Omega_{*,k}),\ v|_{\Gamma_*}=0,$$
$$v^{(k)}|_{\Gamma_*}=u_h|_{\Gamma_*}$$

has a unique solution, and

$$\lim_{k\to\infty}a(v^{(k)},v^{(k)})_{\Omega_*,k}+\xi^kz_k^TK_z^{(0)}z_k=a(u_h,u_h)_{\Omega_*},$$

where z_k and $\bar{z}_k$ are the values of $v^{(k)}$ and v on the nodes of Γ_k respectively.

Proof. The existence and uniqueness of $v^{(k)}$ are obvious. We denote

$$a_0(v^{(k)}, v)=z_k^T K_s^{(0)}\bar{z}_k,$$

and define $u^{(k)}$ as in Theorem 2. Then owing to the minimum property of quadratic functional,

$$a(u^{(k)}, u^{(k)})_{\Omega_{s,k}}+\xi^k a_0(u^{(k)}, u^{(k)}) \geqslant a(v^{(k)}, v^{(k)})_{\Omega_{s,k}}+\xi^k a_0(v^{(k)}, v^{(k)}). \tag{24}$$

By (17)

$$\xi^k a_0(u^{(k)}, u^{(k)})\leqslant C\xi^{-8k}. \tag{25}$$

Define $w^{(k)}$ as in Lemma 3. By the same reason

$$a(w^{(k)}, w^{(k)})_{\Omega_{s,k}}\leqslant a(v^{(k)}, v^{(k)})_{\Omega_{s,k}}\leqslant a(v^{(k)}, v^{(k)})_{\Omega_{s,k}}+\xi^k a_0(v^{(k)}, v^{(k)}). \tag{26}$$

The conclusion of this theorem follows from (20), (23)—(26). Q.E.D.

Using Theorem 3, we can also get an iterative scheme for calculating K_s. Firstly, take any symmetric non-negative matrix $K_s^{(0)}$; secondly, iterate according to the following scheme:

$$z_0^T K_s^{(k+1)} z_0=\min_{y_1}\left\{(z_0^T, y_1^T)\begin{pmatrix} K_1 & -A_1^T \\ -A_1 & K_1' \end{pmatrix}\begin{pmatrix} z_0 \\ y_1 \end{pmatrix}+\xi z_1^T K_s^{(k)} z_1\right\},$$

where z_1 is determined by (13). After some calculation, we get

$$K_s^{(k+1)}=K_1-2C_k^T A_1+C_k^T K_1' C_k+\xi(C_k^T B_1^T+B_1'^T)K_s^{(k)}(B_1C_k+B_1'),$$

where

$$C_k=(K_1'+\xi B_1^T K_s^{(k)} B_1)^{-1}(A-\xi B_1^T K_s^{(k)} B_1').$$

To combine the high speed and stability of these two schemes, it is suggested that $K_s^{(0)}$ is obtained by Theorem 2 at first, and then revised by Theorem 3. The details are in [5].

We get from the above calculation that

$$y_1=C_k z_0.$$

Let $k\to\infty$; we obtain

$$y_1=(K_1'+\xi B_1^T K_s B_1)^{-1}(A-\xi B_1^T K_s B_1')z_0.$$

By (13)

$$z_1=\{B_1(K_1'+\xi B_1^T K_s B_1)^{-1}(A-\xi B_1^T K_s B_1')+B_1'\}z_0.$$

Therefore we obtain

$$X=B_1(K_1'+\xi B_1^T K_s B_1)^{-1}(A-\xi B_1^T K_s B_1')+B_1'.$$

§ 3. Convergence

To prove convergence, we make some more hypotheses on triangulation. Suppose $\max_\theta r_0(\theta)/\min_\theta r_0(\theta)\leqslant C$, and like [1], suppose all angles of all elements possess a positive lower bound.

We prove the following lemmas which are parallel to Lemmas 5—7 of [1]. We denote by s_i, $i=1, 2, 3$, the three sides of any element e, and by $x^{(i)}$, $i=1, \cdots, 6$, its six nodes; $x^{(i)}$ is a vertex as $i\leqslant 3$ and a middle point of one side as $i\geqslant 4$.

Lemma 4. *If $f\in Z^{2,1}(\Omega)$ and f_I is a polynomial of degree $\leqslant 2$ on element e, such* *that*

$$f_I(x^{(i)})=f(x^{(i)}),\quad i=1,2,3,$$

$$\int_{s_i} x_1(f-f_I)dx=0,\quad \text{as } s_i\cap\{x_1=0\}=\emptyset,$$

$$f_I(x^{(j)})=f(x^{(j)}),\quad \text{as } s_i\subset\{x_1=0\},\ x^{(j)}\in s_i,$$

then

$$|f-f_I|_{m,1,\Omega}\leqslant Ch^{2-m}|f|_{2,1,\Omega},\quad m=0,1, \tag{27}$$

where h is the greatest length of all sides of elements in domains Ω_0 and Ω_1.

Proof. By Lemma 5 of [1], let h_e be the greatest length of sides of element e; then we have

$$|f-f_I|_{m,e}\leqslant Ch_e^{2-m}|f|_{2,e},\quad m=0,1.$$

Hence

$$|f-f_I|_{m,\Omega_e}\leqslant Ch^{2-m}|f|_{2,\Omega_e}. \tag{28}$$

Therefore to prove (27), it suffices to estimate $|f-f_I|_{m,1,\Omega_e}$. Noticing that under the above additional hypotheses on triangulation, $\max\limits_e|x|/\min\limits_e|x|\leqslant C$; we have

$$\min_e|x|^{2(m-1)}|f-f_I|^2_{m,e}\leqslant C\max_e|x|^{2(m-1)}h_e^{4-2m}|f|^2_{2,e}.$$

By the similarity of triangulation,

$$h_e\leqslant Ch\max_e|x|.$$

Hence

$$\min_e|x|^{2(m-1)}|f-f_I|^2_{m,e}\leqslant Ch^{4-2m}\max_e|x|^2\cdot|f|^2_{2,e}.$$

The summation over the elements leads to

$$|f-f_I|_{m,1,\Omega_e}\leqslant Ch^{2-m}|f|_{2,1,\Omega_e}.$$

Combining it with (28), we obtain (27). Q.E.D.

We can prove in the same way

Lemma 5. *If $f\in Z_+^{2,1}(\Omega)$, then*

$$\|(f-f_I)/x_1\|_{0,0,\Omega}\leqslant Ch(|f|_{2,1,\Omega}+h\|D^{(0,2)}f/x_1\|_{0,-2,\Omega}).$$

Lemma 6. *If $f\in Z^{1,0}(\Omega)$, then there exists a f_0, which is a constant on each element, such that*

$$\|f-f_0\|_{0,0,\Omega}\leqslant Ch|f|_{1,0,\Omega}.$$

To obtain convergence, we need one more lemma as follows.

Lemma 7. *The restriction of the solution of problem* (7), (8) *coincides with the solution u_h of problem* (10), (11), (8).

Proof. The solution u_h of (7), (8) satisfies

$$a(u_h,u_h)_\Omega=\min_{\substack{v\in V_h(\Omega)\\ v|_\Gamma=u_*}} a(v,v)_\Omega.$$

Take a $v\in V_h(\Omega_0)$ such that $v|_\Gamma=u_*$, and extend v to Ω_* such that v is a solution of problem $W(\Omega_*)$ and $v\in V_h(\Omega)$. This extension is unique. Denote by P the set of all these functions; then the solution of (7), (8), $u_h\in P$. Hence

$$a(u_h,u_h)_\Omega=\min_{v\in P}a(v,v)_\Omega.$$

But as $v\in P$

$$a(v, v)_{\Omega}=a(v, v)_{\Omega_0}+\bar{z}_0^T K_s \bar{z}_0,$$

where $\bar{z}_0$ is the value of v on the nodes of Γ_0, therefore u_h in Ω_0 is the solution of the following problem: $u_h \in V_h(\Omega_0)$,

$$a(u_h, u_h)_{\Omega_0}+z_0^T K_s z_0=\min_{v\in V_h(\Omega_0)}(a(v, v)_{\Omega_0}+\bar{z}_0^T K_s \bar{z}_0).$$

We introduce a Lagrangian multiplier with respect to restriction $b(u_h, q)=0$, $\forall q\in P_h(\Omega_0)$; then we get (10), (11), (8). Q.E.D.

Having the above preparation, we finally obtain an error estimate for the infinite element method.

Theorem 4. *Let u, p be the solution of problem* (3)—(5), *u_h, p_h be the solution of problems* (7), (8) *and* (10), (11), (8), *and $u_1\in Z_+^{2,1}(\Omega)$, $u_2\in Z^{2,1}(\Omega)$, $p\in Z^{1,0}(\Omega)$. Then*

$$\|u-u_h\|_{H(\Omega)}+\|p-p_h+\beta\|_{0,\Omega}$$
$$\leqslant Ch(|u_1|_{2,1,\Omega}+h\|D^{(0,2)}u_1/x_1\|_{0,-2,\Omega}+|u_2|_{2,1,\Omega}+|p|_{1,0,\Omega}),$$

where β is a certain constant.

The proof of Theorem 4 is similar to that of Theorem 2 in [1], and so is omitted here. β appears here because p_h may differ within a constant.

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