ON COLLOCATION METHODS FOR SOLVING THE NEUTRON TRANSPORT EQUATION IN TWO-DIMENSIONAL PROBLEMS*

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Abstract

Collocation methods are considered for solving the time-dependent neutron transport equation in two-dimensional planar geometry. Error estimates and stability are derived. Finally, some numerical results are presented.

Introduction

The neutron transport equation is an integral-differential equation in which the differential part is of hyperbolic type. In solving a neutron transport equation[1], can be as simple and convenient as the DSN method[2], and the logical construction of the program is generally the same except that a lower degree linear algebraic system must be solved on each mesh. In fact, the DSN method is a special type of collocation method. It is a weighted residual method and is equivalent to the discrete Galerkin method. The collocation methods have higher accuracy and faster convergence and requires less operating time to attain the same accuracy than the DSN method.

Many authors have done works of value in using collocation methods to solve partial differential equations, for example[3], [4].

In this paper, we will use the collocation method to solve the time-dependent neutron transport equation in the two-dimensional x, y-plane geometry. Here, the Crank-Nicholson central difference is used to approximate the time variable, and the discrete ordinates approximation is used for the angular variables. An outline of the paper is as follows: the calculation method is given in Section 1. In Sections 2—4, error estimates and stability are derived. In Section 5, we discuss conservation of the method and some relations, such as its comparison with the difference method and the discrete Galerkin method. Finally, in order to explain the effectiveness of the methods, some numerical results are presented.

§ 1. Numerical Method

For the sake of simplicity we consider the initial-boundary value problem for the one-group neutron transport equation as follows:

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$$\begin{cases} A(\Phi)\equiv\frac{1}{v^*}\frac{\partial\Phi}{\partial t}+\mu\frac{\partial\Phi}{\partial x}+\nu\frac{\partial\Phi}{\partial y}+\alpha\Phi=Q(\Phi)+F, \\ \Phi(t,x,y,\mu,\nu)|_{t=0}=\Phi_0(x,y,\mu,\nu), \\ \Phi(t,x,y,\mu,\nu)|_{\Gamma}=0 \quad \text{for} \quad \boldsymbol{\Omega}\cdot\boldsymbol{h}\leqslant 0, \\ \Phi(t,x,y,\mu,\nu)|_{x=0}=\Phi(t,x,y,-\mu,\nu)|_{x=0}, \\ \Phi(t,x,y,\mu,\nu)|_{y=0}=\Phi(t,x,y,\mu,-\nu)|_{y=0}, \end{cases} \tag{1.1}$$

where the function $\Phi(t, x, y, \mu, \nu)$ represents the angular flux of neutrons at the point (t, x, y) and the angular direction $\boldsymbol{\Omega}=(\mu, \nu)$, where

$$\mu=\sin\theta\cos\psi,$$
$$\nu=\sin\theta\sin\psi,$$
$$\xi=\cos\theta.$$

Thus the neutron velocity $\boldsymbol{v}^*=\boldsymbol{v}^*\boldsymbol{\Omega}$ (see Fig. 1). Here, α, β are some nuclear data, and are supposed to be block constants, satisfying

$$0<\alpha_0\leqslant\alpha\leqslant\alpha_1, \quad 0<\beta_0\leqslant\beta\leqslant\beta_1.$$

Denote by B the region in which to solve equation (1.1)

$$B=D_t\times D\times D_{\mu\nu},$$

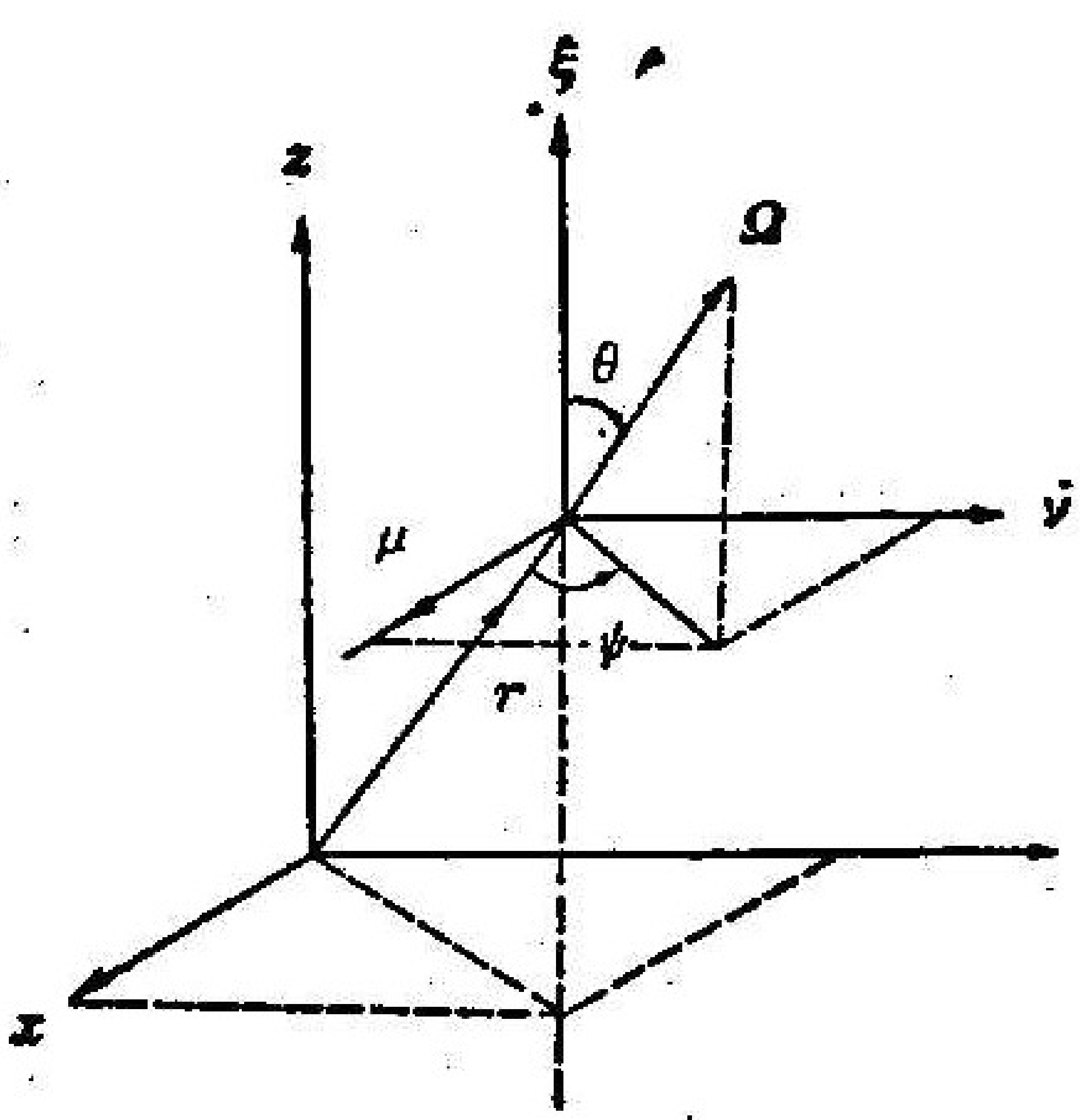

Fig. 1

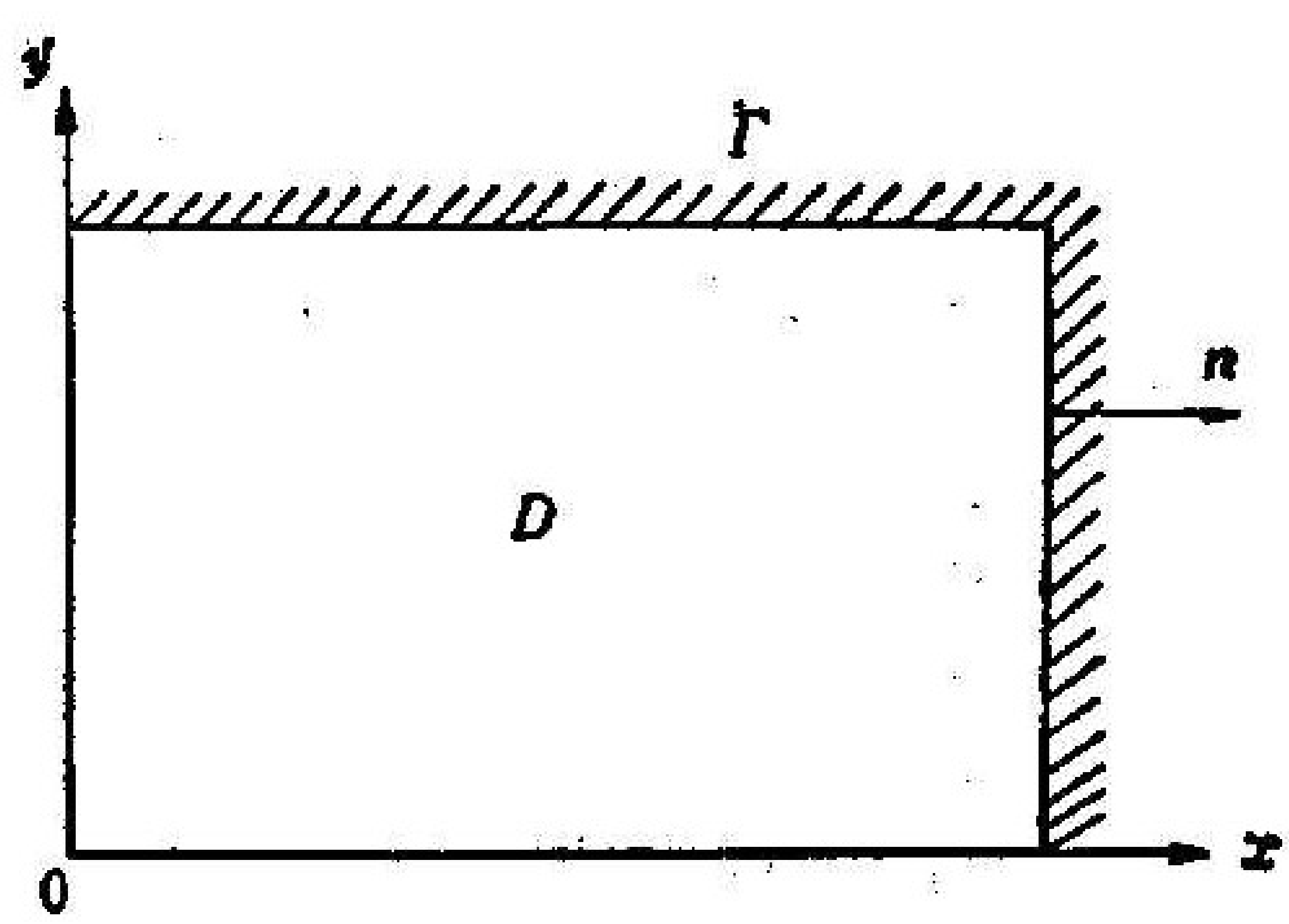

Fig. 2

where D_t: $[0, T]$, $D=D_x\times D_y$, D_x: $[0, X]$, D_y: $[0, Y]$, $D_{\mu\nu}$ is the unit disk in the (μ, ν)-plane: $\mu^2+\nu^2\leqslant 1$. Γ is the boundary of D. Denote by $\boldsymbol{n}$ the unit vector in the direction of outward normal to Γ (see Fig. 2).

$$Q(\Phi)=\beta\int_{\Omega}\Phi\, d\Omega.$$

An outside source term is denoted by $F(t, x, y, \mu, \nu)$. Suppose that Φ_0 is continuous on $D\times D_{\mu\nu}$.

We divide the spatial variables and time into

$$0=x_0<x_1<\cdots<x_I=X;$$
$$0=y_0<y_1<\cdots<y_J=Y;$$
$$0=t_0<t_1<\cdots<t_N=T.$$

Let

$$d_t\Phi^{n+\frac{1}{2}}=\frac{\Phi^{n+1}-\Phi^n}{\Delta t^{n+\frac{1}{2}}},\quad \Phi^{n+\frac{1}{2}}=\frac{1}{2}(\Phi^n+\Phi^{n+1}), \tag{1.2}$$

where $\Phi^n=\Phi(t_n, x, y, \mu, \nu)$. Now, the discrete ordinate approximation is used for angular variables. For example, we can divide the unit circle on (μ, ν)-plane into some blocks of the same area, with ordinates (μ_{ms}, ν_{ms}), $m=1, \cdots, M_s$; $s=1, \cdots, S$. Therefore (1.1) can be written as

$$A_{ms}(\Phi_{ms})\equiv\frac{1}{v^*}d_t\Phi_{ms}^{n+\frac{1}{2}}+\mu_{ms}\frac{\partial\Phi_{ms}^{n+\frac{1}{2}}}{\partial x}+\nu_{ms}\frac{\partial\Phi_{ms}^{n+\frac{1}{2}}}{\partial y}+\alpha\Phi_{ms}^{n+\frac{1}{2}}=Q_{ms}^{n+\frac{1}{2}}+F_{ms}^{n+\frac{1}{2}}, \tag{1.3}$$

where

$$Q_{ms}(t, x, y)=\beta\sum_{s'=1}^{S}\sum_{m'=1}^{M_s}w_{m's'}\Phi_{m's'},$$

$$\Phi_{ms}=\Phi(t, x, y, \mu_{ms}, \nu_{ms}),$$

$$\sum_{ms}w_{ms}=1.$$

Let $\mathscr{T}_h$ denote the set of all spatial meshes for any $t\in D_t$,

$$\mathscr{T}_h=\{D_{ij}: x_{i-1}\leqslant x\leqslant x_i,\ y_{j-1}\leqslant y\leqslant y_j,\ i=1, \cdots, I,\ j=1, \cdots, J\}.$$

It is obvious that the mapping

$$\begin{cases} x=x_{i-1}+\frac{\Delta x_i}{2}(\xi+1), & |\xi|\leqslant1,\\ y=y_{j-1}+\frac{\Delta y_j}{2}(\eta+1), & |\eta|\leqslant1, \end{cases} \tag{1.4}$$

maps D_{ij} onto $\hat{D}$, where $\hat{D}$ is a unit square, $\Delta x_i=x_i-x_{i-1}$, and $\Delta y_j=y_j-y_{j-1}$. Take $h=\max_{ij}\{\Delta x_i, \Delta y_j\}$. Given that ξ_k, η_l are zeros of the Legendre polynomial of degree K and L respectively, we can take the intersections of the family of straight lines

$$\xi=\xi_k,\ \eta=\eta_l,\quad k=1, \cdots, K;\ l=1, \cdots, L$$

to be collocation points in which the corresponding coordinates are (x_{ik}, y_{jl}) on D_{ij}.

Let $P_{KL}(D_{ij})=P_K(D_{xi})\times P_L(D_{yj})$ represent the polynomial space on D_{ij}, where $P_K(D_{xi})$ and $P_L(D_{yj})$ denote the spaces of polynomials of degree K on D_{xi} and L on D_{yj} respectively.

We will not consider the properties of the functions on D_t, $D_{\mu\nu}$, which are always assumed to satisfy our requirement. Let

$$H=\{\varphi: \varphi\in C^{(0)}(D);\ \varphi\in P_{KL}(D_{ij}),\ \forall D_{ij}\in\mathscr{T}_h\}.$$

Our problem is to find a function φ in H such that

$$\left\{\frac{1}{v^*}d_t\varphi_{ms}+\mu_{ms}\frac{\partial\varphi_{ms}}{\partial x}+\nu_{ms}\frac{\partial\varphi_{ms}}{\partial y}+\alpha\varphi_{ms}-(Q_{ms}+F_{ms})\right\}_{x=x_{ik}, y=y_{jl}}=0, \tag{1.5}$$

$$\begin{cases} \varphi(t, x_{ik}, y_{jl}, \mu_{ms}, \nu_{ms})|_{t=0}=\Phi_0(x_{ik}, y_{jl}, \mu_{ms}, \nu_{ms}),\\ \varphi(t, x, y, \mu_{ms}, \nu_{ms})|_{\Gamma}=0, \quad \text{for } \boldsymbol{\Omega}\cdot\boldsymbol{n}<0,\\ \varphi(t, x, y_{jl}, \mu_{ms}, \nu_{ms})|_{x=0}=\varphi(t, x, y_{jl}, -\mu_{ms}, \nu_{ms})|_{x=0},\\ \varphi(t, x_{ik}, y, \mu_{ms}, \nu_{ms})|_{y=0}=\varphi(t, x_{ik}, y, \mu_{ms}, -\nu_{ms})|_{y=0}, \end{cases} \tag{1.6}$$

where the index $n+\frac{1}{2}$ is omitted.

Obviously, the number of undetermined coefficients of $\varphi^{n+\frac{1}{2}}$ is $IJ(K+1)(L+1)MS$, where $M=1+\frac{S}{2}$. The number of the collocation equations is $IJKLMS$, and other equations are just supplied by (1.6) and the continuous conditions.

Solution procedure: suppose that φ^n is given; find out $\varphi^{n+\frac{1}{2}}$; then determine φ^{n+1} by (1.2), $n=0, 1, 2, \cdots, N-1$. Let "•" represent collocation points, "×" represent points determined by continuous or symmetric conditions, and "△" represent points of extrapolation. Calculation begins from the mesh of sign "→" in Fig. 3. φ in source term takes the result of the last iteration. The calculating sequence can be determined by direction Ω, that is:

1. $\mu<0$, $\nu<0$ (see Fig. 3(1)), $s=1, 2, \cdots, \frac{1}{2}S$, $m=1, 2, \cdots, \frac{1}{2}M_s$.
2. $\mu>0$, $\nu<0$ (see Fig. 3(2)), $s=1, 2, \cdots, \frac{1}{2}S$, $m=\frac{1}{2}M_s+1, \cdots, M_s$.
3. $\mu<0$, $\nu>0$ (see Fig. 3(3)), $s=\frac{1}{2}S+1, \cdots, S$, $m=1, 2, \cdots, \frac{1}{2}M_s$.
4. $\mu>0$, $\nu>0$ (see Fig. 3(4)), $s=\frac{1}{2}S+1, \cdots, S$, $m=\frac{1}{2}M_s+1, \cdots, M_s$.

Thus we can use source iteration by direction in Fig. 3 to solve one mesh after another.

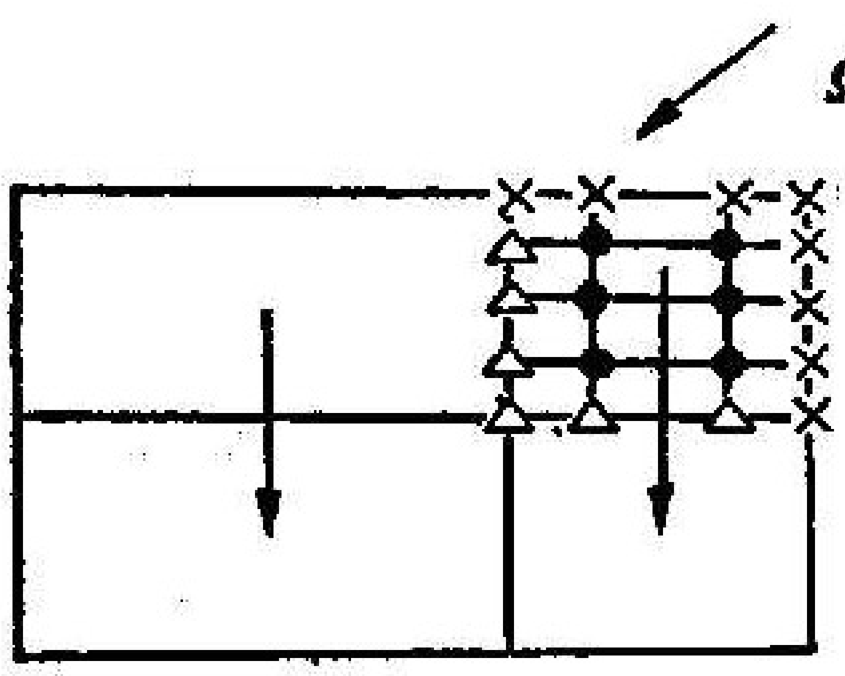

Fig. 3(1)

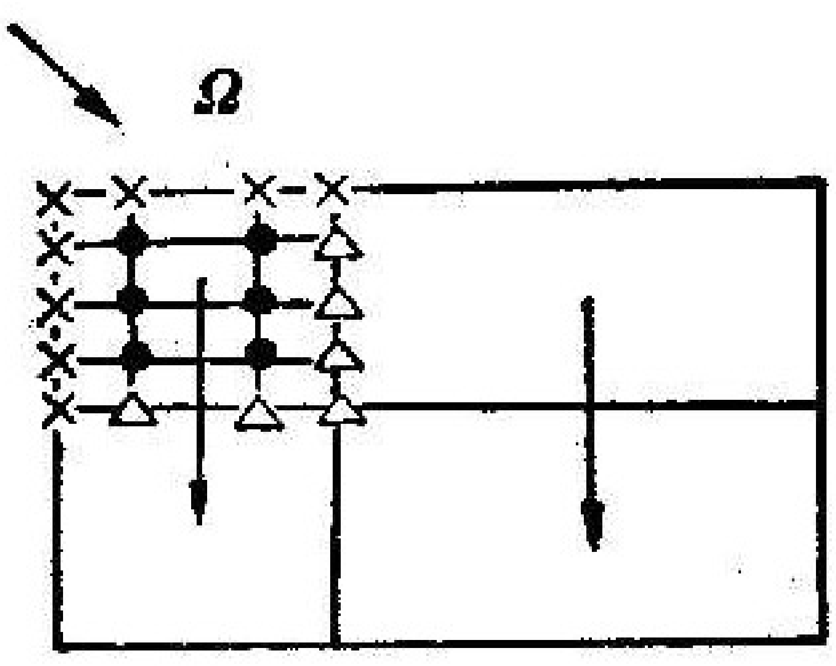

Fig. 3(2)

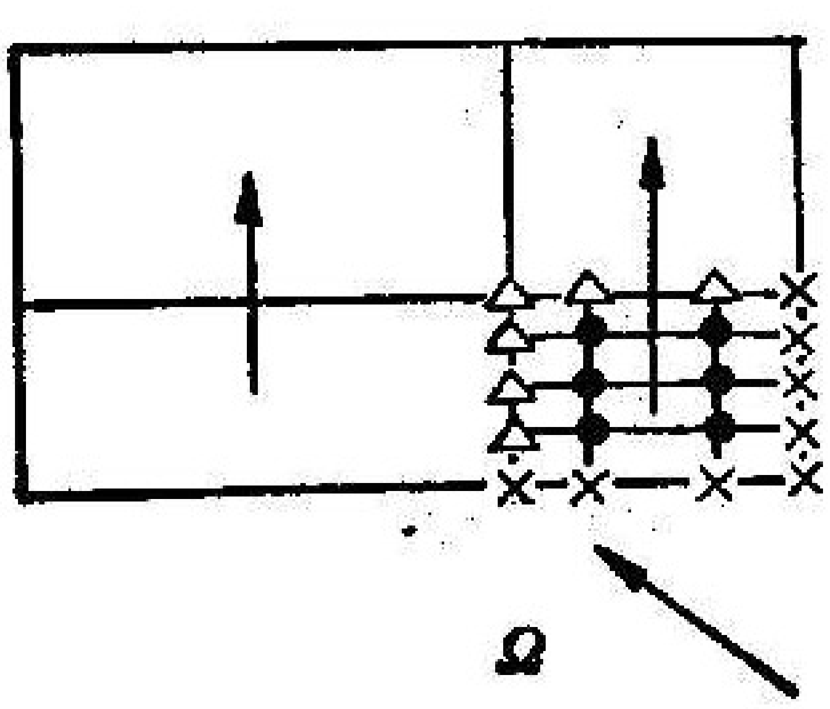

Fig. 3(3)

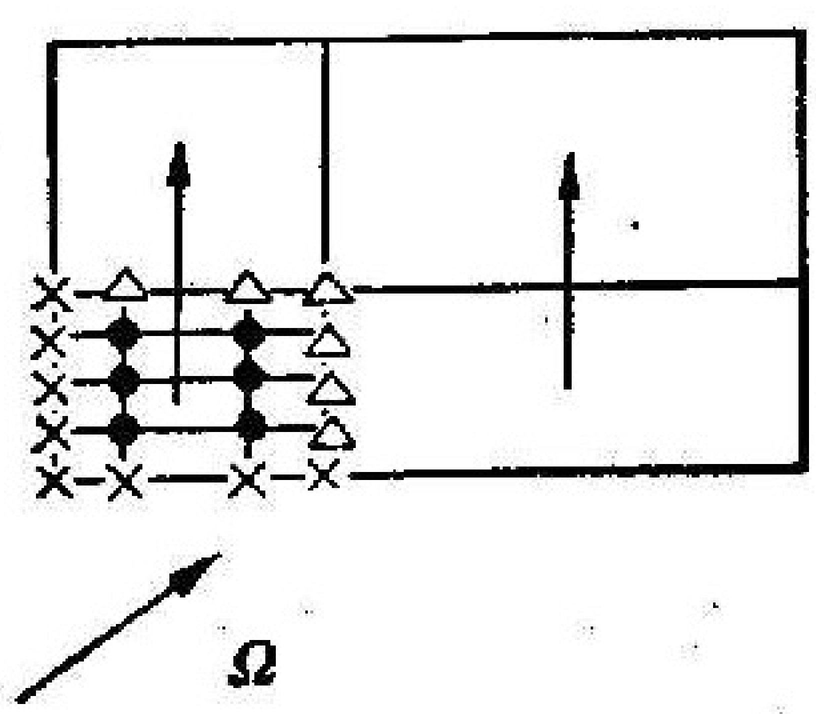

Fig. 3(4)

§ 2. Definition and Lemma

Denote $\frac{\partial^{k+l}v}{\partial x^k \partial y^l}$ by $v_{k,l}$, for $t\in D_t$ and $(\mu, \nu)\in D_{\mu,\nu}$. Define space $W_{K,L}(D)$ as

$$W_{K,L}(D)=\{v: v\in C^{(K,L)}(D_M)\}.$$

If $v\in L_\infty(C^{(0)}(D_t\times D_{\mu\nu})\times W_{K,L}(D))$, define

$$|v_{k,l}|L^\infty(B)=\underset{B}{\text{ess sup}}|v_{k,l}(t, x, y, \mu, \nu)|.$$

In order to simplify the writing we often omit t, μ, ν, for example $\hat{v}(\xi_k, \eta_l)=\hat{v}(t, \xi_k, \eta_l, \mu, \nu)$ (Denote by D_M the material region).

Define the discrete inner product as

$$\langle u, v\rangle=\sum_{n=0}^{N-1}\Delta t^{n+\frac{1}{2}}\langle u, v\rangle_{t^{n+\frac{1}{2}}},$$

where

$$\langle u, v\rangle_{t^{n+\frac{1}{2}}}=\sum_{ms}w_{ms}\langle u_{ms}, v_{ms}\rangle_D|_{t=t^{n+\frac{1}{2}}},$$

$$\langle u, v\rangle_D=\sum_{ij}\langle u, v\rangle_{D_{ij}},$$

$$\langle u, v\rangle_{D_{ij}}=\frac{\Delta x_i}{2}\frac{\Delta y_j}{2}\langle \hat{u}, \hat{v}\rangle_{\hat{D}},$$

$$\langle \hat{u}, \hat{v}\rangle_{\hat{D}}=\sum_{k=1}^{K}\sum_{l=1}^{L}w_k\overline{w}_l\hat{u}(\xi_k, \eta_l)\hat{v}(\xi_k, \eta_l),$$

where ξ_k, η_l; w_k, $\overline{w}_l$ are the Gauss-Legendre abscissae and weights respectively $1\leqslant k\leqslant K$, $1\leqslant l\leqslant L$.

For the sake of simplicity in the next section, we first prove a few estimation formulae in interpolation operator.

Let $$B_K(\xi)=\frac{1}{(2K)!}\frac{d^{k-1}}{d\xi^{k-1}}[(\xi+1)(\xi-1)]^k.$$

If σ_k are zero points of $B_K(\xi)$ in $|\xi|\leqslant 1$, then

$$-1=\sigma_0<\xi_1<\sigma_1<\cdots<\xi_k<\sigma_k=1.$$

Thus $$B_K(\xi)=\frac{1}{(K+1)!}\prod_{k=0}^{K}(\xi-\sigma_k).$$

Similarly, $$B_L(\eta)=\frac{1}{(L+1)!}\prod_{l=0}^{L}(\eta-\delta_l).$$

Define $\hat{\Pi}_\xi$ to be the interpolation operator from $C^{(0)}(\hat{D}_\xi)$ to $P_K(\hat{D}_\xi)$, such that

$$\hat{\Pi}_\xi\hat{v}(\xi, \eta)|_{\xi=\sigma_k}=\hat{v}(\sigma_k, \eta), \quad \forall\hat{v}\in C^{(0)}(\hat{D}_\xi),\ k=0, 1, \cdots, K.$$

Similarly define $\hat{\Pi}_\eta$ to be the interpolation operator from $C^{(0)}(\hat{D}_\eta)$ to $P_L(\hat{D}_\eta)$, such that

$$\hat{\Pi}_\eta\hat{v}(\xi, \eta)|_{\eta=\delta_l}=\hat{v}(\xi, \delta_l), \quad \forall\hat{v}\in C^{(0)}(\hat{D}_\eta),\ l=0, 1, \cdots, L.$$

And define $\hat{\Pi}$ to be the interpolation operator from $C^{(0)}(D)$ to $P_K(\hat{D}_\xi)\times P_L(\hat{D}_\eta)$, such that

$$\hat{\Pi}\hat{v}(\xi, \eta)|_{\xi=\sigma_k, \eta=\eta_l}=\hat{v}(\sigma_k, \delta_l), \quad \forall\hat{v}\in C^{(0)}(\hat{D}),\ k=0, 1, \cdots, K;\ l=0, 1, \cdots, L.$$

It will be seen that $\hat{\Pi}=\hat{\Pi}_\xi\otimes\hat{\Pi}_\eta=\hat{\Pi}_\eta\otimes\hat{\Pi}_\xi$.

If we denote the unit mapping as E, then

$$E-\hat{\Pi}=(E-\hat{\Pi}_\xi)\otimes E+E\otimes(E-\hat{\Pi}_\eta)-(E-\hat{\Pi}_\xi)\otimes(E-\hat{\Pi}_\eta). \tag{2.1}$$

Lemma. *Assume that* $\hat{v}\in C^{(K+2, L+2)}(\hat{D})$; *then*

$$\langle\hat{v}-\hat{\Pi}\hat{v}\rangle_{\hat{D}}\leqslant C(|\hat{v}_{K+1,0}|_{L^\infty(\hat{D})}+|\hat{v}_{0,L+1}|_{L^\infty(\hat{D})}+|\hat{v}_{K+1,L+1}|_{L^\infty(\hat{D})}), \tag{2.2}$$

$$\left\langle\frac{\partial}{\partial\xi}(\hat{v}-\hat{\Pi}\hat{v})\right\rangle_{\hat{D}}\leqslant C(|\hat{v}_{K+2,0}|_{L^\infty(\hat{D})}+|\hat{v}_{1,L+2}|_{L^\infty(\hat{D})}+|\hat{v}_{1,L+1}|_{L^\infty(\hat{D})}+|\hat{v}_{K+1,L+1}|_{L^\infty(\hat{D})}), \tag{2.3}$$

$$\left\langle \frac{\partial}{\partial \eta}(\hat{v}-\hat{\Pi}\hat{v})\right\rangle_{\hat{D}} \leqslant O(|\hat{v}_{0,L+2}|_{L^\infty(\hat{D})}+|\hat{v}_{K+2,1}|_{L^\infty(\hat{D})}+|\hat{v}_{K+1,1}|_{L^\infty(\hat{D})}+|\hat{v}_{K+1,L+1}|_{L^\infty(\hat{D})}). \tag{2.4}$$

Proof. Take $L_\xi\hat{v}=\hat{v}-\hat{\Pi}_\xi\hat{v}$. If $\hat{v}\in P_K(\hat{D}_\xi)$, then we have $\hat{v}-\hat{\Pi}_\xi\hat{v}\in P_K(\hat{D}_\xi)$, and $\sigma_k(k=0, 1, \cdots, K)$ are its zero points. Hence $L_\xi\hat{v}=0$, $\forall\hat{v}\in P_K(\hat{D}_\xi)$. According to the Peano theorem[5] we get

$$L_\xi\hat{v}=\hat{v}-\hat{\Pi}_\xi\hat{v}=\int_{-1}^{1}\hat{v}_{K+1,0}(\lambda, \eta)G(\xi, \lambda)d\lambda,$$

and similarly,

$$L_\eta\hat{v}=\hat{v}-\hat{\Pi}_\eta\hat{v}=\int_{-1}^{1}\hat{v}_{0,L+1}(\xi, \tau)\tilde{G}(\eta, \tau)d\tau,$$

where $G(\xi, \lambda)=L_\xi Q_{K+1}$, $\tilde{G}(\eta, \tau)=L_\eta\tilde{Q}_{L+1}$,

$$Q_k=\begin{cases}(\xi-\lambda)^k, & \xi\geqslant\lambda,\\ 0, & \xi<\lambda,\end{cases}\qquad \tilde{Q}_l=\begin{cases}(\eta-\tau)^l, & \eta\geqslant\tau,\\ 0, & \eta<\tau.\end{cases}$$

Therefore

$$\begin{aligned}(E-\hat{\Pi}_\xi)\otimes(E-\hat{\Pi}_\eta)\hat{v}&=(E-\hat{\Pi}_\xi)(L_\eta\hat{v})\\ &=\int_{-1}^{1}\int_{-1}^{1}\hat{v}_{K+1,L+1}(\lambda, \tau)\tilde{G}(\eta, \tau)G(\xi, \lambda)d\lambda\, d\tau.\end{aligned} \tag{2.5}$$

Then

$$\begin{aligned}\hat{v}-\hat{\Pi}\hat{v}=&\int_{-1}^{1}\hat{v}_{K+1,0}(\lambda, \eta)G(\xi, \lambda)\,d\lambda+\int_{-1}^{1}\hat{v}_{0,L+1}(\xi, \tau)\tilde{G}(\eta, \tau)d\tau\\ &-\int_{-1}^{1}\int_{-1}^{1}\hat{v}_{K+1,L+1}(\lambda, \tau)\tilde{G}(\eta, \tau)G(\xi, \lambda)d\lambda\, d\tau.\end{aligned} \tag{2.6}$$

According to the definition of discrete inner product and since G, $\tilde{G}$ are bounded, (2.2) holds.

If we take $L_\xi\hat{v}=\hat{v}-\hat{\Pi}_\xi\hat{v}-\hat{v}_{K+1,0}(0, \eta)B_K(\xi)$, it is easy to test that

$$L_\xi\hat{v}=0,\quad \forall\hat{v}\in P_{K+1}(\hat{D}_\xi).$$

Thus using the Peano theorem again, we can obtain

$$\hat{v}-\hat{\Pi}_\xi\hat{v}=\hat{v}_{K+1,0}(0, \eta)B_K(\xi)+\int_{-1}^{1}\hat{v}_{K+2,0}(\lambda, \eta)G^*(\xi, \lambda)d\lambda. \tag{2.7}$$

Similarly, taking $L_\eta\hat{v}=\hat{v}-\hat{\Pi}_\eta\hat{v}-\hat{v}_{0,L+1}(\xi, 0)B_L(\eta)$, we get

$$\hat{v}-\hat{\Pi}_\eta\hat{v}=\hat{v}_{0,L+1}(\xi, 0)B_L(\eta)+\int_{-1}^{1}\hat{v}_{0,L+2}(\xi, \tau)\tilde{G}^*(\eta, \tau)d\tau. \tag{2.8}$$

By (2.1), combining (2.5), (2.7) and (2.8), we have

$$\begin{aligned}\hat{v}-\hat{\Pi}\hat{v}=&\int_{-1}^{1}\hat{v}_{K+2,0}(\lambda, \eta)G^*(\xi, \lambda)d\lambda+\int_{-1}^{1}\hat{v}_{0,L+2}(\xi, \tau)\tilde{G}^*(\eta, \tau)d\tau\\ &-\int_{-1}^{1}\int_{-1}^{1}\hat{v}_{K+1,L+1}(\lambda, \tau)\tilde{G}(\eta, \tau)G(\xi, \lambda)d\lambda\, d\tau\\ &+\hat{v}_{K+1,0}(0, \eta)B_K(\xi)+\hat{v}_{0,L+1}(\xi, 0)B_L(\eta).\end{aligned} \tag{2.9}$$

Differentiate (2.9) with respect to ξ, η respectively and then let $\xi=\xi_k$, $\eta=\eta_l$,

$$\frac{\partial}{\partial \xi}(\hat{v}-\hat{\Pi}\hat{v})|_{\xi_k,\eta_l}=\int_{-1}^{1}\hat{v}_{K+2,0}(\lambda,\eta_l)\frac{\partial G^*(\xi,\lambda)}{\partial \xi}\Big|_{\xi=\xi_k}d\lambda+\hat{v}_{1,L+1}(\xi_k,0)B_L(\eta_l)$$
$$+\int_{-1}^{1}\hat{v}_{1,L+2}(\xi_k,\tau)\tilde{G}^*(\eta_l,\tau)d\tau$$
$$-\int_{-1}^{1}\int_{-1}^{1}\hat{v}_{K+1,L+1}(\lambda,\tau)\tilde{G}(\eta_l,\tau)\frac{\partial G(\xi,\lambda)}{\partial \xi}\Big|_{\xi=\xi_k}d\lambda\, d\tau$$

$$\frac{\partial}{\partial \eta}(\hat{v}-\hat{\Pi}\hat{v})|_{\xi_k,\eta_l}=\int_{-1}^{1}\hat{v}_{K+2,1}(\lambda,\eta_l)G^*(\xi_k,\lambda)d\lambda+\hat{v}_{K+1,1}(0,\eta_l)B_K(\xi_k)$$
$$+\int_{-1}^{1}\hat{v}_{0,L+2}(\xi_k,\tau)\frac{\partial \tilde{G}^*(\eta_l,\tau)}{\partial \eta}d\tau$$
$$-\int_{-1}^{1}\int_{-1}^{1}\hat{v}_{K+1,L+1}(\lambda,\tau)\frac{\partial \tilde{G}(\eta_l,\tau)}{\partial \eta}G(\xi_k,\lambda)d\lambda\, d\tau.$$

Also according to the definition of discrete inner product and in consideration of the boundedness of G, $\tilde{G}$, G^*, $\tilde{G}^*$ and their derivatives, (2.3), (2.4) hold. This completes the proof.

By (1.4) and $\hat{v}_{k,l}=2^{-(k+l)}\Delta x_i^k\Delta y_j^l v_{k,l}$, we have the following corollary.

Corollary. If $v\in L_\infty(C^{(0)}(D_t\times D_{\mu\nu})\times w_{K+2,L+2}(D))$, the following inequalities hold:

$$\langle v-\Pi v\rangle\leqslant C(h^{K+1}|v_{K+1,0}|_{L^\infty(B)}+h^{L+1}|v_{0,L+1}|_{L^\infty(B)}+h^{K+L+2}|v_{K+1,L+1}|_{L^\infty(B)}),$$
$$\left\langle\frac{\partial}{\partial x}^{*}(v-\Pi v)\right\rangle\leqslant C(h^{K+1}|v_{K+2,0}|_{L^\infty(B)}+h^{L+1}|v_{1,L+1}|_{L^\infty(B)}$$
$$+h^{L+2}|v_{1,L+2}|_{L^\infty(B)}+h^{K+L+1}|v_{K+1,L+1}|_{L^\infty(B)}), \tag{2.10}$$
$$\left\langle\frac{\partial}{\partial y}(v-\Pi v)\right\rangle\leqslant C(h^{L+1}|v_{0,L+2}|_{L^\infty(B)}+h^{K+1}|v_{K+1,1}|_{L^\infty(B)}$$
$$+h^{K+2}|v_{K+2,1}|_{L^\infty(B)}+h^{K+L+1}|v_{K+1,L+1}|_{L^\infty(B)}).$$

We are going to use the following equality

$$2\sum_n(\varphi_{ms}^{n+1}-\varphi_{ms}^{n})\varphi_{ms}^{n+\frac{1}{2}}e^{-\alpha t^{n+\frac{1}{2}}}$$
$$=(\varphi_{ms}^{N})^2e^{-\alpha T}-(\varphi_{ms}^{0})^2+\sum_n\Delta t^{n+\frac{1}{2}}\left(\frac{1-e^{-\frac{\alpha}{2}\Delta t^{n+\frac{1}{2}}}}{\Delta t^{n+\frac{1}{2}}}\right)\left[(\varphi_{ms}^{n+1})^2+e^{\frac{\alpha}{2}\Delta t^{n+\frac{1}{2}}}(\varphi_{ms}^{n})^2\right]e^{-\alpha t^{n+\frac{1}{2}}}, \tag{2.11}$$

where α is a real constant.

§ 3. Error Estimate

In this section we shall analyse the error of the solution Φ of equation (1.1), which is compared with the corresponding solution φ of the collocation equations (1.5) and (1.6). Let

$$\psi_{ms}=\Phi_{ms}-\varphi_{ms}=\varepsilon_{ms}+\theta_{ms},$$

where

$$\varepsilon_{ms}=\Pi\Phi_{ms}-\varphi_{ms},\quad \theta_{ms}=\Phi_{ms}-\Pi\Phi_{ms}.$$

Then ψ_{ms} satisfy the following equations

$$\frac{1}{v^*} d_t\psi_{ms} + \mu_{ms}\frac{\partial\psi_{ms}}{\partial x} + \nu_{ms}\frac{\partial\psi_{ms}}{\partial y} + \alpha\psi_{ms} = \beta\sum_{m's'} w_{m's'}\psi_{m's'} + R \tag{3.1}$$

on collocation points, where

$$R = \frac{1}{24v^*}\frac{\partial^3\Phi}{\partial t^3}\Big|_{t=\bar{t}}(\Delta t^{n+\frac{1}{2}})^2 + R_1.$$

R_1 is the error of numerical integration of the source term Q and will not be discussed in this paper.

Theorem 1. *Assume that the solution of* (1.1)

$$\Phi \in L_\infty(C^{(0)}(D_t\times D_{\mu\nu})\times W_{K+2,L+2}(D))$$

and

$$\Phi_t \in L_\infty(C^{(0)}(D_t\times D_{\mu\nu})\times W_{K+1,L+1}(D))$$

and φ is the solution of (1.5) *and* (1.6), *when* $\Delta t < \frac{2}{a}$, *where*

$$a = \begin{cases} 4v_0(\beta_1 - \alpha_0) + 1, & \beta_1 - \alpha_0 \geqslant 0, \\ 0, & \beta_1 - \alpha_0 < 0. \end{cases}$$

Then we have

$$\langle\Phi - \varphi\rangle \leqslant O[h^{K+1}A_1(\Phi) + h^{L+1}A_2(\Phi) + h^{K+L+1}A_3(\Phi) + \langle R\rangle], \tag{3.2}$$

where

$$\begin{cases} A_1(\Phi) = |(\Phi_t)_{K+1,0}|_{L^\circ(B)} + |\Phi_{K+2,0}|_{L^\circ(B)} + |\Phi_{K+1,0}|_{L^\circ(B)} + |\Phi_{K+1,1}|_{L^\circ(B)} \\ \qquad + h|\Phi_{K+2,1}|_{L^\circ(B)}, \\ A_2(\Phi) = |(\Phi_t)_{0,L+1}|_{L^\circ(B)} + |\Phi_{0,L+2}|_{L^\circ(B)} + |\Phi_{0,L+1}|_{L^\circ(B)} + |\Phi_{1,L+2}|_{L^\circ(B)} \\ \qquad + h|\Phi_{1,L+2}|_{L^\circ(B)}, \\ A_3(\Phi) = h|(\Phi_t)_{K+1,L+1}|_{L^\circ(B)} + (1+h)|\Phi_{K+1,L+1}|_{L^\circ(B)}. \end{cases} \tag{3.3}$$

Proof. By (3.1)

$$\frac{1}{v^*} d_t\varepsilon_{ms} + \mu_{ms}\frac{\partial\varepsilon_{ms}}{\partial x} + \nu_{ms}\frac{\partial\varepsilon_{ms}}{\partial y} + \alpha\varepsilon_{ms} - \beta\sum_{m's'} w_{m's'}\varepsilon_{m's'}$$
$$= -\Big(\frac{1}{v^*} d_t\theta_{ms} + \mu_{ms}\frac{\partial\theta_{ms}}{\partial x} + \nu_{ms}\frac{\partial\theta_{ms}}{\partial y} + \alpha\theta_{ms}\Big) + \beta\sum_{m's'} w_{m's'}\theta_{m's'} + R. \tag{3.4}$$

Multiplying (3.4) by $\varepsilon_{ms}e^{-at^{n+\frac{1}{2}}}$ and summing up over n, using (2.11) and when $\Delta t < \frac{2}{a}$, we have $\frac{1-e^{-\frac{1}{2}a\Delta t}}{\Delta t} \geqslant \frac{a}{4}$. Thus

$$\frac{1}{2v^*}\Big[(\varepsilon^N_{ms})^2 e^{-aT} - (\varepsilon^0_{ms})^2 + \frac{a}{2}\sum_{n=0}^{N-1}\varepsilon^2_{ms}e^{-at^{n+\frac{1}{2}}}\Delta t\Big]$$
$$+ \sum_{n=0}^{N-1}\Big(\mu_{ms}\varepsilon_{ms}\frac{\partial\varepsilon_{ms}}{\partial x} + \nu_{ms}\varepsilon_{ms}\frac{\partial\varepsilon_{ms}}{\partial y} + \alpha\varepsilon^2_{ms}\Big)\Delta t e^{-at^{n+\frac{1}{2}}} - \sum_{n=0}^{N-1}\beta\varepsilon_{ms}e^{-at^{n+\frac{1}{2}}}\Delta t\sum_{m's'} w_{m's'}\varepsilon_{m's'}$$
$$\leqslant \sum_{n=0}^{N-1}\varepsilon_{ms}e^{-at^{n+\frac{1}{2}}}\Delta t\Big\{\Big[-\Big(\frac{1}{v^*} d_t\theta_{ms} + \mu_{ms}\frac{\partial\theta_{ms}}{\partial x} + \nu_{ms}\frac{\partial\theta_{ms}}{\partial y} + \alpha\theta_{ms}\Big)\Big]$$
$$+ \beta\sum_{m's'} w_{m's'}\theta_{m's'} + R\Big\}\Big|_{x=x_{ki}, y=y_{lj}}. \tag{3.5}$$

Multiplying (3.5) by $w_k \bar{w}_l \dfrac{\Delta x_i \Delta y_j}{4}$ and summing up over i, j, k, l, we have

$$\frac{e^{-aT}}{2v_0}\langle \varepsilon_{ms}^N, \varepsilon_{ms}^N\rangle_D + \sum_{n=0}^{N-1} e^{-at^{n+1}}\Delta t\Big\{\mu_{ms}\Big\langle \varepsilon_{ms}, \frac{\partial \varepsilon_{ms}}{\partial x}\Big\rangle_D + \nu_{ms}\Big\langle \varepsilon_{ms}, \frac{\partial \varepsilon_{ms}}{\partial y}\Big\rangle_D$$
$$+\Big(\frac{a}{4v_0}+\alpha_0\Big)\langle \varepsilon_{ms}, \varepsilon_{ms}\rangle_D - \sum_{m's'} w_{m's'}\langle \varepsilon_{ms}, \beta\varepsilon_{m's'}\rangle_D\Big\}$$
$$\leqslant \frac{1}{2v_1}\langle \varepsilon_{ms}^0, \varepsilon_{ms}^0\rangle_D + \sum_{n=0}^{N-1} e^{-at^{n+1}}\Delta t\Big\{-\Big\langle \frac{d_t\theta_{ms}}{v^*}, \varepsilon_{ms}\Big\rangle_D$$
$$-\Big\langle \varepsilon_{ms}, \mu_{ms}\frac{\partial \theta_{ms}}{\partial x}\Big\rangle - \Big\langle \varepsilon_{ms}, \nu_{ms}\frac{\partial \theta_{ms}}{\partial y}\Big\rangle_D - \langle \varepsilon_{ms}, \alpha\theta_{ms}\rangle_D$$
$$+\sum_{m's'} w_{m's'}\langle \varepsilon_{ms}, \beta\theta_{m's'}\rangle_D + \langle \varepsilon_{ms}, R\rangle_D\Big\}. \tag{3.6}$$

According to the property of Gauss integration, we have

$$\Big\langle \varepsilon_{ms}, \frac{\partial \varepsilon_{ms}}{\partial x}\Big\rangle_D = \sum_{jl} \bar{w}_l \frac{\Delta y_j}{2}\int_0^X \frac{\partial \varepsilon_{ms}^2(x, y_{jl})}{\partial x}dx$$
$$= \sum_{jl}\frac{\Delta y_j}{2}\bar{w}_l[\varepsilon_{ms}^2(X, y_{jl}) - \varepsilon_{ms}^2(0, y_{jl})],$$
$$\Big\langle \varepsilon_{ms}, \frac{\partial \varepsilon_{ms}}{\partial y}\Big\rangle_D = \sum_{ik}\frac{\Delta x_i}{2} w_k[\varepsilon_{ms}^2(x_{ik}, Y) - \varepsilon_{ms}^2(x_{ik}, 0)].$$

Multiplying (3.6) by w_{ms} and summing up over m, s in consideration of the boundary condition and the central symmetrical condition, and also using

$$\sum_{m,s,m',s'} w_{ms}w_{m's'}\langle \varepsilon_{ms}, \beta\varepsilon_{m's'}\rangle_D \leqslant \beta_1\langle \varepsilon, \varepsilon\rangle_{t^{n+1}},$$

we get

$$\frac{e^{-aT}}{2v_0}\langle \varepsilon\rangle_T^2 + \sum_{n=0}^{N-1} e^{-at^{n+1}}\Delta t\Big\{\sum_{ms} w_{ms}\Big[\mu_{ms}\sum_{jl}\frac{\Delta y_j}{2}\bar{w}_l\varepsilon_{ms}^2(X, y_{jl})$$
$$+\nu_{ms}\sum_{ik}\frac{\Delta x_i}{2}w_k\varepsilon_{ms}^2(x_{ik}, Y) + \Big(\frac{a}{4v_0}+\alpha_0-\beta_1\Big)\langle \varepsilon, \varepsilon\rangle_{t^{n+1}}$$
$$\leqslant \frac{1}{2v^*}\langle \varepsilon\rangle_0^2 + \sum_{n=0}^{N-1}\Delta t e^{-at^{n+1}}\Big\{\frac{1}{v_1}\langle d_t\theta\rangle_{t^{n+1}}\langle \varepsilon\rangle_{t^{n+1}} + \langle \varepsilon\rangle_{t^{n+1}}\Big\langle \frac{\partial\theta}{\partial x}\Big\rangle_{t^{n+1}}$$
$$+\langle \varepsilon\rangle_{t^{n+1}}\Big\langle \frac{\partial\theta}{\partial y}\Big\rangle_{t^{n+1}} + \alpha_1\langle \varepsilon\rangle_{t^{n+1}}\langle \theta\rangle_{t^{n+1}} + \beta_1\langle \varepsilon\rangle_{t^{n+1}}\langle \theta\rangle_{t^{n+1}} + \langle R\rangle_{t^{n+1}}\langle \varepsilon\rangle_{t^{n+1}}\Big\}.$$

Hence we get

$$\Big(\frac{a}{4v_0}+\alpha_0-\beta_1\Big)\langle \varepsilon, \varepsilon\rangle \leqslant C\Big(\frac{1}{v_1}\langle d_t\theta, \varepsilon\rangle + \Big\langle \frac{\partial\theta}{\partial x}, \varepsilon\Big\rangle + \Big\langle \frac{\partial\theta}{\partial y}, \varepsilon\Big\rangle$$
$$+\alpha_1\langle \theta, \varepsilon\rangle + \beta_1\langle \theta, \varepsilon\rangle\Big) + \langle R, \varepsilon\rangle.$$

If we only take a to satisfy the given conditions of this theorem, then

$$\langle \varepsilon, \varepsilon\rangle \leqslant C\Big(\langle d_t\theta\rangle^2 + \Big\langle \frac{\partial\theta}{\partial x}\Big\rangle^2 + \Big\langle \frac{\partial\theta}{\partial y}\Big\rangle^2 + \langle \theta\rangle^2 + \langle R\rangle^2\Big).$$

According to the corollary, we have

$$\langle d_t\theta\rangle \leqslant C(h^{K+1}|(\Phi_t)_{K+1,0}|_{L^2(B)} + h^{L+1}|(\Phi_t)_{0,L+1}|_{L^2(B)} + h^{K+L+2}|(\Phi_t)_{K+1,L+1}|_{L^2(B)}).$$

Using our lemma, under conditions of (3.3), we get (3.2). Thus the theorem has been proved.

It is obvious that when $K=L$, if

$$\frac{\partial^3\Phi}{\partial t^3}\in L_\infty(C^{(0)}(D_t\times D_{\mu\nu})\times W_{K+2,\ L+2}(D)),$$

we obtain

$$\langle\Phi-\varphi\rangle\leqslant O[h^{K+1}(A_1(\Phi)+A_2(\Phi)+h^K A_3(\Phi))+\Delta t^2+\langle R_1\rangle].$$

§ 4. Stability

In this section we are going to discuss the stability of the collocation method.

Theorem 2. *Assume that φ is a solution of (1.5) and (1.6) for $\varphi\in H$. Then there is a constant C such that*

$$\langle\varphi\rangle_{t=T}\leqslant C(\langle\varphi\rangle_{t=0}+\langle F\rangle),$$

$$\langle\varphi\rangle\leqslant C(\langle\varphi\rangle_{t=0}+\langle F\rangle),\quad for\quad \Delta t<\frac{2}{a}. \tag{4.1}$$

Proof. Multiplying (1.5) by $\varphi_{ms}e^{-at^{n+\frac12}}$, summing up over n and using (2.11), we obtain

$$\frac{1}{2v^*}\Big[(\varphi^N_{ms})^2e^{-aT}-(\varphi^0_{ms})^2+\frac{a}{2}\sum_n\varphi^2_{ms}e^{-at^{n+\frac12}}\Delta t\Big]$$
$$+\sum_{n=0}^{N-1}\mu_{ms}\varphi_{ms}\frac{\partial\varphi_{ms}}{\partial x}e^{-at^{n+\frac12}}\Delta t+\sum_{n=0}^{N-1}\nu_{ms}\varphi_{ms}\frac{\partial\varphi_{ms}}{\partial y}e^{-at^{n+\frac12}}\Delta t+\sum_{n=0}^{N-1}\alpha\varphi^2_{ms}e^{-at^{n+\frac12}}\Delta t$$
$$\leqslant\sum_{n=0}^{N-1}e^{-at^{n+\frac12}}\varphi_{ms}Q_{ms}+\sum_n e^{-at^{n+\frac12}}\varphi_{ms}F_{ms}\Delta t. \tag{4.2}$$

Multiplying (4.2) by $W_{ms}\frac{\Delta x_i}{2}\frac{\Delta y_j}{2}W_kW_l$, and then summing up over i, j, k, l, m, s, we get

$$\frac{e^{-aT}}{2v_0}\langle\varphi\rangle^2_{t=T}+e^{-aT}\Big(\frac{a}{4v_0}+\alpha_0-\beta_1\Big)\langle\varphi\rangle^2\leqslant\frac{1}{2v_1}\langle\varphi\rangle^2_{t=0}+\langle\varphi\rangle\langle F\rangle.$$

Then

$$\frac{e^{-aT}}{2v_0}\langle\varphi\rangle^2_{t=T}+e^{-aT}\Big(\frac{a}{4v_0}+\alpha_0-\beta_1-\frac12\Big)\langle\varphi\rangle^2\leqslant\frac{1}{2v_1}\langle\varphi\rangle^2_{t=0}+\frac12\langle F\rangle^2.$$

Take a to satisfy $\frac{a}{4v_0}+\alpha_0-\beta_1-\frac12=b>0$. Then we have

$$\langle\varphi\rangle^2_{t=T}\leqslant\frac{v_0}{v_1}e^{aT}\langle\varphi\rangle^2_{t=0}+v_0e^{aT}\langle F\rangle^2,$$

$$\langle\varphi\rangle^2\leqslant\frac{e^{aT}}{2bv_1}\langle\varphi\rangle^2_{t=0}+\frac{e^{aT}}{2b}\langle F\rangle^2.$$

Thus (4.1) holds.

(4.1) can be interpreted as follows: If the Crank-Nicholson difference approximation is used for time, the discrete ordinates approximation is used for the angular direction and the collocation method is used for spatial variables, then the solution continuously depends on the initial value and the outside source. Therefore the method is stable.

§ 5. Conservation and Comparison of the Collocation Method with the Difference Method and the Galerkin Method

1. Conservation.

Multiplying (1.5) by $\Delta t^{n+\frac{1}{2}}\, W_{ms}W_k\overline{W}_l\,\frac{\Delta x_i}{2}\,\frac{\Delta y_j}{2}$, summing up over all indexes, and using the property of Gauss integration and (1.6), we get

$$\begin{aligned}&\frac{1}{v^*}\sum_{ms} w_{ms}\int_D \varphi^N_{ms}\,dx\,dy+\sum_{msn}\Delta t^{n+\frac{1}{2}}\, w_{ms}\Big[\int_0^Y \mu_{ms}\varphi(t^{n+\frac{1}{2}}, X, y, \mu_{ms}, \nu_{ms})\,dy\\ &\quad+\int_0^X \nu_{ms}\varphi(t^{n+\frac{1}{2}}, x, Y, \mu_{ms}, \nu_{ms})\,dx\Big]+\sum_{msn} w_{ms}\,\Delta t^{n+\frac{1}{2}}\int_D \alpha\varphi(t^{n+\frac{1}{2}} x, y, \mu_{ms}, \nu_{ms})\,dx\,dy\\ &=\sum_{\substack{msn\\ i,j,k,l}} w_{ms}\,\Delta t^{n+\frac{1}{2}}\,\frac{\Delta x_i\Delta y_j}{2}\,w_k\bar{w}_l\, f(t^{n+\frac{1}{2}}, x_{ik}, y_{jl}, \mu_{ms}, \nu_{ms})\\ &\quad+\frac{1}{v^*}\sum_{ms} w_{ms}\int_D \varphi^0_{ms}\,dx\,dy,\end{aligned}\tag{5.1}$$

where $f=Q+F$.

It will be seen that if (1.1) is integrated on B, its discrete form is (5.1).

2. Comparison of the Collocation Method with the Difference Method.

Suppose that the space $P^*_{KK}(D_{ij})$ is $P_{KK}(D_{ij})$ with the exception of the term $x^K y^K$. When $K=1$, the polynomials take the form $\varphi=ax+by+c$. For example, in Fig. 4, this collocation method is equivalent to the following difference scheme:

$$\begin{aligned}&\frac{1}{v^*}\,\frac{\varphi^{n+1}_{m,s,i-\frac{1}{2},j-\frac{1}{2}}-\varphi^{n}_{m,s,i-\frac{1}{2},j-\frac{1}{2}}}{\Delta t^{n+\frac{1}{2}}}\\ &\quad+\mu_{ms}\,\frac{\varphi^{n+\frac{1}{2}}_{m,s,i,j-\frac{1}{2}}-\varphi^{n+\frac{1}{2}}_{m,s,i-1,j-\frac{1}{2}}}{\Delta x_i}\\ &\quad+\nu_{ms}\,\frac{\varphi^{n+\frac{1}{2}}_{m,s,i-\frac{1}{2},j}-\varphi^{n+\frac{1}{2}}_{m,s,i-\frac{1}{2},j-1}}{\Delta y_j}\\ &\quad+\alpha\,\varphi^{n+\frac{1}{2}}_{m,s,i-\frac{1}{2},j-\frac{1}{2}}\\ &=f^{n+\frac{1}{2}}_{m,s,i-\frac{1}{2},j-\frac{1}{2}}.\end{aligned}\tag{5.2}$$

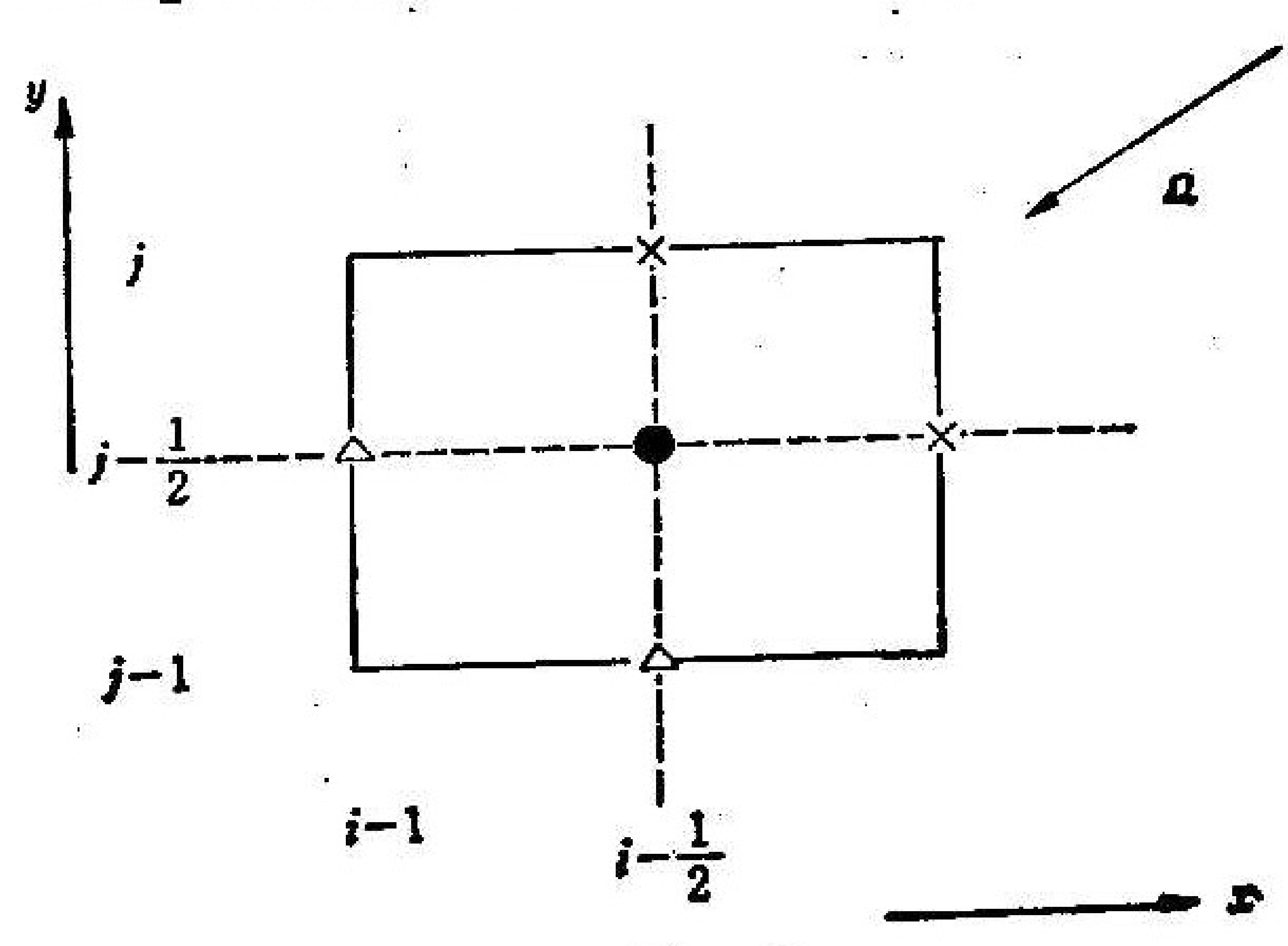

Fig. 4

This is just the diamond scheme of the DSN method. Hence the DSN method is a special type of collocation method.

If we substitute P^*_{KK} for P_{KK} in H, and the angular point of outward trace on D_{ij} is discarded, it is clear that Theorems 1 and 2 hold.

3. Equivalence of the Collocation Method to the Discrete Galerkin Method.

When $t=t^{n+\frac{1}{2}}$, $\mu=\mu_{ms}$, $\nu=\nu_{ms}$, the discrete Galerkin method is represented by

$$\langle A_{ms}(\varphi), u\rangle_D=\langle f, u\rangle_D,\quad \varphi\in H^*,\ \forall u\in H^*,\tag{5.3}$$

where H^* is a set of elements satisfing (1.6) in H. That is to find a function $\varphi\in H^*$, such that (5.3) holds.

Let us introduce the bases $\{u_{ijkl}(t, x, y, \mu_{ms}, \nu_{ms})\}$ for H, which satisfies

$$u_{ijkl}(t^{n+\frac{1}{2}}, x_{i'k'}, y_{j'l'}, \mu_{ms}, \nu_{ms})=\delta_{ii'}\delta_{kk'}\delta_{jj'}\delta_{ll'}, \tag{5.4}$$

where

$$\delta_{ii'}=\begin{cases}0, & \text{for } i\neq i',\\ 1, & \text{for } i=i',\end{cases}$$

$$i=1, \cdots, I;\ j=1, \cdots, J;\ k=1, \cdots, K;\ l=1, \cdots, L.$$

If u is substituted by (5.4) in (5.3), we get collocation equations (1.5).

§ 6. Numerical Results

In this section we describe the results of numerical experiments that were designed to test the effectiveness of the collocation method. The experiments given here were carried out using the procedure in Section 1. We have considered a few simple problems.

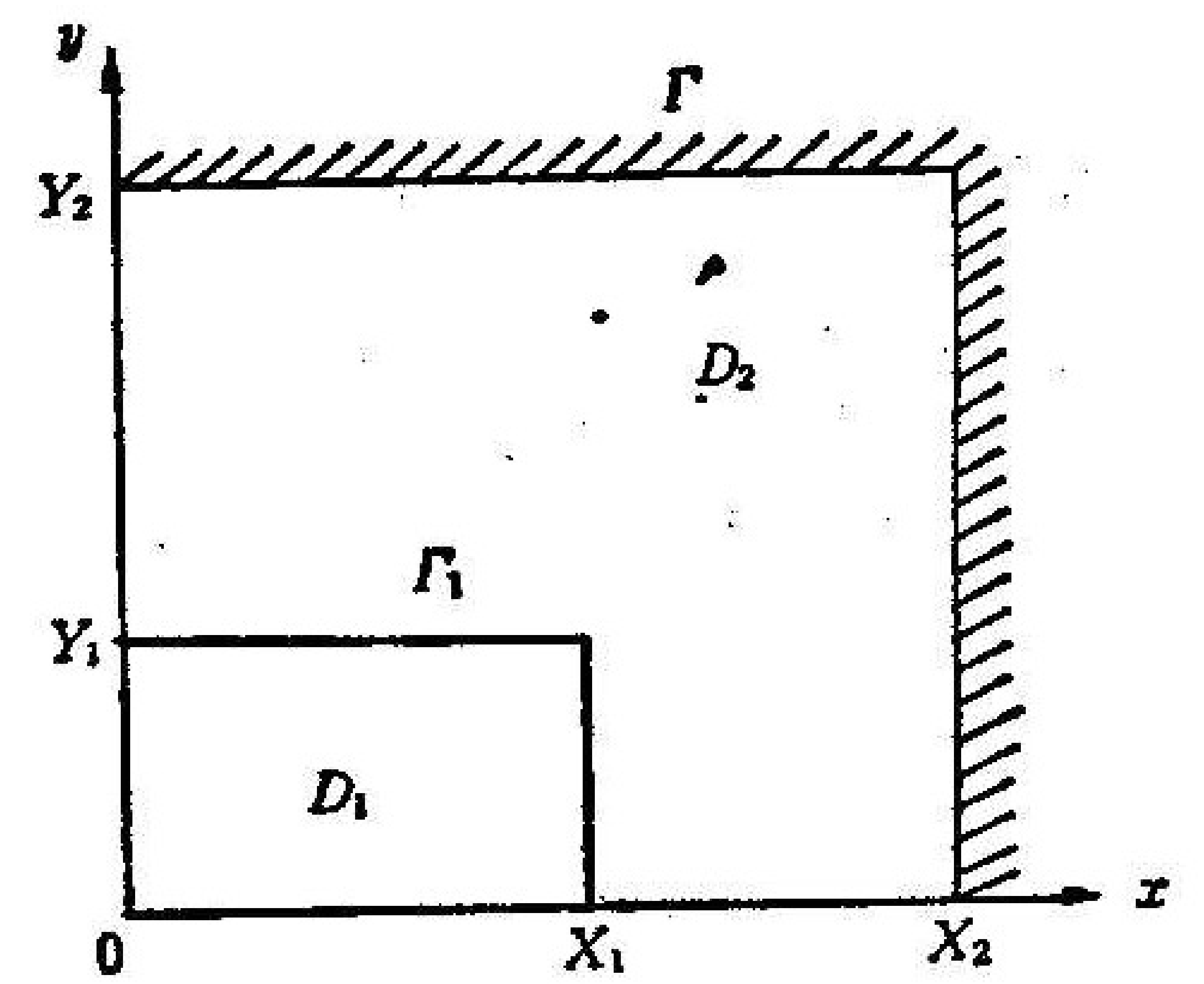

Fig. 5

Example 1. Construct the solution Φ_e of the one-group time-independent neutron transport equation as follows

$$A(\Phi)=\mu\frac{\partial\Phi}{\partial x}+\nu\frac{\partial\Phi}{\partial y}+\alpha\Phi=Q(\Phi)+F,$$

$$\Phi|_\Gamma=\Phi_e|_\Gamma,\quad \text{if}\quad \boldsymbol{\Omega}\cdot\boldsymbol{h}\leqslant 0,$$

$$\Phi(x, y, \mu, \nu)|_{x=0}=\Phi(x, y, -\mu, \nu)|_{x=0},$$

$$\Phi(x, y, \mu, \nu)|_{y=0}=\Phi(x, y, \mu, -\nu)|_{y=0}, \tag{6.1}$$

with Φ being continuous on Γ_1, the boundary face of the inside region D_1 (see Fig. 5), where

$$F=[\mu f'(x)g(y)+\nu f(x)g'(y)+(\alpha-\beta)f(x)g(y)]h(\mu, \nu)$$
$$-\left(1+\frac{8a_9}{3\pi}+\frac{a_{10}}{2}\right)\beta f(x)g(y),$$

$$f=\begin{cases}f_1=a_1(x^3+2x^2+3x+4)+a_2\operatorname{arc\,tg} x+1, & (x, y)\in D_1,\\ f_2=a_3e^{a_4x}, & (x, y)\in D_2,\end{cases}$$

$$g=\begin{cases}g_1=a_5(y^3+2y^2+3y+4)+a_6\operatorname{arc\,tg} y+1, & (x, y)\in D_1,\\ g_2=a_7e^{a_8y}, & (x, y)\in D_2,\end{cases}$$

$$h(\mu, \nu)=1+a_9(|\mu|+|\nu|)+a_{10}(\mu^2+\nu^2),$$

so that the exact solution of problem (6.1) is given by

$$\Phi_e=f(x)g(y)h(\mu, \nu).$$

Let φ denote the collocation solution. To evaluate the accuracy of the method, define the maximum error by

$$E_m = \max_{k,l,i,j,m,s} \left| \frac{\varphi_{k,l,i,j,m,s} - \Phi_{k,l,i,j,m,s}}{\Phi_{k,l,i,j,m,s}} \right|$$

and the "mean square" error by

$$E_s = \sqrt{\frac{1}{M_0} \sum_{k,l,i,j,m,s} (\varphi_{k,l,i,j,m,s} - \Phi_{k,l,i,j,m,s})^2},$$

where $M_0 = KLIJMS$.

For simplicity, we assume that $S=4$, $X_1=Y_1=0.5$, $X_2=Y_2=1$. I and J are the numbers of meshes in x and y direction, respectively. Thus, the total number of meshes is $I \cdot J$. Take $a_1=a_2=a_5=a_6=1$, $a_4=a_8=2$, and $a_9=a_{10}=0.1$; a_3 and a_7 can be derived from the continuity on Γ_1. Here take $\alpha=1$ and $\beta=0$, 0.1, respectively. The errors E_m, E_s are given in Tables 1—2.

When $\beta=0$, errors from discrete ordinates angular quadrature are avoided. It can be seen that when $K=3$, calculation gives the best results and the rate of convergence is the fastest as the size of meshes turns fine. We compare columns $K=1$ and $K=3$. When $K=1$, even if $I \cdot J=64$, the accuracy can not compare with that of $I \cdot J=4$ when $K=3$.

In Table 2, we observe that although the accuracy is improved with the increase of K for the same size of mesh, yet it is unlikely to improve for the same K when the size of mesh turns fine. It is noted that the errors of angular quadrature are important.

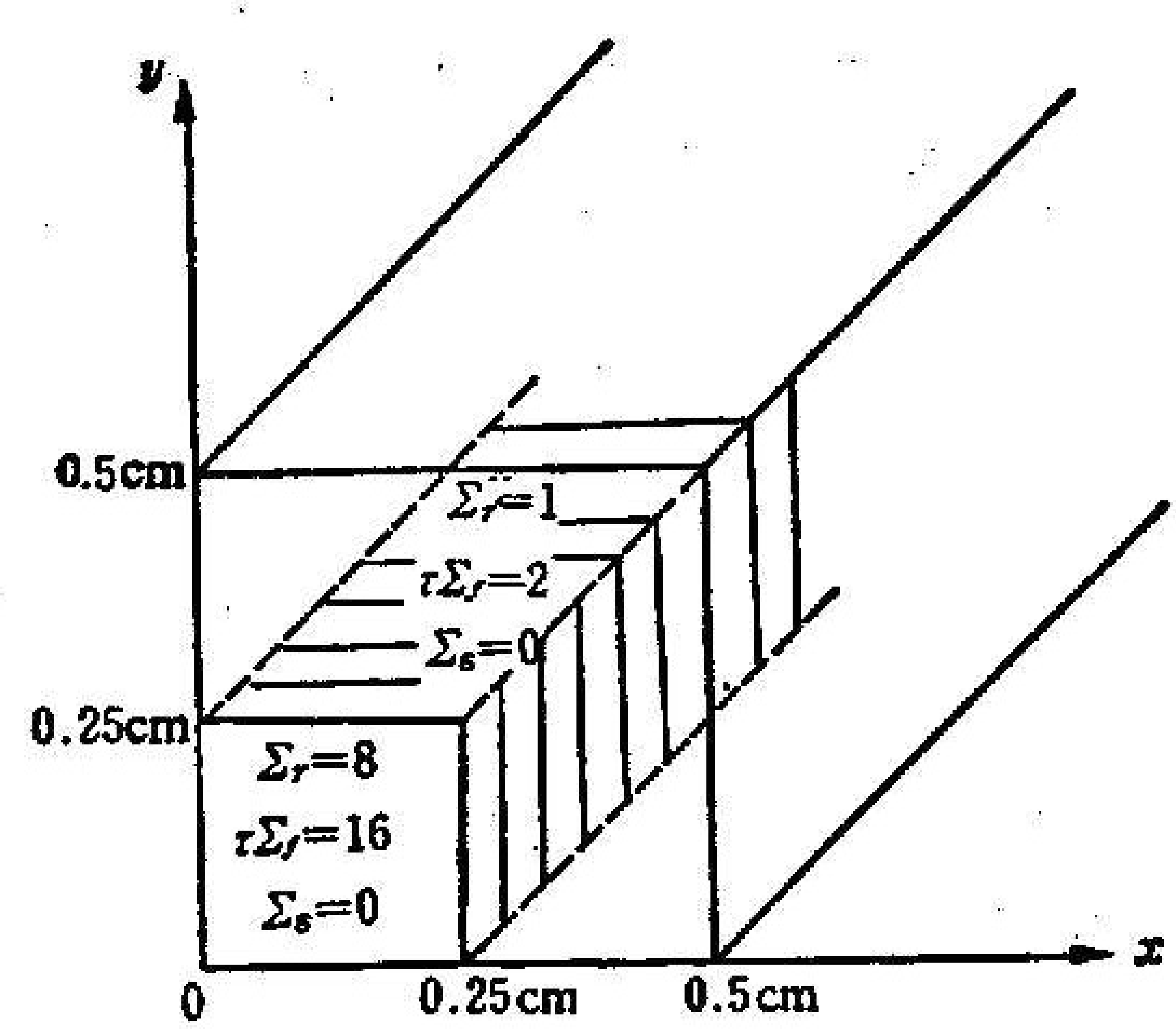

Fig 6

Example 2. The collocation method is readily adapted to solve the eigenvalue problem. In fact, if we take $F=0$ in (1.6), the generalized matrix eigenvalue problem is obtained.

To compare with the existing methods, we have considered the same system as in [6] with a simple geometry (Fig. 6). We have calculated a multiplication rate K_{eff} for the system, where K_{eff} is the greatest eigenvalue of the operator $A^{-1}Q$, and the boundary of the spatial domain is assumed to be a vacuum one. [6] gave the following results

	Regular mesh	Distorted mesh	Monte-Carlo
K_{eff}	1.3217	1.3207	1.3204±0.0050

where the number of meshes is 20×20.

The results of the collocation are given in Table 3.

Table 1 Errors (example 1) $\alpha=1,\ \beta=0$

K \ I, J	1		2		3	
	E_m	E_s	E_m	E_s	E_m	E_s
2	0.25	11.4	0.58×10^{-1}	1.65	0.30×10^{-2}	0.12
4	0.47×10^{-1}	3.04	0.18×10^{-1}	0.46	0.20×10^{-3}	0.84×10^{-2}
8	0.12×10^{-1}	0.78	0.54×10^{-2}	0.12	0.14×10^{-4}	0.53×10^{-3}

Table 2 $\alpha=1,\ \beta=0.1$

K \ I, J	1		2		3	
	E_m	E_s	E_m	E_s	E_m	E_s
2	0.22	11.7	0.64×10^{-1}	1.81	0.55×10^{-2}	0.28
4	0.42×10^{-1}	3.2	0.23×10^{-1}	0.60	0.73×10^{-2}	0.21
8	0.13×10^{-1}	0.92	0.11×10^{-1}	0.28	0.76×10^{-2}	0.21

Table 3 Multiplication rate (example 2)

K \ I, J	1	2	3
	K_{eff}	K_{eff}	K_{eff}
2	1.0844	1.3561	1.3262
4	1.2680	1.3324	1.3191
8	1.3058	1.3223	1.3184
16	1.3153	1.3194	1.3184

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