CANONICAL INTEGRAL EQUATIONS OF STOKES PROBLEM*

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Abstract

The canonical boundary reduction, suggested by Feng Kang, also can be applied to the bidimensional steady Stokes problem. In this paper we first give the representation formula for the solution of the Stokes problem via two complex variable functions. Then by means of complex analysis and the Fourier analysis, we find the expressions of the Poisson integral formulas and the canonical integral equations in three typical domains. From these results the canonical boundary element method for solving the Stokes problem can be developed.

§ 1. Introduction

Since Feng Kang suggested the canonical boundary reduction[1], which reduces a boundary value problem of an elliptic equation to a singular integral equation on the boundary via Green's formula and Green's function, this method already has been applied to the harmonic boundary value problem, the biharmonic boundary value problem and the plane elasticity problem[2,3,6,8]. This kind of reduction conserves the essential characteristics of the original boundary value problem and occupies a particular place in all boundary reductions. From this, a new numerical method, i.e. the canonical boundary element method, has been developed[2,4,6,7]. It also has many distinctive advantages and is fully compatible with the classical FEM[5-7].

Now we consider the steady Stokes problem, which represents the steady flow of an incompressible viscous fluid with a small Reynolds number. We will study the canonical boundary reduction of the bidimensional steady Stokes problem, and find its Poisson formulas and its canonical integral equations in three typical domains.

§ 2. The Principle of the Canonical Boundary Reduction

Consider the boundary value problems in a plane domain Ω with a smooth boundary Γ:

$$\begin{cases} -\nu\Delta \boldsymbol{u}+\operatorname{grad} p=0, & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u}=0, & \\ \boldsymbol{u}=\boldsymbol{u}_0, & \text{on } \Gamma \end{cases} \tag{1}$$

and

$$\begin{cases} -\nu\Delta \boldsymbol{u}+\operatorname{grad} p=0, & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u}=0, & \\ \sum_{j=1}^{2}\sigma_{ij}(\boldsymbol{u}, p)n_j=g_i, \quad i=1, 2, & \text{on } \Gamma, \end{cases} \tag{2}$$

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where the coefficient $\nu>0$ is the dynamic viscosity of the flow, the unknowns are the velocity $\boldsymbol{u}$ of the fluid occupying Ω and its pressure p,

$$(\boldsymbol{u}, p) \in (H^1(\Omega))^2 \times L^2(\Omega)/\mathbb{R} \quad \text{for } \Omega \text{ bounded},$$

or

$$(\boldsymbol{u}, p) \in (W_0^1(\Omega))^2 \times L^2(\Omega) \qquad \text{for } \Omega \text{ unbounded},$$

where

$$W_0^1(\Omega)=\left\{\frac{u}{\sqrt{1+r^2}\ln(2+r^2)} \in L^2(\Omega),\ \frac{\partial u}{\partial x_i} \in L^2(\Omega),\quad i=1,\ 2,\ r=\sqrt{x_1^2+x_2^2}\right\},$$

$$\varepsilon_{ij}(\boldsymbol{u})=\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\Big/2,$$

$$\sigma_{ij}(\boldsymbol{u}, p)=-\delta_{ij}p+2\nu\varepsilon_{ij}(\boldsymbol{u}), \quad i, j=1,\ 2,$$

$\boldsymbol{n}=(n_1, n_2)^T$ is the unit outward normal to Γ. We know that Green's formula for the steady Stokes problem is[9]

$$\begin{aligned} 2\nu \sum_{i,j=1}^{2}\int_\Omega \varepsilon_{ij}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{v})dx & \\ +\sum_{i=1}^{2}\int_\Omega\left(\nu\Delta u_i-\frac{\partial p}{\partial x_i}\right)v_i\,dx-\sum_{i=1}^{2}\int_\Omega p\frac{\partial v_i}{\partial x_i}dx & \\ =\sum_{i,j=1}^{2}\int_\Gamma \sigma_{ij}(\boldsymbol{u}, p)n_j v_i\,ds. & \end{aligned} \tag{3}$$

From this we can obtain the second Green's formula

$$\begin{aligned} \sum_{i=1}^{2}\int_\Omega\left[\left(\nu\Delta u_i-\frac{\partial p}{\partial x_i}\right)v_i-\left(\nu\Delta v_i-\frac{\partial q}{\partial x_i}\right)u_i\right]dx-\sum_{i=1}^{2}\int_\Omega\left(p\frac{\partial v_i}{\partial x_i}-q\frac{\partial u_i}{\partial x_i}\right)dx & \\ =\sum_{i,j=1}^{2}\int_\Gamma[\sigma_{ij}(\boldsymbol{u}, p)n_j v_i-\sigma_{ij}(\boldsymbol{v}, q)n_j u_i]ds. & \end{aligned} \tag{4}$$

Now by putting in (4) $(\boldsymbol{u}, p)=(\boldsymbol{u}(x), p(x))$, the solutions of the Stokes problem, and $(\boldsymbol{v}, q)=(\boldsymbol{G}_1(x, x'), Q_1(x, x'))$ or $(\boldsymbol{G}_2(x, x'), Q_2(x, x'))$ respectively, where $\boldsymbol{G}_1=(G_{11}, G_{12})$, $\boldsymbol{G}_2=(G_{21}, G_{22})$, $G_{ij}(x, x')$ and $Q_i(x, x')$, $i, j=1, 2$, are Green's functions of the Stokes problem in domain Ω, which satisfy

$$\begin{cases} -\nu\Delta G_{ij}(x, x')+\dfrac{\partial}{\partial x_j}Q_i(x, x')=\delta_{ij}\delta(x-x'), & i, j=1,\ 2, \\ \displaystyle\sum_{j=1}^{2}\frac{\partial}{\partial x_j}G_{ij}(x, x')=0, & i=1,\ 2, \\ G_{ij}(x, x')_{x\in\Gamma}=0, & i, j=1,\ 2, \end{cases} \tag{5}$$

where $\delta(x-x')$ is the Dirac delta function, we obtain the Poisson integral formula for the Stokes problem in Ω:

$$\boldsymbol{u}=-\int_\Gamma\begin{bmatrix} \boldsymbol{g}(\boldsymbol{G}_1, Q_1)^T \\ \boldsymbol{g}(\boldsymbol{G}_2, Q_2)^T \end{bmatrix}\boldsymbol{u}_0\,ds, \tag{6}$$

where

$$\boldsymbol{g}(\boldsymbol{G}_i, Q_i)=\begin{bmatrix} \sigma_{11}(\boldsymbol{G}_i, Q_i) & \sigma_{12}(\boldsymbol{G}_i, Q_i) \\ \sigma_{21}(\boldsymbol{G}_i, Q_i) & \sigma_{22}(\boldsymbol{G}_i, Q_i) \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_\Gamma, \quad i=1,\ 2.$$

From (6) and the corresponding formula for p, we can get the relation between $\boldsymbol{g}=(g_1, g_2)^T$ and $\boldsymbol{u}_0$:

$$\boldsymbol{g}=\mathscr{K}\boldsymbol{u}_0, \tag{7}$$

which is called the canonical integral equation for the Stokes problem in Ω. Then the solution of the boundary value problem (1) is given by (6), and the boundary value problem (2) is reduced to the canonical integral equation (7), where $\mathscr{K}$ is the corresponding integral operator.

Let

$$D(\boldsymbol{u}, \boldsymbol{v}) = 2\nu \sum_{i,j=1}^{2} \int_{\Omega} \varepsilon_{ij}(\boldsymbol{u}) \varepsilon_{ij}(\boldsymbol{v}) dx,$$

$$\hat{D}(\boldsymbol{u}_0, \boldsymbol{v}_0) = \int_{\Gamma} \boldsymbol{v}_0 \cdot \mathscr{K} \boldsymbol{u}_0 ds.$$

Then from (3) we immediately obtain an important equality

$$D(\boldsymbol{u}, \boldsymbol{v}) = \hat{D}(\boldsymbol{u}_0, \boldsymbol{v}_0), \tag{8}$$

where $\boldsymbol{u}_0 = \boldsymbol{u}|_\Gamma$, $\boldsymbol{v}_0 = \boldsymbol{v}|_\Gamma$, $(\boldsymbol{u}, p)$ satisfy the Stokes equation, and $\boldsymbol{v}$ satisfies the continuity equation. From (8) we know that through the canonical reduction the value of energy functional is invariable.

§ 3. Representation of Solution via Complex Variable Functions

As known, the harmonic function, the biharmonic function and the solution of a plane elasticity problem have their representations via complex variable functions[10]. The object of this section is to find the representation of solution for the steady Stokes problem via two complex variable functions.

From equations

$$\begin{cases} -\nu \Delta \boldsymbol{u} + \operatorname{grad} p = 0, \\ \operatorname{div} \boldsymbol{u} = 0, \end{cases} \quad \text{in } \Omega \tag{9}$$

we can easily obtain

$$\Delta^2 \boldsymbol{u} = 0, \quad \text{in } \Omega \tag{10}$$

and

$$\Delta p = 0, \quad \text{in } \Omega. \tag{11}$$

Then we can let

$$\begin{cases} u_1(x, y) = \operatorname{Re}[\varphi_1(z)\bar{z} + \psi_1(z)], \\ u_2(x, y) = \operatorname{Re}[\varphi_2(z)\bar{z} + \psi_2(z)], \\ p(x, y) = \operatorname{Re} \chi(z), \end{cases} \tag{12}$$

where $\varphi_1(z)$, $\varphi_2(z)$, $\psi_1(z)$, $\psi_2(z)$, $\chi(z)$ are analytic functions in Ω, $z = x + iy$. Substituting (12) into (9), we get

$$\begin{cases} \operatorname{Re}[\chi'(z) - 4\nu\varphi_1'(z)] = 0, \\ \operatorname{Re}[i\chi'(z) - 4\nu\varphi_2'(z)] = 0, \\ \operatorname{Re}\{[\varphi_1'(z) + i\varphi_2'(z)]\bar{z} + \psi_1'(z) + \varphi_1(z) + i\psi_2'(z) - i\varphi_2(z)\} = 0. \end{cases} \tag{13}$$

Observing that $\chi'(z) - 4\nu\varphi_1'(z)$ and $i\chi'(z) - 4\nu\varphi_2'(z)$ are also analytic functions, from the Cauchy-Riemann conditions, we obtain

$$\begin{cases} \chi'(z) - 4\nu\varphi_1'(z) = i\alpha, \\ i\chi'(z) - 4\nu\varphi_2'(z) = i\beta, \end{cases} \tag{14}$$

where α and β are real constants. From this we get

$$\mathrm{Re}\{[\varphi_1'(z)+i\varphi_2'(z)]\bar{z}\}=\frac{1}{4\nu}\mathrm{Re}[(\beta+i\alpha)z].$$

Then we also have

$$\psi_1'(z)+i\psi_2'(z)+\varphi_1(z)-i\varphi_2(z)+\frac{1}{4\nu}(\beta+i\alpha)z=i\gamma, \tag{15}$$

where γ also is a real constant. From (14) we further get

$$\begin{cases}\varphi_1(z)=\dfrac{1}{4\nu}[\chi(z)-i\alpha z]+a,\\[2mm] \varphi_2(z)=\dfrac{1}{4\nu}[i\chi(z)-i\beta z]+b,\end{cases} \tag{16}$$

where a and b are complex constants. Substituting (16) into (15), we obtain

$$\chi(z)=-2\nu[\psi_1'(z)+i\psi_2'(z)]+2\nu(ib+i\gamma-a). \tag{17}$$

Let

$$\tilde{\varphi}(z)=[\psi_1(z)+i\psi_2(z)]/2,\quad \tilde{\psi}(z)=[-\psi_1(z)+i\psi_2(z)]/2.$$

Combining with (17) and (16), we have

$$\begin{cases}\psi_1(z)=\tilde{\varphi}(z)-\tilde{\psi}(z),\\ \psi_2(z)=-i[\tilde{\varphi}(z)+\tilde{\psi}(z)],\\ \chi(z)=-4\nu\tilde{\varphi}'(z)+2\nu(ib+i\gamma-a),\\ \varphi_1(z)=-\tilde{\varphi}'(z)-i\dfrac{\alpha}{4\nu}z+\dfrac{1}{2}(ib+i\gamma+a),\\[2mm] \varphi_2(z)=-i\tilde{\varphi}'(z)-i\dfrac{\beta}{4\nu}z+\dfrac{1}{2}(b-\gamma-ia).\end{cases} \tag{18}$$

Then from (12) we obtain

$$\begin{cases}u_1(x,y)=\mathrm{Re}\left[-\tilde{\varphi}'(z)\bar{z}+\tilde{\varphi}(z)-\tilde{\psi}(z)+\dfrac{1}{2}(\bar{a}-i\bar{b}-i\gamma)z\right],\\[2mm] u_2(x,y)=\mathrm{Im}\left[\tilde{\varphi}'(z)\bar{z}+\tilde{\varphi}(z)+\tilde{\psi}(z)+\dfrac{1}{2}(i\bar{b}-\bar{a}-i\gamma)z\right],\\[2mm] p(x,y)=-4\nu\,\mathrm{Re}\left[\tilde{\varphi}'(z)-\dfrac{1}{2}(ib-a)\right].\end{cases} \tag{19}$$

At last, let

$$\varphi(z)=\tilde{\varphi}(z)+\left[\frac{1}{2}\mathrm{Re}(a-ib)-\frac{i}{4}\gamma\right]z,$$

$$\psi(z)=\tilde{\psi}(z)-\frac{1}{2}(\bar{a}-i\bar{b})z,$$

we immediately obtain the representation of solution via two complex variable functions

$$\begin{cases}u_1(x,y)=\mathrm{Re}[-\varphi'(z)\bar{z}+\varphi(z)-\psi(z)],\\ u_2(x,y)=\mathrm{Im}[\varphi'(z)\bar{z}+\varphi(z)+\psi(z)],\\ p(x,y)=-4\nu\,\mathrm{Re}\,\varphi'(z),\end{cases} \tag{20}$$

where $\varphi(z)$ and $\psi(z)$ are analytic functions in Ω. It can be easily verified that $\boldsymbol{u}$ and p given by (20) satisfy the system of equations (9).

From (20) we can further obtain

$$\begin{cases} \sigma_{11}(x, y) = 2\nu \operatorname{Re}[2\varphi'(z) - \varphi''(z)\bar{z} - \psi'(z)], \\ \sigma_{22}(x, y) = 2\nu \operatorname{Re}[2\varphi'(z) + \varphi''(z)\bar{z} + \psi'(z)], \\ \sigma_{12}(x, y) = \sigma_{21}(x, y) = 2\nu \operatorname{Im}[\varphi''(z)\bar{z} + \psi'(z)]. \end{cases} \tag{21}$$

(20) and (21) are very similar to the corresponding results for the plane elasticity problem[8,10].

§ 4. The Canonical Integral Equation for the Upper Half-Plane

4.1. The Fourier transform method.

Take the Fourier transform for $x \to \xi$, and let

$$U_i(\xi, y) = \mathscr{F}[u_i(x, y)], \quad i = 1, 2, \quad P(\xi, y) = \mathscr{F}[p(x, y)],$$

the system (9) becomes

$$\begin{cases} \dfrac{d^2}{dy^2} U_1 - \xi^2 U_1 - i\xi \dfrac{P}{\nu} = 0, \\ \dfrac{d^2}{dy^2} U_2 - \xi^2 U_2 - \dfrac{1}{\nu} \dfrac{\partial P}{\partial y} = 0, \\ i\xi U_1 + \dfrac{dU_2}{dy} = 0. \end{cases}$$

Solving this system of ODE with parameter ξ and observing that $y > 0$, we can obtain

$$\begin{cases} U_1(\xi, y) = \dfrac{i}{\xi}\{\beta(\xi) - |\xi|[\alpha(\xi) + \beta(\xi)y]\}e^{-|\xi|y}, \\ U_2(\xi, y) = [\alpha(\xi) + \beta(\xi)y]e^{-|\xi|y}, \\ P(\xi, y) = 2\nu\beta(\xi)e^{-|\xi|y}, \end{cases} \tag{22}$$

where $\alpha(\xi)$ and $\beta(\xi)$ remain to be determined. Substituting $y = 0$ into (22), we get

$$\begin{cases} \alpha(\xi) = U_2(\xi, 0), \\ \beta(\xi) = -i\xi U_1(\xi, 0) + |\xi| U_2(\xi, 0). \end{cases} \tag{23}$$

Then

$$\begin{cases} U_1(\xi, y) = [U_1(\xi, 0) - |\xi| y U_1(\xi, 0) - i\xi y U_2(\xi, 0)]e^{-|\xi|y}, \\ U_2(\xi, y) = [-i\xi y U_1(\xi, 0) + U_2(\xi, 0) + |\xi| y U_2(\xi, 0)]e^{-|\xi|y}, \\ P(\xi, y) = 2\nu[-i\xi U_1(\xi, 0) + |\xi| U_2(\xi, 0)]e^{-|\xi|y}. \end{cases} \tag{24}$$

Using the Fourier transform formulas

$$\begin{cases} \mathscr{F}\left[\dfrac{y}{\pi(x^2+y^2)}\right] = e^{-|\xi|y}, \\ \mathscr{F}\left[-\dfrac{2xy}{\pi(x^2+y^2)^2}\right] = i\xi e^{-|\xi|y}, \\ \mathscr{F}\left[\dfrac{x^2-y^2}{\pi(x^2+y^2)^2}\right] = -|\xi| e^{-|\xi|y}, \end{cases}$$

taking the Fourier inverse transform of (24), we immediately obtain the Poisson integral formula

$$\begin{cases} u_1(x,\ y)=\dfrac{2x^2y}{\pi(x^2+y^2)^2}*u_1(x,\ 0)+\dfrac{2xy^2}{\pi(x^2+y^2)^2}*u_2(x,\ 0), \\ u_2(x,\ y)=\dfrac{2xy^2}{\pi(x^2+y^2)^2}*u_1(x,\ 0)+\dfrac{2y^3}{\pi(x^2+y^2)^2}*u_2(x,\ 0); \qquad y>0, \\ p(x,\ y)=2\nu\left[\dfrac{2xy}{\pi(x^2+y^2)^2}*u_1(x,\ 0)+\dfrac{y^2-x^2}{\pi(x^2+y^2)^2}*u_2(x,\ 0)\right], \end{cases} \tag{25}$$

Moreover, observing

$$\begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix}=\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix}_\Gamma=\begin{bmatrix} -\nu\left(\dfrac{\partial u_1}{\partial y}+\dfrac{\partial u_2}{\partial x}\right) \\ p-2\nu\dfrac{\partial u_2}{\partial y} \end{bmatrix}_\Gamma,$$

and taking its Fourier transform and substituting (24) into it, we obtain

$$\begin{bmatrix} \mathscr{F}[g_1] \\ \mathscr{F}[g_2] \end{bmatrix}=2\nu\begin{bmatrix} |\xi|U_1(\xi,\ 0) \\ |\xi|U_2(\xi,\ 0) \end{bmatrix}. \tag{26}$$

Using the Fourier transform formula

$$\mathscr{F}\left[-\frac{1}{\pi x^2}\right]=|\xi|,$$

taking the Fourier inverse transform of (26), we get the canonical integral equation

$$\begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix}=2\nu\begin{bmatrix} -\dfrac{1}{\pi x^2} & 0 \\ 0 & -\dfrac{1}{\pi x^2} \end{bmatrix}*\begin{bmatrix} u_1(x,\ 0) \\ u_2(x,\ 0) \end{bmatrix}. \tag{27}$$

Applying the limit formulas of generalized functions when $y\to 0_+$[11], equation (27) also can be obtained from (25) by differentiation.

4.2. The method of complex analysis.

Substituting $y=0$ into (20), the representation of solution for the Stokes problem, we obtain

$$\begin{cases} u_1(x,\ 0)=\mathrm{Re}\,[-\varphi'(x)x+\varphi(x)-\psi(x)]=\mathrm{Re}\,F(z)\,|_{y=0}, \\ u_2(x,\ 0)=\mathrm{Im}\,[\varphi'(x)x+\varphi(x)+\psi(x)]=\mathrm{Im}\,G(z)\,|_{y=0}, \\ p(x,\ 0)=-4\nu\,\mathrm{Re}\,\varphi'(x), \end{cases}$$

where $\qquad F(z)=-\varphi'(z)z+\varphi(z)-\psi(z),\quad G(z)=\varphi'(z)z+\varphi(z)+\psi(z).$

Moreover, from (21) we have

$$\begin{cases} g_1(x)=-\sigma_{12}(x,\ 0)=-2\nu\,\mathrm{Im}[\varphi''(x)x+\psi'(x)], \\ g_2(x)=-\sigma_{22}(x,\ 0)=-2\nu\,\mathrm{Re}[\varphi''(x)x+2\varphi'(x)+\psi'(x)]. \end{cases} \tag{28}$$

Since $F(z)$ and $G(z)$ are analytic functions in Ω, we can use the harmonic canonical integral equation[1,2] and obtain

$$-\frac{1}{\pi x^2}*u_1(x,\ 0)=\left[\frac{\partial}{\partial n}\mathrm{Re}\,F(z)\right]_{y=0}=-\mathrm{Re}\frac{\partial}{\partial y}F(z)\,|_{y=0}=-\mathrm{Re}\,iF'(z)\,|_{y=0}$$
$$=\mathrm{Im}\,F'(z)\,|_{y=0}=-\mathrm{Im}[\varphi''(x)x+\psi'(x)],$$

$$-\frac{1}{\pi x^2}*u_2(x,\ 0)=\left[\frac{\partial}{\partial n}\mathrm{Im}\,G(z)\right]_{y=0}=-\mathrm{Re}\frac{\partial}{\partial x}G(z)\,|_{y=0}=-\mathrm{Re}\,G'(z)\,|_{y=0}$$
$$=-\mathrm{Re}[\varphi''(x)x+2\varphi'(x)+\psi'(x)].$$

Comparing with (28), we immediately get the canonical integral equation (27).

§ 5. The Canonical Integral Equation for the Exterior Circular Domain

For the sake of simplicity we let $R=1$, where R is the radius of the circle.

5.1. The Fourier expansion method.

From Section 3 we know that the solution of the steady Stokes problem can be represented by (20). Since $\varphi(z)$ and $\psi(z)$ are analytic functions in the exterior unit circular domain, we can let

$$\begin{cases} \varphi(z)=\sum_{0}^{\infty}\alpha_{-n}z^{-n}, \\ \psi(z)=\sum_{0}^{\infty}\beta_{-n}z^{-n}, \end{cases} \tag{29}$$

where α_{-n}, β_{-n} are complex coefficients, and let $\alpha_n=\bar{\alpha}_{-n}$, $\beta_n=\bar{\beta}_{-n}$, $n=1, 2, \cdots$. Substituting (29) into (20), we have

$$\begin{cases} u_1(r,\theta)=\mathrm{Re}(\alpha_0-\beta_0)+\frac{1}{2r}[(\alpha_{-1}-\beta_{-1})e^{-i\theta}+(\alpha_1-\beta_1)e^{i\theta}] \\ \qquad +\frac{1}{2}\sum_{\substack{-\infty \\ n\neq 0,\pm1}}^{\infty}[(|n|-2)\alpha_{(|n|-2)\mathrm{sign}\, n}r^2+(\alpha_n-\beta_n)]r^{-|n|}e^{in\theta}, \\ u_2(r,\theta)=\mathrm{Im}(\alpha_0+\beta_0)+\frac{i}{2r}[(\alpha_1+\beta_1)e^{i\theta}-(\alpha_{-1}+\beta_{-1})e^{-i\theta}] \\ \qquad +\frac{i}{2}\sum_{\substack{-\infty \\ n\neq 0,\pm1}}^{\infty}[(2-|n|)\alpha_{(|n|-2)\mathrm{sign}\, n}r^2+\alpha_n+\beta_n](\mathrm{sign}\, n)r^{-|n|}e^{in\theta}, \\ p(r,\theta)=2\nu\sum_{\substack{-\infty \\ n\neq 0}}^{\infty}(|n|-1)\alpha_{(|n|-1)\mathrm{sign}\, n}r^{-|n|}e^{in\theta}. \end{cases} \tag{30}$$

Then

$$\begin{cases} u_1(1,\theta)=\mathrm{Re}(\alpha_0-\beta_0)+\frac{1}{2}[(\alpha_{-1}-\beta_{-1})e^{-i\theta}+(\alpha_1-\beta_1)e^{i\theta}] \\ \qquad +\sum_{\substack{-\infty \\ n\neq 0,\pm1}}^{\infty}\frac{1}{2}[(|n|-2)\alpha_{(|n|-2)\mathrm{sign}\, n}+\alpha_n-\beta_n]e^{in\theta}, \\ u_2(1,\theta)=\mathrm{Im}(\alpha_0+\beta_0)+\frac{i}{2}[(\alpha_1+\beta_1)e^{i\theta}-(\alpha_{-1}+\beta_{-1})e^{-i\theta}] \\ \qquad +\sum_{\substack{-\infty \\ n\neq 0,\pm1}}^{\infty}\frac{i\,\mathrm{sign}\, n}{2}[(2-|n|)\alpha_{(|n|-2)\mathrm{sign}\, n}+\alpha_n+\beta_n]e^{in\theta}. \end{cases} \tag{31}$$

Moreover, substituting (29) into (21) and observing $\boldsymbol{n}=(-\cos\theta, -\sin\theta)^T$, we obtain

$$\begin{cases} g_1(\theta)=\nu\left\{[(\alpha_{-1}-\beta_{-1})e^{-i\theta}+(\alpha_1-\beta_1)e^{i\theta}]+\sum_{\substack{-\infty \\ n\neq 0,\pm1}}^{\infty}|n|[(|n|-2)\alpha_{(|n|-2)\mathrm{sign}\, n}+\alpha_n-\beta_n]e^{in\theta}\right\}, \\ g_2(\theta)=i\nu\left\{(\alpha_1+\beta_1)e^{i\theta}-(\alpha_{-1}+\beta_{-1})e^{-i\theta}+\sum_{\substack{-\infty \\ n\neq 0,\pm1}}^{\infty}n[(2-|n|)\alpha_{(|n|-2)\mathrm{sign}\, n}+\alpha_n+\beta_n]e^{in\theta}\right\}. \end{cases}$$

Comparing with (31), we find the canonical integral equation

$$\begin{bmatrix} g_1(\theta) \\ g_2(\theta) \end{bmatrix} = 2\nu \begin{bmatrix} -\dfrac{1}{4\pi \sin^2 \dfrac{\theta}{2}} & 0 \\ 0 & -\dfrac{1}{4\pi \sin^2 \dfrac{\theta}{2}} \end{bmatrix} * \begin{bmatrix} u_1(1, \theta) \\ u_2(1, \theta) \end{bmatrix}. \tag{32}$$

Let

$$\begin{aligned} u_1(1, \theta) &= \sum_{-\infty}^{\infty} a_n e^{in\theta}, \quad a_{-n} = \bar{a}_n, \\ u_2(1, \theta) &= \sum_{-\infty}^{\infty} b_n e^{in\theta}, \quad b_{-n} = \bar{b}_n, \end{aligned} \qquad n = 0, 1, 2, \cdots$$

where a_0, b_0 are real and a_i, b_i, $i \neq 0$, are complex. Comparing with (31), we have

$$\begin{cases} \operatorname{Re}(\alpha_0 - \beta_0) = a_0, \quad \operatorname{Im}(\alpha_0 + \beta_0) = b_0, \\ \alpha_1 = a_1 - ib_1, \quad \beta_1 = -(a_1 + ib_1), \\ \alpha_n = a_n - ib_n, \\ \beta_n = [(n-2)a_{n-2} - a_n] - i[b_n + (n-2)b_{n-2}], \quad n = 2, 3, \cdots. \end{cases} \tag{33}$$

Substitute it into (30); then

$$\begin{cases} u_1(r, \theta) = \sum_{-\infty}^{\infty} a_n r^{-|n|} e^{in\theta} + \left(1 - \dfrac{1}{r^2}\right)\left\{\dfrac{\cos 2\theta}{2} \sum_{-\infty}^{\infty} (|n| a_n - inb_n) r^{-|n|} e^{in\theta} \right. \\ \qquad \left. + \dfrac{\sin 2\theta}{2} \sum_{-\infty}^{\infty} (ina_n + |n| b_n) r^{-|n|} e^{in\theta} \right\}, \\ u_2(r, \theta) = \sum_{-\infty}^{\infty} b_n r^{-|n|} e^{in\theta} + \left(1 - \dfrac{1}{r^2}\right)\left\{\dfrac{\sin 2\theta}{2} \sum_{-\infty}^{\infty} (|n| a_n - in b_n) r^{-|n|} e^{in\theta} \right. \\ \qquad \left. - \dfrac{\cos 2\theta}{2} \sum_{-\infty}^{\infty} (ina_n + |n| b_n) r^{-|n|} e^{in\theta} \right\}, \\ p(r, \theta) = \dfrac{2\nu}{r}\left\{\cos\theta \sum_{-\infty}^{\infty} (|n| a_n - in b_n) r^{-|n|} e^{in\theta} \right. \\ \qquad \left. + \sin\theta \sum_{-\infty}^{\infty} (ina_n + |n| b_n) r^{-|n|} e^{in\theta} \right\}. \end{cases} \tag{34}$$

At last, using the formula

$$\frac{1}{2\pi} \sum_{-\infty}^{\infty} r^{-|n|} e^{in\theta} = \frac{r^2 - 1}{2\pi(1 + r^2 - 2r\cos\theta)} \equiv P(r, \theta), \quad r > 1,$$

where $P(r, \theta)$ is the Poisson integral kernel for the harmonic equation in the exterior unit circular domain, we obtain the Poisson integral formula for the Stokes problem in Ω

$$\begin{cases} u_1(r, \theta) = P(r, \theta) * u_1(1, \theta) + \dfrac{r^2 - 1}{2r^2}\left\{ \cos 2\theta \left[\left(-r \dfrac{\partial}{\partial r} P(r, \theta)\right) * u_1(1, \theta) \right.\right. \\ \qquad \left. - \dfrac{\partial}{\partial \theta} P(r, \theta) * u_2(1, \theta)\right] + \sin 2\theta \left[\dfrac{\partial}{\partial \theta} P(r, \theta) * u_1(1, \theta) \right. \\ \qquad \left.\left. + \left(-r \dfrac{\partial}{\partial r} P(r, \theta)\right) * u_2(1, \theta)\right]\right\}, \\ u_2(r, \theta) = P(r, \theta) * u_2(1, \theta) + \dfrac{r^2 - 1}{2r^2}\left\{ \sin 2\theta \left[\left(-r \dfrac{\partial}{\partial r} P(r, \theta)\right) * u_1(1, \theta) \right.\right. \end{cases} \tag{35}$$

$$
\begin{cases}
\quad -\frac{\partial}{\partial\theta}P(r,\theta)*u_2(1,\theta)\Big]-\cos 2\theta\Big[\frac{\partial}{\partial\theta}P(r,\theta)*u_1(1,\theta) \\
\quad +\Big(-r\frac{\partial}{\partial r}P(r,\theta)\Big)*u_2(1,\theta)\Big]\Big\}, \\
p(r,\theta)=\frac{2\nu}{r}\Big\{\cos\theta\Big[\Big(-r\frac{\partial}{\partial r}P(r,\theta)\Big)*u_1(1,\theta)-\frac{\partial}{\partial\theta}P(r,\theta)*u_2(1,\theta)\Big] \\
\quad +\sin\theta\Big[\frac{\partial}{\partial\theta}P(r,\theta)*u_1(1,\theta)+\Big(-r\frac{\partial}{\partial r}P(r,\theta)\Big)*u_2(1,\theta)\Big]\Big\}.
\end{cases}
$$

5.2. The method of complex analysis.

From the representation (20) we can obtain

$$
\begin{cases}
u_1(1,\theta)=\mathrm{Re}\{-\varphi'(e^{i\theta})e^{-i\theta}+\varphi(e^{i\theta})-\psi(e^{i\theta})\}=\mathrm{Re}\,F(z)\,|_{r=1}, \\
u_2(1,\theta)=\mathrm{Im}\{\varphi'(e^{i\theta})e^{-i\theta}+\varphi(e^{i\theta})+\psi(e^{i\theta})\}=\mathrm{Im}\,G(z)\,|_{r=1}, \\
p(1,\theta)=-4\nu\,\mathrm{Re}\,\varphi'(e^{i\theta}),
\end{cases}
$$

where $\quad F(z)=-\varphi'(z)\frac{1}{z}+\varphi(z)-\psi(z),\qquad G(z)=\varphi'(z)\frac{1}{z}+\varphi(z)+\psi(z).$

Moreover, from (21) and $\boldsymbol{n}=(-\cos\theta,\ -\sin\theta)^T$, we have

$$
\begin{cases}
g_1(\theta)=2\nu\,\mathrm{Re}[\varphi''(e^{i\theta})-e^{-i\theta}\varphi'(e^{i\theta})-e^{i\theta}\varphi'(e^{i\theta})+e^{i\theta}\psi'(e^{i\theta})], \\
g_2(\theta)=-2\nu\,\mathrm{Im}[\varphi''(e^{i\theta})-e^{-i\theta}\varphi'(e^{i\theta})+e^{i\theta}\varphi'(e^{i\theta})+e^{i\theta}\psi'(e^{i\theta})].
\end{cases} \tag{36}
$$

Since $F(z)$ and $G(z)$ are analytic functions in Ω, we can use the harmonic canonical integral equation[1,2] and get

$$
-\frac{1}{4\pi\sin^2\frac{\theta}{2}}*u_1(1,\theta)=-\frac{\partial}{\partial r}\mathrm{Re}\,F(z)\Big|_{r=1}=-\mathrm{Re}\frac{\partial}{\partial r}F(z)\Big|_{r=1}
$$

$$
=-\mathrm{Re}[e^{i\theta}F'(z)]_{r=1}=\mathrm{Re}[\varphi''(e^{i\theta})-e^{-i\theta}\varphi'(e^{i\theta})-e^{i\theta}\varphi'(e^{i\theta})+e^{i\theta}\psi'(e^{i\theta})],
$$

$$
-\frac{1}{4\pi\sin^2\frac{\theta}{2}}*u_2(1,\theta)=-\frac{\partial}{\partial r}\mathrm{Im}\,G(z)\Big|_{r=1}=-\mathrm{Im}\frac{\partial}{\partial r}G(z)\Big|_{r=1}
$$

$$
=-\mathrm{Im}[e^{i\theta}G'(z)]_{r=1}=-\mathrm{Im}[\varphi''(e^{i\theta})-e^{-i\theta}\varphi'(e^{i\theta})+e^{i\theta}\varphi'(e^{i\theta})+e^{i\theta}\psi'(e^{i\theta})].
$$

Comparing with (36), we immediately find the canonical integral equation (32).

5.3. The canonical integral equation and the Poisson integral formula in polar decomposition.

Let

$$
\boldsymbol{u}=u_1\boldsymbol{e}_x+u_2\boldsymbol{e}_y=u_r\boldsymbol{e}_r+u_\theta\boldsymbol{e}_\theta,
$$

$$
\boldsymbol{g}=g_1\boldsymbol{e}_x+g_2\boldsymbol{e}_y=g_r\boldsymbol{e}_r+g_\theta\boldsymbol{e}_\theta.
$$

Using the formulas

$$
u_r=u_1\cos\theta+u_2\sin\theta,
$$

$$
u_\theta=-u_1\sin\theta+u_2\cos\theta,
$$

and

$$
g_r=g_1\cos\theta+g_2\sin\theta,
$$

$$
g_\theta=-g_1\sin\theta+g_2\cos\theta,
$$

we can easily obtain the canonical integral equation and the Poisson integral formula for the Stokes problem in polar decomposition from (32) and (35)

$$\begin{bmatrix} g_r(\theta) \\ g_\theta(\theta) \end{bmatrix} = 2\nu \begin{bmatrix} -\dfrac{1}{4\pi \sin^2 \dfrac{\theta}{2}} + \dfrac{1}{2\pi} & -\dfrac{1}{2\pi} \operatorname{ctg} \dfrac{\theta}{2} \\ \dfrac{1}{2\pi} \operatorname{ctg} \dfrac{\theta}{2} & -\dfrac{1}{4\pi \sin^2 \dfrac{\theta}{2}} + \dfrac{1}{2\pi} \end{bmatrix} * \begin{bmatrix} u_r(1, \theta) \\ u_\theta(1, \theta) \end{bmatrix} \tag{37}$$

and

$$\left\{ \begin{aligned} u_r(r, \theta) &= \Big\{ \cos\theta\, P(r, \theta) + \frac{r^2-1}{2r^2} \Big[\cos\theta \Big(-r \frac{\partial}{\partial r} P(r, \theta) \Big) \\ &\quad + \sin\theta \frac{\partial}{\partial \theta} P(r, \theta) \Big] \Big\} * u_r(1, \theta) \\ &\quad + \Big\{ \sin\theta\, P(r, \theta) + \frac{r^2-1}{2r^2} \Big[\sin\theta \Big(-r \frac{\partial}{\partial r} P(r, \theta) \Big) \\ &\quad - \cos\theta \frac{\partial}{\partial \theta} P(r, \theta) \Big] \Big\} * u_\theta(1, \theta), \\ u_\theta(r, \theta) &= \Big\{ -\sin\theta\, P(r, \theta) + \frac{r^2-1}{2r^2} \Big[\sin\theta \Big(-r \frac{\partial}{\partial r} P(r, \theta) \Big) \\ &\quad - \cos\theta \frac{\partial}{\partial \theta} P(r, \theta) \Big] \Big\} * u_r(1, \theta) \\ &\quad + \Big\{ \cos\theta\, P(r, \theta) - \frac{r^2-1}{2r^2} \Big[\cos\theta \Big(-r \frac{\partial}{\partial r} P(r, \theta) \Big) \\ &\quad + \sin\theta \frac{\partial}{\partial \theta} P(r, \theta) \Big] \Big\} * u_\theta(1, \theta), \\ p(r, \theta) &= \frac{2\nu}{r} \Big\{ \Big[\cos\theta \Big(-r \frac{\partial}{\partial r} P(r, \theta) \Big) + \sin\theta \frac{\partial}{\partial \theta} P(r, \theta) \Big] * u_r(1, \theta) \\ &\quad + \Big[\sin\theta \Big(-r \frac{\partial}{\partial r} P(r, \theta) \Big) - \cos\theta \frac{\partial}{\partial \theta} P(r, \theta) \Big] * u_\theta(1, \theta) \Big\}, \end{aligned} \right. \tag{38}$$

where
$$P(r, \theta) = \frac{r^2-1}{2\pi(1+r^2-2r\cos\theta)}.$$

§ 6. The Canonical Integral Equation for the Interior Circular Domain

Still let $R=1$. By the same methods as used in the last section, we can obtain the following results:

$$\begin{bmatrix} g_1(\theta) \\ g_2(\theta) \end{bmatrix} = 2\nu \begin{bmatrix} -\dfrac{1}{4\pi \sin^2 \dfrac{\theta}{2}} & 0 \\ 0 & -\dfrac{1}{4\pi \sin^2 \dfrac{\theta}{2}} \end{bmatrix} * \begin{bmatrix} u_1(1, \theta) \\ u_2(1, \theta) \end{bmatrix}$$

$$+ \begin{bmatrix} \dfrac{\nu \sin\theta}{\pi} \displaystyle\int_0^{2\pi} (-u_1(1, \theta)\sin\theta + u_2(1, \theta)\cos\theta)\, d\theta - \dfrac{\cos\theta}{2\pi} \displaystyle\int_0^{2\pi} p(1, \theta)\, d\theta \\ \dfrac{\nu \cos\theta}{\pi} \displaystyle\int_0^{2\pi} (u_1(1, \theta)\sin\theta - u_2(1, \theta)\cos\theta)\, d\theta - \dfrac{\sin\theta}{2\pi} \displaystyle\int_0^{2\pi} p(1, \theta)\, d\theta \end{bmatrix}, \tag{39}$$

$$
\begin{cases}
u_1(r,\theta)=P(r,\theta)*u_1(1,\theta)+\dfrac{1-r^2}{2r^2}\Big\{\cos 2\theta\Big[\Big(r\dfrac{\partial}{\partial r}P(r,\theta)\Big)*u_1(1,\theta) \\
\qquad +\dfrac{\partial}{\partial\theta}P(r,\theta)*u_2(1,\theta)\Big] \\
\qquad +\sin 2\theta\Big[-\dfrac{\partial}{\partial\theta}P(r,\theta)*u_1(1,\theta)+\Big(r\dfrac{\partial}{\partial r}P(r,\theta)\Big)*u_2(1,\theta)\Big]\Big\} \\
\qquad -\dfrac{1-r^2}{2\pi r}\displaystyle\int_0^{2\pi}[u_1(1,\theta')\cos(\theta+\theta')+u_2(1,\theta')\sin(\theta+\theta')]d\theta', \\
u_2(r,\theta)=P(r,\theta)*u_2(1,\theta)+\dfrac{1-r^2}{2r^2}\Big\{\sin 2\theta\Big[\Big(r\dfrac{\partial}{\partial r}P(r,\theta)\Big)*u_1(1,\theta) \\
\qquad +\dfrac{\partial}{\partial\theta}P(r,\theta)*u_2(1,\theta)\Big] \\
\qquad -\cos 2\theta\Big[-\dfrac{\partial}{\partial\theta}P(r,\theta)*u_1(1,\theta)+\Big(r\dfrac{\partial}{\partial r}P(r,\theta)\Big)*u_2(1,\theta)\Big]\Big\} \\
\qquad +\dfrac{1-r^2}{2\pi r}\displaystyle\int_0^{2\pi}[-u_1(1,\theta')\sin(\theta+\theta')+u_2(1,\theta')\cos(\theta+\theta')]d\theta', \\
p(r,\theta)=-\dfrac{2\nu}{r}\Big\{\cos\theta\Big[\Big(r\dfrac{\partial}{\partial r}P(r,\theta)\Big)*u_1(1,\theta)+\dfrac{\partial}{\partial\theta}P(r,\theta)*u_2(1,\theta)\Big] \\
\qquad +\sin\theta\Big[-\dfrac{\partial}{\partial\theta}P(r,\theta)*u_1(1,\theta)+\Big(r\dfrac{\partial}{\partial r}P(r,\theta)\Big)*u_2(1,\theta)\Big]\Big\},
\end{cases}
\tag{40}
$$

where $P(r,\theta)=\dfrac{1-r^2}{2\pi(1+r^2-2r\cos\theta)}$, $r<1$, is the Poisson integral kernel for the harmonic equation in the interior unit circular domain, and

$$
\begin{bmatrix} g_r(\theta) \\ g_\theta(\theta)\end{bmatrix}=2\nu\begin{bmatrix} -\dfrac{1}{4\pi\sin^2\dfrac{\theta}{2}}+\dfrac{1}{2\pi} & -\dfrac{1}{2\pi}\operatorname{ctg}\dfrac{\theta}{2} \\ \dfrac{1}{2\pi}\operatorname{ctg}\dfrac{\theta}{2} & -\dfrac{1}{4\pi\sin^2\dfrac{\theta}{2}}\end{bmatrix}*\begin{bmatrix} u_r(1,\theta) \\ u_\theta(1,\theta)\end{bmatrix}
-\begin{bmatrix}\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}p(1,\theta)d\theta \\ 0\end{bmatrix},
\tag{41}
$$

$$
\begin{cases}
u_r(r,\theta)=\Big\{\cos\theta P(r,\theta)+\dfrac{1-r^2}{2r^2}\Big[\cos\theta\Big(r\dfrac{\partial}{\partial r}P(r,\theta)\Big)-\sin\theta\dfrac{\partial}{\partial\theta}P(r,\theta)\Big] \\
\qquad -\dfrac{1-r^2}{2\pi r}\Big\}*u_r(1,\theta) \\
\qquad +\Big\{\sin\theta P(r,\theta)+\dfrac{1-r^2}{2r^2}\Big[\sin\theta\Big(r\dfrac{\partial}{\partial r}P(r,\theta)\Big) \\
\qquad +\cos\theta\dfrac{\partial}{\partial\theta}P(r,\theta)\Big]\Big\}*u_\theta(1,\theta), \\
u_\theta(r,\theta)=\Big\{-\sin\theta P(r,\theta)+\dfrac{1-r^2}{2r^2}\Big[\sin\theta\Big(r\dfrac{\partial}{\partial r}P(r,\theta)\Big) \\
\qquad +\cos\theta\dfrac{\partial}{\partial\theta}P(r,\theta)\Big]\Big\}*u_r(1,\theta)
\end{cases}
\tag{42}
$$

$$
\begin{cases}
\quad +\left\{\cos\theta\, P(r,\theta)-\dfrac{1-r^2}{2r^2}\left[\cos\theta\left(r\dfrac{\partial}{\partial r}P(r,\theta)\right)\right.\right. \\
\quad \left.\left.-\sin\theta\dfrac{\partial}{\partial\theta}P(r,\theta)\right]+\dfrac{1-r^2}{2\pi r}\right\}*u_\theta(1,\theta), \\
p(r,\theta)=-\dfrac{2\nu}{r}\left\{\left[\cos\theta\left(r\dfrac{\partial}{\partial r}P(r,\theta)\right)-\sin\theta\dfrac{\partial}{\partial\theta}P(r,\theta)\right]*u_r(1,\theta)\right. \\
\quad \left.+\left[\sin\theta\left(r\dfrac{\partial}{\partial r}P(r,\theta)\right)+\cos\theta\dfrac{\partial}{\partial\theta}P(r,\theta)\right]*u_\theta(1,\theta)\right\}.
\end{cases}
$$

The results of this paper can be applied to numerical calculations. From these results the canonical boundary element method for solving steady Stokes problems can be developed. The detail will appear in a forthcoming paper.

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