Vol. 4 No. 1 JOURNAL OF COMPUTATIONAL MATHEMATICS January 1986

NUMERICAL ANALYSIS OF BIFURCATION PROBLEMS OF NONLINEAR EQUATIONS*1)

LI KAI-TAI (李开泰) MEI ZHEN (梅 甄) ZHANG CHENG-DIAN (张承钿)
(Xi'an Jiaotong University, Xi'an, China)

Abstract

The paper presents some essential results of branch solutions of nonlinear problems and their numerical approximation. The general theory is applied to the bifurcation problems of the Navier-Stokes equations.

§ 1. Introduction

The purpose of this paper is to study the bifurcation problems of the nonlinear equation

$$F(\lambda, u)=u+T(\lambda)G(\lambda, u)=0 \tag{1.1}$$

and its discretized form

$$F_h(\lambda, u)=u+T_h(\lambda)G(\lambda, u)=0, \tag{1.2}$$

where we assume that for some Banach spaces V and W, $\{T(\lambda);\ \lambda\in\Lambda\}$ and $\{T_h(\lambda);\ \lambda\in\Lambda\}$ are two families of linear bounded mappings from W into V, h is the discrete parameter which tends to 0, and $G(\lambda, u)$ is a nonlinear mapping from $\Lambda\times V$ into W, Λ being a subset of a Banach space.

We consider the bifurcation of the continuous problem (1.1) and the convergence of its numerical approximations. The outline of the paper is as follows.

Section 2 is devoted to general analysis of singular points of nonlinear mapping F and parameterization of its branch solutions. In Section 3 we discuss the approximation of simple limit points of F. Section 4 deals with the numerical prediction of a singular point of F. The bifurcation problem of the Navier-Stokes equations is considered in Section 5 and Section 6 provides a numerical method for computing its branch solutions.

§ 2. Simple Singular Points

Let V, W be Banach spaces, and Λ a subset of a Banach space. Suppose that

1) G: $\Lambda\times V\to W$ is a C^m $(m\geq 2)$ bounded mapping;

2) T, T_h: $\Lambda\times W\to V$ are C^m bounded mappings with respect to λ and for any fixed $\lambda\in\Lambda$, $T(\lambda)$, $T_h(\lambda)\in L(W, V)$.

Define the mappings F, F_h: $\Lambda\times V\to V$ at follows:

* Received June 1, 1984.

1) Projects Supported by the Science Fund of the Chinese Academy of Sciences.

$$F(\lambda, u)=u+T(\lambda)\cdot G(\lambda, u)+u^*, \qquad (2.1)$$
$$F_h(\lambda, u)=u+T_h(\lambda)\cdot G(\lambda, u)+u^*,$$

where u^* is a given point in V.

Theorem 2.1.[7] *Let Λ be a compact set and $u(\lambda)$: $\Lambda\to V$ be a nonsingular solution of F, i.e.*

1) $F(\lambda, u(\lambda))=0, \quad \forall\lambda\in\Lambda;$

2) $D_uF(\lambda, u(\lambda))$ *is an isomorphism on* V;

3) $u(\lambda)$ *is a* C^m *mapping.*

If in addition the following conditions are satisfied:

$$\text{i)} \ \limsup_{h\to 0,\lambda\in\Lambda}\|D^l_\lambda T_h(\lambda)-D^lT(\lambda)\|=0, \quad 0\leqslant l\leqslant m, \qquad (2.2)$$

$$\text{ii)} \ \sup_{\lambda\in\Lambda}\|D^m_\lambda T_h(\lambda)\|\leqslant C, \quad C \text{ is independent of } h, \qquad (2.3)$$

then there exist constants a, h_0, $K>0$, such that if $h\leqslant h_0$, there is a unique C^m mapping $u_h(\lambda)$: $\Lambda\to V$ satisfying:

$$F_h(\lambda, u_h(\lambda))=0, \quad \forall\lambda\in\Lambda, \qquad (2.4)$$
$$\|u_h(\lambda)-u(\lambda)\|\leqslant a,$$

and

$$\|D^l_\lambda u_h(\lambda^*)-D^lu(\lambda)\|\leqslant K\Big\{|\lambda^*-\lambda|+\sum_{i=0}^{l}\Big\|\frac{d^i}{d\lambda^i}[(T_h(\lambda)-T(\lambda))\cdot G(\lambda, u(\lambda))]\Big\|\Big\},$$
$$\forall\lambda^*, \lambda\in\Lambda, \quad 0\leqslant l\leqslant m-1, \qquad (2.5)$$

where $|\cdot|$ stands for the norm of the Banach space that contains Λ.

Definition. *A pair of $(\lambda_0, u_0)\in\Lambda\times V$ is called a simple singular point of F if (λ_0, u_0) satisfies:*

$$\text{1)} \ F^0=F(\lambda_0, u_0)=0, \qquad (2.6)$$

2) $T(\lambda_0)D_uG(\lambda_0, u_0)$ *is a compact operator and* -1 *is one of its eigenvalues with algebraic multiplicity* 1.

Denote $D_uF^0=D_uF(\lambda_0, u_0)$, and in the sequel V' stands for the dual space of V and $\langle\,,\rangle$ represents the dual pairing between them.

Lemma 2.1. *Let (λ_0, u_0) be a simple singular point of F. Then there are $\{\varphi_i\}_{i=1}^p\subset V$, $\{\varphi_i^*\}_{i=1}^p\subset V'$ ($p\geqslant 1$ integer) such that*

$$D_uF^0\varphi_i=0, \quad \|\varphi_i\|=1, \quad 1\leqslant i\leqslant p,$$
$$V_1\equiv \mathrm{Ker}(D_uF^0)=[\varphi_1, \varphi_2, \cdots, \varphi_p],$$
$$(D_uF)^*\varphi_i^*=0, \quad \langle\varphi_i, \varphi_j^*\rangle=\delta_{ij},$$

and

$$V_2\equiv \mathrm{Range}(D_uF^0)=[\varphi_1^*, \varphi_2^*, \cdots, \varphi_p^*]^\perp,$$
$$V=V_1\dot{+}V_2,$$
$$D_uF^0 \text{ is an isomorphism from } V_2 \text{ onto } V_2,$$

where $[\varphi_1, \varphi_2, \cdots, \varphi_p]$ is a linear space spanned by $\varphi_1, \varphi_2, \cdots, \varphi_p$.

The proof can be found in [10].

For simplicity, we shall write $L=(D_uF^0/V_2)$ as the inverse isomorphism of D_uF^0 on V_2. Let us now define a projection Q: $V\to V_2$ by

$$Qv=v-\sum_{i=1}^{p}\langle v, \varphi_i^*\rangle\varphi_i.$$

Then the equation

$$F(\lambda, u)=0 \tag{2.7}$$

is equivalent to the system

$$\begin{aligned} &QF(\lambda, u)=0, \\ &(I-Q)F(\lambda, u)=0. \end{aligned} \tag{2.8}$$

Given any $(\lambda, u)\in \Lambda\times V$, there exists a unique decomposition of the form:

$$\begin{aligned} &\lambda=\lambda_0+\xi, \quad \xi\in[\Lambda], \\ &u=u_0+\alpha_i\varphi_i+v, \quad \alpha=(\alpha_1, \cdots, \alpha_p)\in R^p, \ v\in V, \end{aligned} \tag{2.9}$$

where $[\Lambda]$ stands for the linear space spanned by Λ. In the sequel we shall use the Einstein convention. The first equation of (2.7) becomes

$$\tilde{F}(\xi, \alpha, v)=0,$$

where F: $[\Lambda]\times R^p\times V_2\to V_2$ is defined by

$$\tilde{F}(\xi, \lambda, v)=QF(\lambda_0+\xi, u_0+\alpha_i\varphi_i+v)=v+\tilde{T}(\xi)\tilde{G}(\xi, \alpha, v)+v^*,$$

$v^*=Q(u+u^*)$, while $\tilde{T}$, $\tilde{G}$ are defined as

$$\begin{aligned} &\tilde{T}(\xi)g=QT(\lambda_0+\xi)g, \\ &\tilde{G}(\xi, \alpha, v)=G(\lambda_0+\xi, u_0+\alpha_i\varphi_i+v). \end{aligned}$$

By the definition of $\tilde{T}$, $\tilde{G}$, $\tilde{F}$, it is clear that all of them are C^m mappings and $\tilde{F}(0, 0, 0)=0$,

$$D_v\tilde{F}(0, 0, 0)=D_uF^0|_{v_2} \quad \text{is an isomorphism of } V_2.$$

Here by applying the implicit function theorem, we get

Lemma 2.2. *Assume (λ_0, u_0) is a simple singular point of F. Then there exist two positive constants δ, $r>0$ and a unique C^m mapping v: $S(0;\delta|[\Lambda])\times S(0;r|R^p)\to V_2$ such that*

$$\begin{aligned} &\tilde{F}(\xi, \alpha, v(\xi, \alpha))=0, \\ &v(0, 0)=0, \end{aligned}$$

where $S(0;\delta|[\Lambda])=\{g\in[\Lambda], |\xi|\leqslant\delta\}$, $S(0;r|R^p)=\{\alpha\in R^p, |\alpha|\leqslant r\}$ and if there is no confusion, $|\cdot|$ represents the norm in R^p.

Now, solving equation (2.7) in a neighborhood of the singular point (λ_0, u_0) amounts to the following equation in a neighborhood of $(0, 0)\in[\Lambda]\times R^p$:

$$\begin{aligned} &(I-Q)F(\lambda_0+\xi, u_0+\alpha_i\varphi_i+v(\xi, \alpha))=0 \\ \text{or} \quad &F(\lambda_0+\xi, u_0+\alpha_i\varphi_i+v(\xi, \alpha))=QF(\lambda_0+\xi, u_0+\alpha_i\varphi+v(\xi, \alpha)). \end{aligned} \tag{2.10}$$

It shows that (ξ, α) is a solution of (2.10) if and only if

$$F(\lambda_0+\xi, u_0+\alpha_i\varphi_i+v(\xi, \alpha))\in V_2.$$

Let

$$f(\xi, \alpha)=\begin{pmatrix} f_1(\xi, \alpha) \\ \vdots \\ f_p(\xi, \alpha)\end{pmatrix}=\begin{pmatrix} \langle F(\lambda_0+\xi, u_0+\alpha_i\varphi_i+v(\xi, \alpha)), \varphi_1^*\rangle \\ \vdots \\ \langle F(\lambda_0+\xi, u_0+\alpha_i\varphi_i+v(\xi, \alpha)), \varphi_p^*\rangle \end{pmatrix}.$$

Then equation (2.10) is equivalent to

$$f(\xi, \alpha)=0.$$

Clearly, elementary calculation shows that

$$f(0, 0)=0, \quad D_a f(0, 0)=0. \tag{2.11}$$

We shall consider the approximate problem in the sequel. It is equivalent to the system:

$$\begin{cases} QF_h(\lambda, u)=0, \\ (I-Q)F_h(\lambda, u)=0. \end{cases} \tag{2.12}$$

By the uniqueness of decomposition (2.9), the first equation of the above system becomes

$$\tilde{F}_h(\xi, \alpha, v)=0,$$

where $\tilde{F}_h$: $[\Lambda]\times R^p\times V_2\to V_2$ is defined by

$$\tilde{F}_h(\xi, \alpha, v)=QF_h(\lambda_0+\xi, u_0+\alpha_i\varphi_i+v)=v+\tilde{T}_h(\xi)\tilde{G}(\xi, \alpha, v)+v^*$$

and G, v^* are the same as above, while $\tilde{T}_h$: $[\Lambda]\times W\to V_2$ changes into

$$Tg(\lambda)g=QT_h(\lambda_0+\xi)g.$$

Obviously, T_h is a C^m bounded mapping. If the statement

$$\lim_{h\to 0}\sup_{\lambda\in S(\lambda_0;\delta)}\|D^l_\lambda(T_h(\lambda)-T(\lambda))\|=0, \quad 0\leq l\leq m-1 \tag{2.13}$$

holds in a neighborhood $S(\lambda_0;\ \delta)$ of λ_0, then the statement

$$\lim_{h\to 0}\sup_{\xi\in S(0;\delta)}\|D^l_\xi(\tilde{T}_h(\xi)-T(\xi))\|=0, \quad 0\leq l\leq m-1 \tag{2.14}$$

also holds in a neighborhood of $\xi=0$.

We assume Λ is bounded in the sequel and by using Theorem 2.1, we derive

Theorem 2.2. *Let (λ_0, u_0) be a simple singular point of F and T, T_h satisfy* (2.13). *Then there exist constants δ, h_0, K, a, $r>0$, such that for $h\leq h_0$ small enough, there is a unique C^m mapping v_h: $S(0;\ \delta\,|\,[\Lambda])\times S(0;\ r\,|\,R^p)\to V_2$ satisfying*

$$F_h(\xi, \alpha, v_h(\xi, \alpha))=0,$$

$$\|v_h(\xi, \alpha)-v(\xi, \alpha)\|\leq a,$$

and

$$\|D^l v_h(\xi^*, \alpha^*)-D^l v(\xi, \alpha)\|$$
$$\leq K\Big\{|\xi^*-\xi|+|\alpha^*-\alpha|+\sum_{i=0}^{l}\|D^i[H_h(\xi, \alpha)-H(\xi, \alpha)]\|\Big\}, \tag{2.15}$$
$$0\leq l\leq m-1,\ \forall(\xi^*, \alpha^*),\ (\xi, \alpha)\in S(0;\ \delta)\times S(0;\ r),$$
$$\sup_{(\xi,\alpha)\in S(0;\delta)\times S(0;r)}\|D^m v_h(\xi, \alpha)\|\leq K,$$

where $H_h(\xi, \alpha)=\tilde{T}_h(\xi)\tilde{G}(\xi, \alpha, v(\xi, \alpha))$, $H(\xi, \alpha)=\tilde{T}(\xi)\tilde{G}(\xi, \alpha, v(\xi, \alpha))$.

Proof. Using Theorem 2.1 and Lemma 2.2 directly shows the desired results.

Hence, system (2.12) is equivalent to the equation

$$(I-Q)F_h(\lambda_0+\xi, u_0+\alpha_i\varphi_i+v_h(\xi, \alpha))=0,$$

which holds if and only if $F_h(\lambda_0+\xi, u_0+\alpha_i\varphi_i+v_h(\xi, \alpha))\in V_2$.

Let us define $f_h(\xi, \alpha)$ in such a way:

$$f_h(\xi, \alpha)=\begin{pmatrix} f_{1h}(\xi, \alpha) \\ \vdots \\ f_{ph}(\xi, \alpha)\end{pmatrix}=\begin{pmatrix}\langle F_h(\lambda_0+\xi, u_0+\alpha_i\varphi_i+v_h(\xi, \alpha)), \varphi_1^*\rangle \\ \vdots \\ \langle F_h(\lambda_0+\xi, u_0+\alpha_i\varphi_i+v_h(\xi, \alpha)), \varphi_p^*\rangle\end{pmatrix}.$$

Then the approximate problem amounts to solving

$$f_h(\xi, \alpha)=0.$$

Lemma 2.3. *Under the hypothesis of Theorem 2.2 we have the following inequalities:*

$$\|D^l f_h(\xi^*, \alpha^*)-D^l f(\xi, \alpha)\| \leqslant K\Big\{|\xi^*-\xi|+|\alpha^*-\alpha|+\sum_{i=0}^{l}\|D^i[H_h(\xi, \alpha)-H(\xi, \alpha)]\|\Big\},$$

$$0\leqslant l\leqslant m-1, \ \forall(\xi^*, \alpha^*), \ (\xi, \alpha)\in S(0; \delta)\times S(0; r),$$

$$\sup_{(\xi,\alpha)\in S(0;\delta)\times S(0;r)}\|D^m f_h(\xi, \alpha)\|\leqslant K.$$

Assume there are pairs of C^m functions $(\xi(t), \alpha(t))$, $(\xi_h^*(t), \alpha_h^*(t))$, from R^q ($q\geqslant 1$ interger) into $[A]\times R^p$, and there exists a constant $\varepsilon>0$, such that

$$(\xi(t), \alpha(t)), \ (\xi_h^*(t), \alpha_h^*(t))\in S(0; \delta)\times S(0; r), \ \forall t\in S(0; \varepsilon|R^q).$$

Lemma 2.4. *Assume the functions $(\xi(t), \alpha(t))$, $(\xi_h^*(t), \alpha_h^*(t))$ are as above, and*

1) $\lim\limits_{h\to 0}\sup\limits_{|t|\leqslant\varepsilon}\Big(\Big\|\dfrac{d^l}{dt^l}(\xi_h^*(t)-\xi(t))\Big\|+\Big\|\dfrac{d^l}{dt^l}(\alpha_h^*(t)-\alpha(t))\Big\|\Big)=0, \quad 0\leqslant l\leqslant m-1$

2) $\sup\limits_{|t|\leqslant\varepsilon}\Big(\Big\|\dfrac{d^m}{dt^m}\xi_h^*(t)\Big\|+\Big\|\dfrac{d^m}{dt^m}\alpha_h^*(t)\Big\|\Big)\leqslant C$ *(independent of h).*

Then under the hypothesis of Theorem 2.1, the following hold:

$$\Big\|\frac{d^l}{dt^l}(v_h(\xi_h^*(t), \alpha_h^*(t))-v(\xi(t), \alpha(t))\Big\|$$
$$\leqslant K\sum_{i=0}^{l}\Big\{\Big\|\frac{d^i}{dt^i}(\xi_h^*(t)-\xi(t))\Big\|+\Big\|\frac{d^i}{dt^i}(\alpha_h^*(t)-\alpha(t))\Big\|$$
$$+\Big\|\frac{d^i}{dt^i}(H_h(\xi(t), \alpha(t))-H(\xi(t), \alpha(t))\Big\|\Big\}, \tag{2.16}$$
$$0\leqslant l\leqslant m-1, \quad \forall t\in S(0; \varepsilon|R^q),$$

where H_h, H are defined in (2.15), $h\leqslant h_0$ is small enough, and K is a constant independent of h.

Lemma 2.5. *Under the hypotheses of Theorem 2.2 and Lemma 2.4, the following statement holds:*

$$\Big\|\frac{d^l}{dt^l}(f_h(\xi_h^*(t), \alpha_h^*(t))-f(\xi(t), \alpha(t))\Big\|$$
$$\leqslant K\sum_{i=0}^{l}\Big\{\Big\|\frac{d^i}{dt^i}(\xi_h^*(t)-\xi(t))\Big\|+\Big\|\frac{d^i}{dt^i}(\alpha_h^*(t)-\alpha(t))\Big\|$$
$$+\Big\|\frac{d^i}{dt^i}(H_h(\xi(t), \alpha(t))-H(\xi(t), \alpha(t)))\Big\|\Big\},$$
$$0\leqslant l\leqslant m-1, \quad \forall t\in S(0; \varepsilon|R^q).$$

The proofs of the above lemmas are similar to those in [2].

§ 3. Simple Limit Point

Definition. *Let $[A]=R^p$ (or $[A]$ be an isomorphism to R^p), and (λ_0, u_0) be a simple singular point of F. If in addition*

$$\text{matrix } A\equiv\langle D_{\lambda_i}F(\lambda_0, u_0), \varphi_j^*\rangle \textit{ is nonsingular,} \tag{3.1}$$

where $D_{\lambda_i} F$ represents the i-th partial derivative of F to $\lambda=(\lambda_1, \cdots, \lambda_p)^T$, then (λ_0, u_0) is called a simple limit point of F.

Lemma 3.1. *Let (λ_0, u_0) be a simple limit point of F. Then there exists a constant $r>0$ and a unique C^m mapping $\xi(\alpha): S(0; r|R^p)\to R^p$ such that*

$$f(\xi(\alpha), \alpha)=0,$$
$$\xi(0)=0.$$

So in a neighborhood of (λ_0, u_0), there is a unique branch solution $\{(\lambda(\alpha), u(\alpha)): \alpha\in S(0; r/R^p)\}$ such that

$$F(\lambda(\alpha), u(\alpha))=0, \quad \forall\alpha\in S(0; r|R^p),$$

where $\lambda(\alpha)$, $u(\alpha)$ are C^m mappings given by

$$\begin{cases}\lambda(\alpha)=\lambda_0+\xi(\alpha), \\ u(\alpha)=u_0+\alpha_i\varphi_i+v(\xi(\alpha), \alpha).\end{cases} \tag{3.2}$$

Proof. Using Lemmas 2.1, 2.2 and (3.1), we get that

$$D_\xi f(0, 0)=\langle D_{\lambda_i}F(\lambda_0, u_0), \varphi_j^*\rangle=A$$

is nonsingular. By (2.11) we know $f(0, 0)=0$. Applying the local implicit function theorem, we complete the proof.

Lemma 3.2. *Assume the hypothesis of Theorem 2.2. If (λ_0, u_0) is a simple limit point of F, then there exist constants r, b, h, $K>0$, such that for $h\le h_0$ small enough, there is a unique C^m mapping $\xi_h(\alpha)$: $S(0; r|R^p)\to R^p$ such that*

$$f_h(\xi_h(\alpha), \alpha)=0,$$
$$|\xi_h(\alpha)-\xi(\alpha)|\le b, \tag{3.3}$$

and

$$\left\|\frac{d^l}{d\alpha^l}(\xi_h(\alpha)-\xi(\alpha))\right\|\le K\sum_{i=0}^{l}\left\|\frac{d^i}{d\alpha^i}[F_h(\lambda(\alpha), u(\alpha))-F(\lambda(\alpha), u(\alpha))]\right\|,$$
$$0\le l\le m-1, \quad \forall|\alpha|\le r; \tag{3.4}$$
$$\sup_{|\alpha|\le r}\left\|\frac{d^m}{d\alpha^m}\xi_h(\alpha)\right\|\le K,$$

where $\lambda(\alpha)$, $u(\alpha)$ are defined by (3.3).

Proof. We shall use Theorem 1 in [2] to prove the conclusion. Using Lemmas 2.3, 3.1 and (3.14), we can apply Theorem 1 in [2] to get r, h_0, b, $K>0$, such that, if $h\le h_0$, there exists a unique C^m function $\xi_h(\alpha)$ satisfying (3.3) and the second statement of (3.4), and

$$\left\|\frac{d^l}{d\alpha^l}(\xi_h(\alpha)-\xi(\alpha))\right\|\le K\sum_{i=0}^{l}\left\|\frac{d^i}{d\alpha^i}(f_h(\xi(\alpha), \alpha)-f(\xi(\alpha), \alpha))\right\|.$$

Letting $t=\alpha$, $\alpha(t)=\alpha_h^*(t)=\alpha$, $\xi(t)=\xi_h^*(t)=\xi(\alpha)$ in Lemma 2.5 and combining the above inequality, we get the first statement of (3.4), which completes the proof.

Theorem 3.1. *Suppose the hypotheses of Lemma 3.2 hold. Then the approximate problem*

$$F_h(\lambda, u)=0 \tag{3.5}$$

has a unique branch solution $\{(\lambda_h(\alpha), u_h(\alpha)); |\alpha|\le r\}$ in a neighborhood of the branch solution $\{(\lambda(\alpha), u(\alpha)); |\alpha|\le r\}$ of the continuous problem for $h\le h_0$ sufficiently small.

Moreover, $\lambda_h(\alpha)$, $u_h(\alpha)$ are of class C^m and we can obtain the following error *estimates*

$$\|D^l\lambda_h(\alpha)-D^l\lambda(\alpha)\|+\|D^lu_h(\alpha)-D^lu(\alpha)\|\leqslant K\sum_{i=0}^{l}\left\|\frac{d^i}{d\alpha^i}F_h(\lambda(\alpha),u(\alpha))\right\|,$$
$$0\leqslant l\leqslant m-1,\quad \forall|\alpha|\leqslant r, \tag{3.6}$$
$$\sup_{|\alpha|\leqslant r}(\|D^m(\lambda_h(\alpha)-\lambda(\alpha))\|+\|D^m(u_h(\alpha)-u(\alpha))\|)\leqslant K.$$

Proof. Decomposing $\lambda_h(\alpha)$, $u_h(\alpha)$ as

$$\lambda_h(\alpha)=\lambda_0+\xi_h(\alpha),$$
$$u_h(\alpha)=u_0+\alpha_i\varphi_i+v_h(\xi_h(\alpha),\alpha),$$

from Lemma 3.2 we know these functions are of class C^m and satisfy (3.5). Furthermore, the following error estimates hold:

$$\|D^l(\lambda_h(\alpha)-\lambda(\alpha))\|+\|D^l(u_h(\alpha)-u(\alpha))\|$$
$$\leqslant\left\|\frac{d^l}{d\alpha^l}(\xi_h(\alpha)-(\alpha))\right\|+\left\|\frac{d^l}{d\alpha^l}(v_h(\xi_h(\alpha),\alpha)-v(\xi(\alpha),\alpha))\right\|,$$
$$0\leqslant l\leqslant m-1,\quad \forall|\alpha|\leqslant r.$$

Using (3.4), (2.16) (Let $\alpha(t)=\alpha_h^*(t)=\alpha$, $\xi(t)=\xi(\alpha)$, $\xi_h^*(t)=\xi_h(\alpha)$) it is easy to derive the first inequality of (4.8). The second is obvious.

Now, we shall consider the derivative of $\xi(\alpha)$ at $\alpha=0$. Simple calculation shows

$$D_\alpha\xi(0)=0,$$
$$D_\alpha^2\xi(0)=-A^{-1}\langle D_u^2F^0\Phi,\Phi^*\rangle. \tag{3.7}$$

Definition. *Let (λ_0,u_0) be a simple limit point of F. If in addition $A^{-1}\langle D^2F^0\Phi,\varphi_i^*\rangle$, $1\leqslant i\leqslant p$, is a certain definite matrix, then (λ_0,u_0) is called a normal singular point of F.*

Clearly, if (λ_0,u_0) is a normal singular point of F, then matrixes $\frac{d^2}{d\alpha^2}\xi_i(0)$, $1\leqslant i\leqslant p$ are all definite.

Lemma 3.3. *Assume the hypotheses of Lemma 3.2 and $m\geqslant3$. Then if (λ_0,u_0) is a normal singular point of F, there exist r_1, $h_0>0$, such that if $h\leqslant h_0$, there is a unique $\alpha_h^j\in S(0;r_1|R^p)$, $1\leqslant j\leqslant p$, satisfying*

$$\begin{cases}\lim_{h\to0}\alpha_h^j=0,\\ \frac{d}{d\alpha}\xi_{jh}(\alpha_h^j)=0,\quad 1\leqslant j\leqslant p,\end{cases}$$

$\frac{d^2}{d\alpha^2}\xi_j(\alpha)$, $\frac{d^2}{d\alpha^2}\xi_{jh}(\alpha)$ are all definite matrixes and their signs are the same as those of

$$\frac{d^2}{d\alpha^2}\xi_j(0),\quad \forall\alpha\in S(0;r_1|R^p),\quad 1\leqslant j\leqslant p. \tag{3.8}$$

Definition. *Let $\lambda_h^j=\lambda_h(\alpha_h^j)$, $u_h^j=u_h(\alpha_h^j)$. We call (λ_h^j,u_h^j) the j-th normal singular point of F_h.*

Theorem 3.2. *Under the hypotheses of Lemma 3.4, we have*

$$|\lambda_h^j-\lambda_0|+\|u_h^j-u_0\|\leqslant K\sum_{i=0}^{l}\left\|\frac{d^i}{d\alpha^i}[F_h(\lambda(\alpha),u(\alpha))-F(\lambda(\alpha),u(\alpha))]\right\|_{\alpha=0}, \tag{3.9}$$

where K is a constant independent of h. Furthermore

$$|\lambda_h^l-\lambda_0|\leqslant K\Big\{|\langle F_h(\lambda_0, u_0)-F(\lambda_0, u_0), \Phi^*\rangle| + \|F_h(\lambda_0, u_0)-F(\lambda_0, u_0)\|\cdot\|(D_uF_h(\lambda_0, u_0)-D_uF(\lambda_0, u_0))^*\Phi^*\| + \sum_{i=0}^{l}\Big\|\frac{d^i}{d\alpha^i}[F_h(\lambda(\alpha), u(\alpha))-F(\lambda(\alpha), u(\alpha))]\Big\|_{\alpha=0}^2\Big\}. \tag{3.10}$$

Proof. According to (3.8) and the C^m property of $\xi(\alpha)$, there exists a constant M, such that

$$\Big\|\frac{d^2}{d\alpha^2}\xi_{jh}(\alpha)^{-1}\Big\|\leqslant M,\quad \forall|\alpha|\leqslant r_1,\ 1\leqslant j\leqslant p.$$

Hence

$$|\alpha_h^l|\leqslant\max_{|\alpha|\leqslant r_1}\Big\|\frac{d^2}{d\alpha^2}\xi_{jh}(\alpha)^{-1}\Big\|\cdot\Big\|\frac{d}{d\alpha}\xi_{jh}(\alpha_h^l)-\frac{d}{d\alpha}\xi_{jh}(0)\Big\| \leqslant M\Big\|\frac{d}{d\alpha}\xi_{jh}(0)\Big\|\leqslant M\Big\|\frac{d}{d\alpha}\xi_h(0)-\frac{d}{d\alpha}\xi(0)\Big\| \tag{3.11}$$

and

$$|\lambda_h^l-\lambda_0|\leqslant|\xi_h(\alpha_h^l)-\xi_h(0)|+|\xi_h(0)|\leqslant C_1|\alpha_h^l|+|\xi_h(0)-\xi(0)|. \tag{3.12}$$

Here we have used the mean value theorem and the fact $\xi(0)=0$. In the same way,

$$\|u_h^l-u_0\|\leqslant\|u_h(\alpha_h^l)-u_h(0)\|+\|u_h(0)\|\leqslant C_2|\alpha_h^l|+\|u_h(0)-u(0)\|. \tag{3.13}$$

Combining inequalities (3.12) and (3.13), and using (3.11), (3.4) and (3.6), we get (3.9).

As previous, we have

$$|\lambda_h^l-\lambda_0|\leqslant|\xi_h(0)|+\Big\|\frac{d}{d\alpha}\xi_h(0)\Big\|\cdot|\alpha_h^l|+C_3|\alpha_h^l|^2 \leqslant|\xi_h(0)|+C_4\Big\|\frac{d}{d\alpha}(\xi_h(0)-\xi(0))\Big\|^2. \tag{3.14}$$

According to (3.11) and (3.4), we see:

$$|\xi_h(0)|\leqslant C_5|f_h(0, 0)-f(0, 0)| \leqslant C_5|\langle F_h(\lambda_0, u_0)-F(\lambda_0, u_0), \Phi^*\rangle| +C_5|\langle T_h(\lambda_0)\cdot[G(\lambda_0, u_0+v_h(0, 0))-G(\lambda_0, u_0)], \Phi^*\rangle|. \tag{3.15}$$

But on the other hand, by the C^m boundedness of G, it holds that

$$G(\lambda_0, u_0+v_h(0, 0))-G(\lambda_0, u_0)=D_uG(\lambda_0, u_0)V_h(0, 0)+D_u^2G^0\cdot(V_h(0, 0))^2.$$

Thus

$$|\langle T_h(\lambda_0)(G(\lambda_0, u_0+v_h(0, 0))-G(\lambda_0, u_0)), \Phi^*\rangle| \leqslant\|(D_uF_h(\lambda_0, u_0)-D_uF(\lambda_0, u_0))^*\Phi^*\|\,\|v_h(0, 0)\|+C_6\|v_h(0, 0)\|^2.$$

We have used the fact that $\langle D_uF^0\cdot v(0, 0), \Phi^*\rangle=0$. Substituting the above inequality and (3.15) into (3.14), we obtain

$$|\lambda_h^l-\lambda_0|\leqslant C_5|\langle F_h(\lambda_0, u_0)-F(\lambda_0, u_0), \Phi^*\rangle|+C_5\|(D_uF_h^0-D_uF^0)^*\Phi^*\|\cdot\|v_h(0, 0)\| +C_5C_6\|v_h(0, 0)\|^2+C_6\Big\|\frac{d}{d\alpha}(\xi_h(\alpha)-\xi(\alpha))\Big\|_{\alpha=0}^2.$$

By applying (2.15) again with $l=0$, $\xi=\xi^*=0$, $\alpha^*=\alpha=0$, and (3.4) in the above inequality, (3.10) is proved.

§ 4. Numerical Prediction of Bifurcation Points

In this section we shall predict bifurcation points of the original problem by numerical methods.

Lemma 4.1. *Let* (λ_0, u_0) *be a solution of problem* (1.1), *and* $D_uF(\lambda_0, u_0)$ *be an isomorphism operator on* V. *Then as* $h \leqslant h_0$ *is small enough, there exist two unique functions* $u(\lambda)$ $(u(\lambda_0)=u_0)$ *and* $u_h(\lambda)$ *satisfying* (1.1) *and* (1.2) *for* $\lambda \in S(\lambda_0; \delta)$ (δ *sufficient small*) *respectively.*

Furthermore, there is a constant $d>0$ independent of h, such that

$$\begin{cases} \|D_uF(\lambda, u(\lambda))\cdot v\| \geqslant d\|v\|, \\ \|D_uF_h(\lambda, u_h(\lambda))\cdot v\| \geqslant d\|v\|, \quad \forall \lambda \in S(\lambda_0; \delta),\ \forall v \in V, \end{cases} \tag{4.1}$$

$$\lim_{h\to 0,\ \lambda\in S(\lambda_0,\delta)} \sup \|u_h(\lambda) - u(\lambda)\| = 0. \tag{4.2}$$

The proof can be easily accomplished by using the implicit function theorem.

Definition. (λ_{0h}, u_{0h}) *is called an asymptotic solution of equation* (1.2) *if for any* $\varepsilon>0$, *there exists a real number* $h_0>0$, *such that*

$$\|F_h(\lambda_{0h}, u_{0h})\| \leqslant \varepsilon, \quad \forall h \leqslant h_0.$$

Theorem 4.1. *Assume* (λ_{0h}, u_{0h}) *is an asymptotic solution of equation* (1.2), *and* $(\lambda_{0h}, u_{0h}) \to (\lambda_0, u_0)$. *Let*

$$d_h = \sup\{e;\ e>0,\ \|D_uF_h(\lambda_{0h}, u_{0h})\cdot v\| \geqslant e\|v\|,\quad \forall v\in V\},$$
$$d = \sup\{e;\ e>0,\ \|D_uF(\lambda_0, u_0)\cdot v\| \geqslant e\|v\|,\quad \forall v\in V\}.$$

Then i) (λ_0, u_0) *is a solution of equation* (1.1).

ii) $\lim_{h\to 0} d_h = d.$

Proof. i) It is easy to see that (λ_0, u_0) is indeed a solution of equation (1.1) from the following inequality

$$\|F(\lambda_0, u_0)\| \leqslant \|F(\lambda_0, u_0) - F_h(\lambda_0, u_0)\| + \|F_h(\lambda_0, u_0) - F_h(\lambda_{0h}, u_{0h})\| + \|F_h(\lambda_{0h}, u_{0h})\|.$$

ii) By the definition of d_h or d, we have

$$d_h\|v\| \leqslant \|D_uF(\lambda_{0h}, u_{0h})\cdot v\| \leqslant \|D_uF(\lambda_0, u_0)\cdot v\| + \|(D_uF(\lambda_0, u_0) - D_uF_h(\lambda_{0h}, u_{0h}))\cdot v\|,$$

and for any $\varepsilon>0$, there exists an element $v\neq 0$ such that

$$\|D_uF(\lambda_0, u_0)\cdot v\| \leqslant (d+\varepsilon)\|v\|.$$

Hence $\qquad d_h \leqslant d+3\varepsilon.$

Similarly $\qquad d \leqslant d_h+3\varepsilon.$

Finally, we get $\qquad \lim_{h\to 0} d_h = d.$

Remark 4.1. 1) As a direct consequence of this theorem, we know that (λ_0, u_0) is a singular solution of quation (1.1) if and only if $d_h \to 0$.

2) If (λ_0, u_0) is a solution of equation (1.1), equation (1.2) always possesses asymptotic solutions.

Let V_1 and V_{1h} be two closed subspaces of V which can be decomposed as

$$V=V_1\dot{+}V_2=V_{1h}\dot{+}V_{2h}. \tag{4.3}$$

Assume P, P_h are projectors from V onto V_1 and V_{1h} along V_2 and V_{2h} respectively, and

$$\lim_{h\to 0}\|P_h-P\|=0. \tag{4.4}$$

Then $Q=I-P$, $Q_h=I-P_h$ are projectors from V onto V_2 and V_{2h} respectively,

$$\lim_{h\to 0}\|Q_h-Q\|=0. \tag{4.5}$$

Theorem 4.2. *Assume the hypotheses of Theorem 4.1 and (4.4), (4.5). If in addition the following hold:*

1) *There is a constant $d>0$ independent of h, such that*

$$\|D_uF_h(\lambda_{0h}, u_{0h})\cdot w\|\geqslant d\|w\|, \ \forall w\in V_{2h}, \quad h\leqslant h_0 \text{ small enough},$$

2) $\lim\limits_{h\to 0} d_h\equiv\lim\limits_{h\to 0}\sup\limits_{\substack{v\in V_{1h}\\ \|v\|=1}}\|D_uF_h(\lambda_{0h}, u_{0h})\cdot v\|=0,$

then, we can get

i) *(λ_0, u_0) is a singular point of F.*

ii) *There exists a constant $d_0>0$, such that*

$$\|D_uF(\lambda_0, u_0)\cdot w\|\geqslant d_0\|w\|, \quad \forall w\in V_2.$$

iii) $$D_uF(\lambda_0, u_0)\cdot v=0, \quad \forall v\in V_1.$$

Proof. First, according to Remark 4.1, i) is obvious. Secondly, with condition 1) we have

$$\begin{aligned}\|D_uF(\lambda_0, u_0)\cdot w\|&=\|D_uF(\lambda_0, u_0)w-D_uF_h(\lambda_{0h}, u_{0h})w+D_uF_h(\lambda_{0h}, u_{0h})w\|\\ &\geqslant\|D_uF_h(\lambda_{0h}, u_{0h})w\|-\|D_uF(\lambda_0, u_0)-D_uF_h(\lambda_{0h}, u_{0h})\|\cdot\|w\|\\ &\geqslant\tilde{d}\|w\|-\varepsilon\|w\| \quad (\forall h<h_0 \text{ small enough})\\ &=(\tilde{d}-\varepsilon)\|w\|, \quad \forall w\in V_{2h}.\end{aligned}$$

Hence

$$\begin{aligned}\|D_uF(\lambda_0, u_0)w\|&\geqslant\|D_uF(\lambda_0, u_0)Q_hw\|-\|D_uF(\lambda_0, u_0)(w-Q_hw)\|\\ &\geqslant(\tilde{d}-\varepsilon)\|Q_hw\|-\|D_uF(\lambda_0, u_0)\|\cdot\|Q-Q_h\|\cdot\|w\|\\ &\geqslant(\tilde{d}-\varepsilon)(\|w\|-\|Q-Q_h\|\cdot\|w\|)-\|D_uF(\lambda_0, u_0)\|\cdot\|Q-Q_h\|\cdot\|w\|\\ &\geqslant(\tilde{d}-\varepsilon+(\tilde{d}-\varepsilon-\|D_uF(\lambda_0, u_0)\|)\|Q-Q_h\|)\cdot\|w\|\\ &\geqslant d_0\|w\|, \quad \forall h\leqslant h_0, \ w\in V_2.\end{aligned}$$

So the conclusion ii) is proved. Finally, we know

$$\begin{aligned}\|D_uF(\lambda_0, u_0)v\|&\leqslant\|D_uF(\lambda_0, u_0)-D_uF_h(\lambda_{0h}, u_{0h})\|\cdot\|v\|+\|D_uF_h(\lambda_{0h}, u_{0h})v\|\\ &\leqslant\|D_uF(\lambda_0, u_0)-D_uF_h(\lambda_{0h}, u_{0h})\|\cdot\|v\|+d_h\|v\|.\end{aligned}$$

By using the properties of T, T_h and condition 2), the proof is completed.

Remark 4.2. 1) Let $d=\sup\limits_{\substack{v\in V_1\\ \|v\|=1}}\|D_uF(\lambda_0, u_0)v\|$. Then $\lim\limits_{h\to 0}d_h=d$.

2) The statement $\mathrm{Ker}(D_uF(\lambda_0, u_0))=V_1$ is true if and only if condition 2) in Theorem 4.2 holds.

3) Theorem 4.2 can be written in a more general form. For simplicity, we denote

$$E_h=(D_uF_h(\lambda_{0h}, u_{0h}))^r, \quad E=(D_uF(\lambda_0, u_0))^r, \quad r\geqslant 1,$$
$$d_h=\sup_{\substack{v\in V_{1h}\\ \|v\|=1}}\|E_hv\|, \quad d=\sup_{\substack{v\in V_1\\ \|v\|=1}}\|Ev\|.$$

Theorem 4.2'. *Assume that hypotheses (2.2) and (4.4) and the following hold:*

1) (λ_{0h}, u_{0h}) *is an asymptotic solution of* (1.2) *and*

$$\lim_{h\to 0}(\lambda_{0h}, u_{0h})=(\lambda_0, u_0).$$

2) *There is a constant* $\tilde{d}>0$ *independent of* h, *such that*

$$\|E_hw\|\geqslant\tilde{d}\|w\|, \quad \forall w\in V_{2h}, \quad h<h_0 \text{ (small enough)}.$$

Then,

i) (λ_0, u_0) *is a solution of equation* (1.1).

ii) $\lim_{h\to 0} d_h=d$.

iii) *There exists a constant* $d_0>0$, *such that*

$$\|E\cdot w\|\geqslant d_0\|w\|, \ \forall w\in V_2.$$

iv) (λ_0, u_0) *is a singular solution of* (1.1) *if and only if* $d=0$.

v) $\mathrm{Ker}(E)=V_1$ *if and only if* $d=0$.

Theorem 4.2 shows that V_1 is the null space of $D_uF(\lambda_0, u_0)$. In the sequel of this section we shall consider conditions which make

$$\mathrm{Ker}(D_uF(\lambda_0, u_0)^2)=V.$$

Lemma 4.2. *Let* $S\in\mathscr{L}(V, V)$, $V_1=\mathrm{Ker}(S)$. *Then the following statements are equivalent:*

i) $\mathrm{Ker}(S^2)=V_1$,

ii) $\|Sv_2-v_1\|>0, \ \forall v_1\in V_1, \ \forall v_2\notin V,$

iii) $V_1\cap\mathrm{Rang}(S)=\{0\}$.

Proof. i)$\Rightarrow$iii): For $v\in V_1\cap\mathrm{Rang}(S)$ arbitrary, we know $Sv=0$; on the other hand one can find an element w such that $v=Sw$. Hence $S^2w=Sv=0$ which shows $w\in V$, $v=Sw=0$.

iii)$\Rightarrow$ii): Otherwise, there exists at least a pair of $v_1\in V_1$, $v_2\notin V_1$ such that $\|Sv_2-v_1\|=0$, i.e. $Sv_2=v_1\in V_1$. Hence $Sv_2\in V_1\cap\mathrm{Rang}(S)$, $Sv_2=0$ or $v_2\in V_1$, which contradicts $v_2\notin V_1$.

ii)$\Rightarrow$i): Take $v\in\mathrm{Ker}(S^2)$ to be arbitrary. Then $Sv\in V_1$. If $v\notin V_1$, let $v_1=Sv$, $v_2=v$. We then get a contradication

$$0=\|Sv_2-v_1\|>0.$$

This completes the proof.

Lemma 4.3. *Let* $S\in\mathscr{L}(V, V)$ *be a clcsed range operator and* $V_1=\mathrm{Ker}(S)$ *a finite dimensional space. Then* $\mathrm{Ker}(S^2)=V_1$ *if and only if the following hold:*

$$\inf\{\|v_1-Sv_2\|, \ v_1\in V_1, \ v_2\in V_2, \ 0<a\leqslant\|v_1\|, \ \|v_2\|\leqslant b\}\geqslant C>0, \tag{4.6}$$

where $C=C(a, b)$, *and* $V_2\subset V$ *such that* $V=V_1\dot{+}V_2$.

Lemma 4.4. *Let* $S, S_h\in\mathscr{L}(V, V)$ *be closed range operators which satisfy*

$$\lim_{h\to 0}\|S_h-S\|=0. \tag{4.7}$$

If (4.4) *holds, we get* (4.6) *if and only if*

$$\inf\{\|v_{1h}-S_hv_{2h}\|, \ v_{1h}\in V_{1h}, \ v_{2h}\in V_{2h}, \ 0<a'\leqslant\|v_{1h}\|, \ \|v_{2h}\|\leqslant b'\}\geqslant C'>0, \tag{4.8}$$

where $C'=C'(a', b')$ is a constant independent of h.

Proof. Assume (4.6) holds. Then from

$$v_{1h}-S_h v_{2h}=(P_h-P)v_{1h}+Pv_{1h}-SQv_{2h}+(S-S_h)Qv_{2h}+S_h(Q-Q_h)v_{2h},$$

we have

$$\|v_{1h}-S_h v_{2h}\|\geqslant -\|P_h-P\|\cdot\|v_{1h}\|+\|Pv_{1h}-SQv_{2h}\|-\|S-S_h\|\cdot\|Q\|\cdot\|v_{2h}\|$$
$$-\|S_h\|\cdot\|Q-Q_h\|\cdot\|v\|\geqslant C'>0,\quad 0<a'\leqslant\|v_{1h}\|,\ \|v_{2h}\|\leqslant b'.$$

The other side may be proved in the same way.

Now, let us return to problem (1.1) and (1.2). For simplicity, we shall assume that T, T_h are compact and (2.2) holds.

Theorem 4.3. *Under hypotheses (4.4), the following propositions are equivalent:*

i) *(λ_0, u_0) is a solution of (1.1) and*

$$\mathrm{Ker}(D_uF(\lambda_0, u_0))=\mathrm{Ker}((D_uF(\lambda_0, u_0))^2)=V_1; \tag{4.9}$$

ii) *(λ_{0h}, u_{0h}) is an asymptotic solution of (1.2), $d_h\to 0$ (d_h is defined as in Theorem 4.2'), $(\lambda_{0h}, u_{0h})\to(\lambda_0, u_0)$ and*

$$\inf_{\substack{\|v_{1h}\|=\|v_{2h}\|=1\\ v_{1h}\in V_{1h}\\ v_{2h}\in V_{2h}}}\|v_{1h}-D_uF_h(\lambda_{0h}, u_{0h})v_{2h}\|\geqslant C>0, \tag{4.10}$$

where C is a constant independent of h.

Proof. i)⇒ii): According to Remark 4.1 there exists an asymptotic solution (λ_{0h}, u_{0h}) converging to (λ_0, u_0). Using Lemmas 4.3, 4.4 we can obtain (4.10) from (4.9). As for $d_h\to 0$, it is natural by Theorem 4.2.

ii)⇒i): Applying Theorem 4.2 we know that (λ_0, u_0) is a solution of (1.1) and $\mathrm{Ker}(D_uF(\lambda_0, u_0))=V_1$. Using Lemmas 4.4, 4.3, we get (4.9).

Remark 4.3. i) This theorem also holds in the case of $V_1=\{0\}$.

ii) It can be generalized as Theorem 4.2.

§ 5. Application to the Navier–Stokes Equations

We consider the steady viscous incompressible Navier–Stokes equations:

$$\begin{cases}-\nu u+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-\nabla p=\boldsymbol{f}, & \text{in } \Omega\subset R,\\ \nabla\cdot\boldsymbol{u}=0,\end{cases} \tag{5.1}$$
$$\boldsymbol{u}=\boldsymbol{O} \text{ on } \partial\Omega;\ \int_\Omega p\,dx=0,$$

where $\boldsymbol{f}\in(L^{\frac{4}{3}}(\Omega))^n$, $n\leqslant 4$, $\Omega\subset R$ bounded conic domain.

Denote $V=H_0^1(\Omega)^n\times L_0^2(\Omega)$; $W=L^{\frac{4}{3}}(\Omega)^n\times L^2(\Omega)$, $W_0=L^{\frac{4}{3}}(\Omega)\times\{0\}$. Obviously, $V\subset W\subset V'=H^{-1}(\Omega)^n\times L^2(\Omega)$, and $W_0\subset W$.

Now we define G and a:

$$G(\lambda, (\boldsymbol{u}, p))=((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-\boldsymbol{f}, 0): R_+\times V\to W_0,$$

$$a(\lambda; (\boldsymbol{u}, p), (\boldsymbol{v}, q))=\frac{1}{\lambda}\int_\Omega\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,dx+\int_\Omega(p\nabla\cdot\boldsymbol{v}+q\nabla\cdot\boldsymbol{u})dx:\ R_+\times V\times V\to R.$$

The weak form of problem (5.1) is the following

$$a(\lambda; (\boldsymbol{u}, p), (\boldsymbol{v}\cdot q))+\langle(\boldsymbol{v}, q), G(\lambda, (\boldsymbol{u}, p))\rangle=0,\quad\forall(\boldsymbol{v}, q)\in V, \tag{5.2}$$

where $\lambda=1/\nu$, and its approximate problem is

$$a(\lambda;\ (\boldsymbol{u}_h,\ p_h),\ (\boldsymbol{v},\ q))+\langle(\boldsymbol{v},\ q),\ G(\lambda,\ (\boldsymbol{u}_h,\ p_h))\rangle=0,\quad \forall(\boldsymbol{v},\ q)\in V_h, \tag{5.3}$$

where $V_h=X_h\times L_h$, $X_h\subset H_0^1(\Omega)^n\cap Y_h$, $Y_h\in H^2(\Omega)^n$, $L_h\subset L_0^2(\Omega)$ are discrete spaces and satisfy conditions (H1), (H2), (H3) in [4].

The variational form of the Stokes problem is

$$a(\lambda;\ (\boldsymbol{u},\ p),\ (\boldsymbol{v},\ q))=\langle(\boldsymbol{v},\ q),\ (\boldsymbol{g},\ 0)\rangle,\quad \forall(\boldsymbol{v},\ q)\in V. \tag{5.4}$$

We introduce the operator T: $R_+\times W\to V$ as follows:

$$T(\lambda)\cdot(\boldsymbol{g},\ 0)=(\boldsymbol{u},\ p)\quad \text{satisfying (5.4)}.$$

It follows from the Sobolev imbedded theorem that $T(\lambda)$ is compact.

Let $(\boldsymbol{u}^*,\ p^*)$ denote the solution of (5.4) when $\lambda=1$, i.e.

$$(\boldsymbol{u}^*,\ p^*)=T(1)\cdot(\boldsymbol{g},\ 0).$$

We define the mappings S: $W_0\to V$ and Q: $W_0\to V$ by

$$S(\boldsymbol{g},\ 0)=(\boldsymbol{u}^*,\ 0);\quad Q(\boldsymbol{g},\ 0)=(\boldsymbol{O},\ p^*).$$

Lemma 5.1. $T(\lambda)=\lambda S+Q$, $\forall\lambda>0$.

Now, let us introduce the operator T_h: $R_+\times W\to V$ as follows:

$$a(\lambda;\ (\boldsymbol{u}_h,\ p_h),\ (\boldsymbol{v},\ q))=\langle(\boldsymbol{v},\ q),\ (\boldsymbol{g},\ 0)\rangle,\quad \forall(\boldsymbol{v},\ q)\in V_h. \tag{5.5}$$

Under hypotheses (H1)—(H3), problem (5.5) has a unique solution for any $\lambda\in R_+$, $(\boldsymbol{g},\ 0)\in W_0$.

The proof is in [4].

Let $(\boldsymbol{u}_h^*,\ p_h^*)$ be a solution of (5.5) when $\lambda=1$, i.e.

$$(\boldsymbol{u}_h^*,\ p_h^*)=T_h(1)(\boldsymbol{g},\ 0).$$

Define S_h, Q_h: $W_0\to V$ as follows

$$S_h(\boldsymbol{g},\ 0)=(\boldsymbol{u}_h^*,\ 0);\quad Q_h(\boldsymbol{g},\ 0)=(\boldsymbol{O},\ p_h^*).$$

Like Lemma 5.1 we have the following result:

Lemma 5.2. $T_h(\lambda)=\lambda S_h+Q_h$, $\forall\lambda>0$.

Lemma 5.3. *Under hypotheses* (H1)—(H3), *we can get*

$$\lim_{h\to 0}\|S_h-S\|=0;\quad \lim_{h\to 0}\|Q_h-Q\|=0.$$

Furthermore, if $T(1)(\boldsymbol{g},\ 0)\in H^{m+1}(\Omega)^n\times H^m(\Omega)$, *we have*

$$\|T_h(1)(\boldsymbol{g},\ 0)-T(1)(\boldsymbol{g},\ 0)\|\leqslant Ch^m\|\boldsymbol{g}\|_{m-1},$$

where C is a constant independent of h.

The proof can be found in [4].

Concluding the above discussions, we easily get

i) $$T,\ T_h,\ G \text{ are of class } C^\infty; \tag{5.6}$$

ii) $\lim\limits_{h\to 0}\sup\limits_{\lambda\in\Lambda}\|D_\lambda^l(T_h(\lambda)-T(\lambda))\|=0$, $l=0,\ 1,\ 2,\ \cdots$, if $\Lambda\subset R_+$ is a bounded set. (5.7)

Theorem 5.1. *Let* $\{(\boldsymbol{u}(\lambda),\ p(\lambda));\ \lambda\in\Lambda\}$ *be a nonsingular solution of* (5.2). *Then there exists a constant* h_0, *such that for* $h\leqslant h_0$ *small enough, there is a unique* C^∞ *mapping* $(\boldsymbol{u}_h(\lambda),\ p_h(\lambda))$: $\Lambda\to V$ *satisfying*

i) $(u_h(\lambda),\ p_h(\lambda))$ *is a solution of* (5.3);

ii) $\lim_{h\to 0}(\|u_h^{(l)}(\lambda)-u^{(l)}(\lambda)\|+\|p_h^{(l)}(\lambda)-p^{(l)}(\lambda)\|)=0,\quad l\geq 0;$

iii) *if* $u(\lambda)\in H^m(\Omega)^n$, $f\in H^{m-1}(\Omega)^n$, *then there exists a constant* K_m *independent of* h *such that*

$$\|u_h^{(l)}(\lambda)-u^{(l)}(\lambda)\|+\|p_h^{(l)}(\lambda)-p^{(l)}(\lambda)\|\leq K_m h^m,\quad 0\leq l\leq m.$$

Proof. By Theorem 2.1, conclusion i) is easily proved, and there exist constants $\tilde{K}_l(0\leq l\leq m)$ independent of h, such that

$$\|u_h^{(l)}(\lambda)-u^{(l)}(\lambda)\|+\|p_h^{(l)}(\lambda)-p^{(l)}(\lambda)\|$$
$$\leq \tilde{K}_l\sum_{i=0}^{l}\left\|\frac{d^i}{d\lambda^i}[(T_h(\lambda)-T(\lambda))G(\lambda,u(\lambda),p(\lambda))]\right\|.$$

By calculating

$$\left\|\frac{d^i}{d\lambda^i}[(T_h(\lambda)-T(\lambda))G(\lambda,u(\lambda),p(\lambda))]\right\|$$
$$\leq C\left\{\left\|(S_h-S)\frac{d^i}{d\lambda^i}G\right\|+\left\|(Q_h-Q)\frac{d^i}{d\lambda^i}G\right\|+\left\|(S_h-S)\frac{d^{i-1}}{d\lambda^{i-1}}G\right\|\right\}.$$

Again by (5.6), (5.7) and Lemma 5.3 we complete the proof.

Theorem 5.2. *If* $(\lambda_0,(u_0,p_0))$ *is a simple limit point of* (5.2), *then there exist constants* r, $h_0>0$ *and unique* C^∞ *mappings:*

$$(\lambda(\alpha),(u(\alpha),p(\alpha))),\ (\lambda_h(\alpha),(u_h(\alpha),p_h(\alpha))):(-r,r)\to\Lambda\times V,\ h\leq h_0,$$

such that

i) *they are solutions of* (5.2) *and* (5.3) *respectively;*

ii) $\lim_{h\to 0}\sup_{|\alpha|<r}\left(\left|\frac{d^l}{d\alpha^l}(\lambda_h(\alpha)-\lambda(\alpha))\right|+\|u_h^{(l)}(\lambda)-u^{(l)}(\lambda)\|+\|p_h^{(l)}(\lambda)-p^{(l)}(\lambda)\|\right)=0;$

iii) *if* $u(\alpha)\in H^m(\Omega)^n$, $f\in H^{m-1}(\Omega)^n$, *there is a constant* K_m *independent of* h, *such that*

$$\sup_{|\alpha|<r}\{|\lambda_h^{(l)}(\alpha)-\lambda^{(l)}(\alpha)|+\|u_h^{(l)}(\alpha)-u^{(l)}(\alpha)\|+\|p_h^{(l)}(\alpha)-p^{(l)}(\alpha)\|\}\leq K_m h^m,$$
$$0\leq l\leq m.$$

The proof is similar to that of Theorem 5.1.

Theorem 5.3. *If* $(\lambda_0,(u_0,p_0))$ *is a normal limit point of* (5.2), *then as* $h\leq h_0$, *we have*

i) *problem* (5.3) *possesses a normal limit point* $(\lambda_h^0,(u_h^0,p_h^0))$;

ii) $\lim_{h\to 0}(|\lambda_h^0-\lambda_0|+\|u_h^0-u_0\|_1+\|p_h^0-p_0\|_0)=0;$

iii) *if* $u(\alpha)\in H^m(\Omega)^n$, $f\in H^{m-1}(\Omega)^n$, *there exists a constant* K_m *independent of* h *such that*

$$|\lambda_h^0-\lambda_0|+\|u_h^0-u_0\|_1+\|p_h^0-p_0\|_0\leq K_m h^m.$$

Proof. Applying Theorem 3.2 in the proof of Theorem 5.1 we can easily complete the proof.

§ 6. The Penalty Approximation of Branch Solution

We still consider the stationary incompressible Navier-Stokes equations with the same notations and operators as in § 5. Now we shall choose $H=H_0^1(\Omega)^n$,

$V=\{v\in H,\ \nabla\cdot v=0\}$, $W=L^2(\Omega)^n$. Then problem (2.1) is equivalent to

Find $u\in V$, such that $a_0(u,v)+\lambda\langle(u\cdot\nabla)u-f,\ v\rangle=0,\quad\forall v\in V$,

where $a_0(u,v)=\int_\Omega\nabla u\cdot\nabla v\,dx$, $\lambda=1/\nu$. Define G: $\Lambda\times H\to W$ as $G(\lambda,u)=(u\cdot\nabla)u-f$ and S: $W\to V$ by $a_0(Sg,v)=\langle g,v\rangle$, $\forall v\in V$, $g\in W$. It is easy to see that S is compact. Thus, if we define the operator $T(\lambda)$: $W\to V$ for all $\lambda\in R_+$ as $T(\lambda)=\lambda Sg$, $\forall g\in W$, we know problem (5.1) becomes

Find $u\in H$ such that $F(\lambda,u)=u+T(\lambda)G(\lambda,u)=0$.

Introduce the finite element spaces $X_h\subset H$ and $L_h\subset L^2(\Omega)$ as in § 5 and assume hypotheses (H1), (H2), (H3) hold.

Let $V_h=\{v_h\in X_h,\ (q_h,\ \nabla v_h)=0,\ \forall q_h\in L_h\}$, and let ρ_h be the orthogonal projection from $L^2(\Omega)$ to L_h.

Lemma 6.1. *For all $g\in W$, $\lambda\in\Lambda$, the following equation has a unique solution $u_h^\varepsilon\in X_h$:*

$$a_0(u_h^\varepsilon,\ v_h)+\frac{1}{\varepsilon}(\rho_h(\nabla\cdot u_h),\ \rho_h(\nabla\cdot v_h))=\lambda\langle g,\ v_h\rangle,\quad\forall v_h\in X_h\tag{6.1}$$

and we have the estimates

$$|u-u_h^\varepsilon|_{1,\Omega}\leqslant C(h+\varepsilon)\|g\|_W.$$

Furthermore, if $u\in H^{m+1}(\Omega)^n$, we get

$$|u-u_h^\varepsilon|_{1,\Omega}\leqslant C(h^m+\varepsilon)\|g\|_{m-1},$$

where $1\leqslant m\leqslant l$, l is the interger in (H1) *and* (H2), *and C is a constant independent of λ, h, ε.*

The proof of this lemma can be found in [4].

Define the operator $T_h(\lambda)$: $(0,\ g)\in W\to T_h(\lambda)g=u_h^\varepsilon\in X_h$ as the solution of (6.1), i.e.

$$a_0(u_h^\varepsilon,\ v_h)+\frac{1}{\varepsilon}(\rho_h(\nabla\cdot u_h^\varepsilon),\ \rho_h(\nabla\cdot v_h))=\lambda\langle g,\ v_h\rangle,\quad\forall v_h\in X_h,$$

Then, since $|(\rho_h(\nabla\cdot u_h^\varepsilon),\ \rho_h(\nabla\cdot v_h))|\leqslant C|u_h^\varepsilon|_{1,\Omega}|v_h|_{1,\Omega}$, we can introduce the operator B_h: $X_h\to X_h$ as

$$(\rho_h(\nabla\cdot u_h),\ \rho_h(\nabla\cdot v_h))=a_0(B_hu_h,\ v_h),\quad\forall v_h\in X_h$$

and S_h: $W\to X_h$ as

$$a_0(S_hg,\ v_h)=\langle g,\ v_h\rangle,\quad\forall v_h\in X_h.$$

So (6.1) becomes

$$a_0\Big(u_h^\varepsilon+\frac{1}{\varepsilon}B_hu_h^\varepsilon,\ v_h\Big)=a_0(\lambda S_hg,\ v_h),\ \forall v_h\in X_h.$$

Hence

$$\Big(I+\frac{1}{\varepsilon}B_h\Big)u_h^\varepsilon=\lambda S_hg.$$

According to Lemma 6.1, u_h^ε is uniquely decided and we get

$$T_h^\varepsilon(\lambda)=\lambda\Big(I+\frac{1}{\varepsilon}B_h\Big)^{-1}S_h.$$

Lemma 6.2. *Under the above hypotheses, there exists a constant C independent of λ, h, ε such that*

$$|T(\lambda)\boldsymbol{g}-T_h^\varepsilon(\lambda)\boldsymbol{g}|_{1,\Omega}\leqslant C(h+\varepsilon)\|\boldsymbol{g}\|_W,\quad \forall\lambda\in\Lambda. \tag{6.2}$$

Furthermore, if $T(\lambda)\boldsymbol{g}\in H^{m+1}(\Omega)^n$, *we have*

$$|T(\lambda)\boldsymbol{g}-T_h^\varepsilon(\lambda)\boldsymbol{g}|_{1,\Omega}\leqslant C(h^m+\varepsilon)\|\boldsymbol{g}\|_{m-1},\quad 1\leqslant m\leqslant l. \tag{6.3}$$

Proof. The whole conclusion follows from Lemma 6.1 naturally. Given a function $\varepsilon(h)$: $R_+\to R_+$ satisfying $\lim\limits_{h\to 0}\varepsilon(h)=0$, let $\tilde{T}_h(\lambda)=T_h^{\varepsilon(h)}(\lambda)$ and

$$F_h(\lambda,\boldsymbol{u})\equiv\boldsymbol{u}+\tilde{T}_h(\lambda)G(\lambda,\boldsymbol{u}). \tag{6.4}$$

Theorem 6.1. *Let* $\{\boldsymbol{u}(\lambda);\lambda\in\Lambda\}$ *be a nonsingular solution of* (5.1). *Then there exists a constant* $h_0>0$, *such that if* $h\leqslant h_0$ *is small enough, there is a unique* C^∞ *mapping* $\boldsymbol{u}_h(\lambda)$: $\Lambda\to H$ *satisfying*

$$F_h(\lambda,\boldsymbol{u}_h(\lambda))=0,$$
$$\|D^m\boldsymbol{u}(\lambda)-D^m\boldsymbol{u}_h(\lambda)\|\leqslant K_m(h+\varepsilon(h)). \tag{6.5}$$

Furthermore, if $\boldsymbol{f}\in H^{m-1}(\Omega)^n$, $\boldsymbol{u}(\lambda)\in H^m(\Omega)^n$, *we get the estimate*

$$\|D^m\boldsymbol{u}(\lambda)-D^m\boldsymbol{u}_h(\lambda)\|\leqslant K_m(h^m+\varepsilon(h)),$$

where K_m *is a constant independent of* λ, h.

Proof. By the definitions of $T(\lambda)$, $\tilde{T}_h(\lambda)$, $G(\boldsymbol{u})$ and Lemma 6.2, we see that the hypotheses in Theorem 2.1 hold. So there exists $\boldsymbol{u}_h(\lambda)$ satisfying (6.5) and

$$\begin{aligned}\|D^m\boldsymbol{u}(\lambda)-D^m\boldsymbol{u}_h(\lambda)\|&\leqslant K_m^1\sum_{i=0}^m\left\|\frac{d^i}{d\lambda^i}[T(\lambda)G(\boldsymbol{u}(\lambda))-\tilde{T}_h(\lambda)G(\boldsymbol{u}(\lambda))]\right\|\\&\leqslant K_m^1\sum_{i=0}^3\left\{\left\|(T(\lambda)-\tilde{T}_h(\lambda))\frac{d^i}{d\lambda^i}G(\boldsymbol{u}(\lambda))\right\|\right.\\&\quad\left.+\left\|(T(1)-\tilde{T}_h(1))\frac{d^i}{d\lambda^i}G(\boldsymbol{u}(\lambda))\right\|\right\}.\end{aligned}$$

Using (6.2) we derive (6.4) and if $\boldsymbol{f}\in H^{m-1}(\Omega)^n$, $\boldsymbol{u}(\lambda)\in H^m(\Omega)^n$, then $G(\boldsymbol{u}(\lambda))\in H^{m-1}(\Omega)^n$, which in turn shows $T(\lambda)G(\boldsymbol{u}(\lambda))\in H^{m+1}(\Omega)$. So we complete the proof by using (6.3).

Theorem 6.2. *Let* (λ_0, u_0) *be a simple limit point of* (6.1). *Then there exist constants* α_0, h_0, *such that for* $h\leqslant h_0$ *small enough, there are two families of* C^∞ *mappings* $\{(\lambda(\alpha),\boldsymbol{u}(\alpha));|\alpha|\leqslant\alpha_0\}$, $\lambda(\alpha)=\lambda_0$, $\boldsymbol{u}(0)=\boldsymbol{O}$, *and* $\{(\lambda_h(\alpha),\boldsymbol{u}_h(\alpha));|\alpha|\leqslant\alpha_0\}$ *satisfying*

$$F(\lambda(\alpha),\boldsymbol{u}(\alpha))=0,\quad F(\lambda_h(\alpha),\boldsymbol{u}_h(\alpha))=0,\quad\forall|\alpha|\leqslant\alpha_0$$

and we have the estimates

$$|\lambda_h^{(m)}(\alpha)-\lambda^{(m)}(\alpha)|+\|\boldsymbol{u}_h^{(m)}(\alpha)-\boldsymbol{u}^{(m)}(\alpha)\|\leqslant K_m(h+\varepsilon(h)).$$

Furthermore, if $\boldsymbol{f}\in H^{m-1}(\Omega)^n$, $\boldsymbol{u}(\alpha)\in H^m(\Omega)^n$, *we get*

$$|\lambda_h^{(m)}(\alpha)-\lambda^{(m)}(\alpha)|+\|\boldsymbol{u}_h^{(m)}(\alpha)-\boldsymbol{u}^{(m)}(\alpha)\|\leqslant K_m(h^m+\varepsilon(h)),$$

where K_m *is a constant independent of* h *and* λ.

Proof. It is obvious that all conditions in Theorem 2.2 are satisfied. So the branch solutions exist and we have

$$\begin{aligned}&|\lambda_h^{(m)}(\alpha)-\lambda^{(m)}(\alpha)|+\|\boldsymbol{u}_h^{(m)}(\alpha)-\boldsymbol{u}^{(m)}(\alpha)\|\\&\leqslant K_m\sum_{i=0}^m\left\|\frac{d^i}{d\alpha^i}[(T(\lambda(\alpha))-T_h(\lambda(\alpha))G(\boldsymbol{u}(\alpha))]\right\|.\end{aligned}$$

In the same way as the proof of Theorem 6.1 the proof can be completed.

References

[1] F. Brezzi, J. Rappaz, P. A. Raviart, Finite dimensional approximation of nonlinear problems. Part I: Branches of nonsingular solutions, *Numer. Math.*, **36** (1980), 1—25.

[2] F. Brezzi, J. Rappaz, P. A. Raviart, Finite dimensional approximation of nonlinear problems. Part II: Limit points, *Numer. Math.*, **37** (1981), 1—28.

[3] P. G. Cialet, The Finite Element Method for Elliptic Problems, Amsterdam, North-Holland, 1978.

[4] V. Girault, P. A. Raviart, Finite element approximation of the Navier-Stokes equations, Lecture Notes in Math., Vol. 749, 1979.

[5] Guan Zhao-zhi, et al., Linear functional analysis (in Chinese), 1980.

[6] F. Kikuchi, Finite element of approximation to bifurcation problems of turning point type, I. N. R. I. A Meeting, "Méthodes de calcul scientifique et technique III", 1977.

[7] Li Kai-tai, et al., Bifurcation problems of the Navier-Stokes equations, *Scientific Reports of Xi'an Jiaotong University*, No. 195, 1983.

[8] R. Scholz, Computation of turning points of the stationary Navier-Stokes equations using mixed finite element, ISNM54, Bifurcation problems and their numerical solution, 1980.

[9] R. Temam, Navier-Stokes equations, Amsterdam, North-Holland, 1977.

[10] F. Riesz, B. Sz-Nagy, Lecons D'analyse Fonctionelle, Arad'emiai Riado' Budapest, 1955.