ERROR ESTIMATES OF TWO NONCONFORMING FINITE ELEMENTS FOR THE OBSTACLE PROBLEM*

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Abstract

The linear nonconforming element and Wilson's element for the obstacle problem are considered. Optimal error bounds for both elements are obtained in the case of regular subdivisions of domain Ω in R^2.

§ 1. Abstract Error Estimate

There are a number of works in the analysis of finite element methods for variational inequalities (c.f. [7] and the references therein). Particularly the analysis of F.E.M for the obstacle problem has been studied more or less completely (c.f. [2]—[9]). All these analyses, however, are related to conforming finite elements, except for the mixed type[3].

In this paper, we will analyse two nonconforming finite elements for the obstacle problem. We first show an abstract error estimate, which is similar to the Second Strang Lemma[5]. Next, in § 2, we will analyse the linear nonconforming element approximation to the obstacle problem. Finally, in § 3, Wilson's element will be considered for the obstacle problem.

Let Ω be a convex domain in R^2 with piecewise smooth boundary $\partial\Omega$, X a Hilbert space of functions defined on Ω with norm $\|\cdot\|$, K a nonempty convex closed subset in X, and $a(\cdot, \cdot)$ a continuous, X-elliptic, bilinear from on $X\times X$, $f\in X'$——the dual space of X, with the duality pairing $\langle\cdot, \cdot\rangle$ between X' and X. The abstract variational inequality considered is the following:

$$\begin{cases}\text{find } u\in K, \text{ such that} \\ a(u, v-u)\geqslant\langle f, v-u\rangle \quad \forall v\in K.\end{cases} \tag{1.1}$$

The solution of (1.1) will be approximated by the finite element method for a regular subdivision. For each $h>0$, let $\mathscr{T}_h$ be a regular subdivision on Ω[5], $\Omega^h=\bigcup_{\tau\in\mathscr{T}_h}\tau$, X_h be a finite element approximate space of X with norm $\|\cdot\|_h$ (either conforming or nonconforming, i.e., $X_h\subset X$ or $X_h\not\subset X$ respectively), and K_h be a convex closed subset in X_h, as an approximation of K. Then the approximate problem of (1.1) is the following:

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$$\begin{cases} \text{find } u_h \in K_h, \text{ such that} \\ a_h(u_h, v_h - u_h) \geqslant \langle f, v_h - u_h \rangle_h \quad \forall v_h \in K_h, \end{cases} \tag{1.2}$$

where

$$a_h(u_h, v_h) = \sum_{\tau \in \mathscr{T}_h} a(u_h|_\tau, v_h|_\tau),$$

$$\langle f, v_h \rangle_h = \sum_{\tau \in \mathscr{T}_h} \langle f|_\tau, v_h|_\tau \rangle,$$

and $u_h|_\tau$, $v_h|_\tau$ and $f|_\tau$ are the restrictions of u, v and f on the element τ respectively.

Throughout this paper we will use the notations of Sobolev spaces $H^m(\Omega)$ as in [1] and we assume that C is a generic constant, which may have different values in different places, if not specifically indicated.

We have an abstract error estimate, similar to Second Strang Lemma[5], as follows:

Theorem 1. *Assume that $a_h(\cdot, \cdot)$ is a continuous, X_h-elliptic, bilinear form on $X_h \times X_h$, and u and u_h are the solutions of problems (1.1) and (1.2) respectively. Then there exists a constant C independent of X_h such that*

$$\|u - u_h\|_h < C \inf_{v_h \in K_h} \left\{ \|u - v_h\|_h + \frac{a_h(u, v_h - u_h) - \langle f, v_h - u_h \rangle_h}{\|u_h - v_h\|_h} \right\}. \tag{1.3}$$

The proof is easy and similar to that in [5], so it is omitted.

§ 2. The Linear Nonconforming Element

To begin with, we consider the obstacle problem:

$$\begin{cases} \text{find } u \in K, \text{ such that} \\ a(u, v-u) \geqslant \langle f, v-u \rangle \quad \forall v \in K, \end{cases} \tag{2.1}$$

where

$$K = \{v \in H^1(\Omega) : v \geqslant \psi \text{ a.e. in } \Omega,\ v = g \text{ on } \partial\Omega\}, \tag{2.2}$$

$$\begin{cases} a(u, v) = \int_\Omega (\nabla u \cdot \nabla v + u \cdot v) dx, \\ \langle f, v \rangle = \int_\Omega f \cdot v \, dx \end{cases} \tag{2.3}$$

and $f \in L^2(\Omega)$, g, $\psi \in H^2(\Omega)$, $g \geqslant \psi$ on $\partial\Omega$.

We now solve problem (2.1) using the linear nonconforming finite element approximation. Let B_i $(1 \leqslant i \leqslant 3)$ be the midpoints of edges of triangle $\tau \in \mathscr{T}_h$, and X_h be a space consisting of the piecewise linear functions with nodes at B_i, which is Crouzeix-Raviart's element space $(r = 1)$. Let K_h be a convex subset of X_h as follows:

$$K_h = \{v^h \in X_h : v^h \geqslant \psi \text{ at the nodes in } \Omega^h,\ v^h = g(P_m) \text{ at the nodes } m \text{ on } \partial\Omega^h\}, \tag{2.4}$$

where P_m is the intersection point of $\partial\Omega$ with the outer normal at the node m on $\partial\Omega^h$ (Fig. 1).

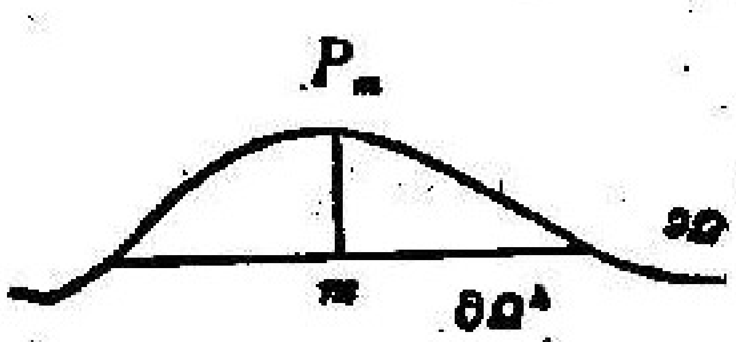

Fig. 1

The linear nonconforming finite element approximation of (2.1) is the following:

$$\begin{cases} \text{find } u^h \in K_h, \text{ such that} \\ a(u^h, v^h - u^h) \geqslant \langle f, v^h - u^h \rangle_h \quad \forall v^h \in K_h, \end{cases} \tag{2.5}$$

where

$$\begin{cases} a_h(u^h, v^h) = \sum_{\tau \in \mathscr{T}_h} \int_\tau (\nabla u^h \cdot \nabla v^h + u^h \cdot v^h)\, dx \\ \langle f, v^h \rangle_h = \int_{\Omega^h} f \cdot v^h\, dx. \end{cases} \tag{2.6}$$

The norm $\|\cdot\|_h$ in X_h is defined as follows: $\forall v^h \in X_h$,

$$\|v^h\|_h^2 = \sum_{\tau \in \mathscr{T}_h} \|v^h\|_{1,\tau}^2. \tag{2.7}$$

Let Π: $H^2(\Omega) \to X_h$ be an interpolation operator defined as follows: for any given $v \in H^2(\Omega)$, we define $\Pi v \in X_h$ such that

$$(\Pi v)(B_i) = v(B_i), \quad 1 \leqslant i \leqslant 3, \ \forall \tau \in \mathscr{T}_h. \tag{2.8}$$

Then due to the well-known results in [5], we have the following error estimates:

Lemma 1. $v \in H^l(\tau)$,

$$|v - \Pi_\tau v|_{m,\tau} \leqslant C h^{l-m} |v|_{l,\tau}, \quad m = 0, 1, 2. \ l = 2, \tag{2.9}$$

and

$$\|v - \Pi v\|_h \leqslant Ch |v|_{2,\Omega} \quad \forall v \in H^2(\Omega), \tag{2.10}$$

where $\Pi_\tau v = (\Pi v)|_\tau$.

Let v^I be another modified linear interpolation of $v \in H^2(\Omega)$, which is the same as Πv except for the boundary nodes $m \in \partial\Omega^h$:

$$v^I(m) = v(P_m). \tag{2.11}$$

Then $v^I \in K_h$ if $v \in K$. Since for $v \in K$,

$$(\Pi v)(m) = v(m) \ \forall \text{ boundary nodes } m \in \partial\Omega^h, \tag{2.12}$$

then in general $\Pi v \bar{\in} K_h$.

Using the technique as in [12] for the analysis of Crouzeix–Raviart's element $(r=1)$, we have the following

Lemma 2. *There exis's a constant C independent of h, such that*

$$\left| \sum_{\tau \in \mathscr{T}_h} \int_{\partial\tau} \partial_{\nu_\tau} u \cdot (u^I - v^h)\, d\gamma \right| < Ch |u|_{2,\Omega} \cdot \|u^I - v^h\|_h, \quad \forall v^h \in K_h, \tag{2.13}$$

where ∂_{ν_τ} *denotes the outer normal derivative operator.*

We now will show the error estimate of the linear nonconforming element for the obstacle problem.

Theorem 2. *Assume that* $f \in L^2(\Omega)$, ψ, $g \in H^2(\Omega)$, *u and* u^h *are the solutions of* (2.1) *and* (2.5) *respectively, with* $u \in H^2(\Omega)$. *Let* Ω *be a convex domain in* R^2 *with Lipschitz boundary* $\partial\Omega$, *and* $\mathscr{T}_h$ *be a regular triangulation of* Ω. *Then there exists a constant C dependent on u,* ψ *and f, but independent of h, such that*

$$\|u - u^h\|_h \leqslant Ch. \tag{2.14}$$

Proof. By a well-known result, the solution u of (2.1) satisfies the following problem:

$$\begin{cases} -\Delta u + u - f \geqslant 0, \ (-\Delta u + u - f)(\psi - u) = 0, \quad \text{a.e. in } \Omega, \\ u = g \text{ on } \partial\Omega. \end{cases} \tag{2.15}$$

Since $u^I\in K_h$, from Theorem 1 we have

$$\|u-u^h\|_h\leqslant C\left\{\|u-u^I\|_h+\frac{a_h(u,\ u^I-u^h)-\langle f,\ u^I-u^h\rangle_h}{\|u^I-u^h\|_h}\right\}. \tag{2.16}$$

Let us estimate the first term on the right side of (2.16):

$$\|u-u^I\|_h\leqslant\|u-\Pi u\|_h+\|\Pi u-u^I\|_h. \tag{2.17}$$

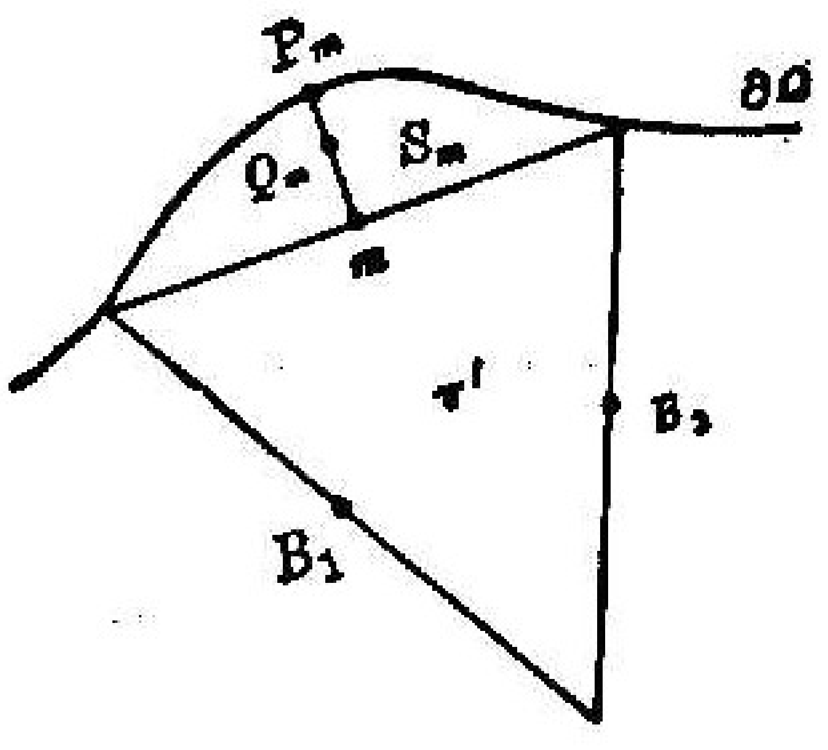

Fig. 2

From Lemma 1 (c.f. (2.10)), we have

$$\|u-\Pi u\|_h\leqslant Ch|u|_{2,\Omega}. \tag{2.18}$$

Let $\mathscr{T}'_h$ be the set of boundary elements τ' (c.f. Fig. 2). Then from the definitions of Πu and u^I, we have

$$\|\Pi u-u^I\|_h^2=\sum_{\tau'\in\mathscr{T}'_h}\|\Pi u-u^I\|_{1,\tau'}^2. \tag{2.19}$$

On τ', we have

$$\begin{aligned}(\Pi u-u^I)(x)&=(\Pi u-u^I)(m)\cdot\mu_m(x)\\ &=(u(m)-u(P_m))\cdot\mu_m(x),\end{aligned} \tag{2.20}$$

where μ_m is a linear interpolation basic function on τ', such that $\mu_m(m)=1$, $\mu_m(B_i)=0$, $i=1, 2$. Since the boundary $\partial\Omega$ is piecewise smooth, $|\overline{P_m m}|\leqslant Ch^2$ (c.f. Fig. 2); then from the imbedding theorem [1; Ch. 5.4, case B]

$$\begin{aligned}|(\Pi u-u^I)(x)|&\leqslant C\int_{\overline{mP_m}}|\nabla u|dt\leqslant C|P_m m|^{3/4}\left(\int_{\overline{mP_m}}|\nabla u|^4dt\right)^{1/4}\\ &\leqslant Ch^{3/2}\cdot|u|_{W^{1,4}(\overline{P_m m})}\leqslant Ch^{3/2}\|u\|_{2,\Omega},\end{aligned}$$

from which,

$$\|\Pi u-u^I\|_{0,\tau'}^2\leqslant Ch^5\|u\|_{2,\Omega}^2. \tag{2.21}$$

Similarly, from (2.20), we have

$$\begin{aligned}|\nabla(\Pi u-u^I)(x)|&\leqslant|u(m)-u(P_m)|\cdot|\nabla\mu_m(x)|\\ &\leqslant Ch^{-1}\int_{\overline{mP_m}}|\nabla u|dt\leqslant Ch^{-1}|\overline{P_m m}|^{3/4}\cdot|u|_{W^{1,4}(\overline{P_m m})}\leqslant Ch^{1/2}\|u\|_{2,\Omega},\end{aligned}$$

from which,

$$|\Pi u-u^I|_{1,\tau'}^2\leqslant Ch^3\|u\|_{2,\Omega}^2. \tag{2.22}$$

From (2.19), (2.21) and (2.22), we have

$$\|\Pi u-u^I\|_h\leqslant Ch\|u\|_{2,\Omega}. \tag{2.23}$$

By (2.17), (2.18) and (2.23), we have

$$\|u-u^I\|_h\leqslant Ch|u|_{2,\Omega}. \tag{2.24}$$

We now estimate the second term on the right side of (2.16), by use of Green's formula, as follows:

$$\begin{aligned}&a_h(u,\ u^I-u^h)-\langle f,\ u^I-u^h\rangle_h\\ &=\sum_{\tau\in\mathscr{T}_h}\int_\tau[\nabla u\cdot\nabla(u^I-u^h)+u\cdot(u^I-u^h)]dx-\int_{\Omega_h}f\cdot(u^I-u^h)dx\\ &=\int_{\Omega_h}(-\Delta u+u-f)(u^I-u^h)dx+\sum_{\tau\in\mathscr{T}_h}\int_{\partial\tau}\partial_{\nu_\tau}u\cdot(u^I-u^h)d\gamma.\end{aligned} \tag{2.25}$$

Taking Lemma 2 into account, we need to estimate only the first term on the right side of (2.25). Let $w=-\Delta u+u-f\in L^2(\Omega)$; then

$$\int_{\Omega^h}(-\Delta u+u-f)(u^I-u^h)dx$$
$$=\langle w,\ u^I-u^h\rangle_h=\langle w,\ \Pi u-u^h\rangle_h+\langle w,\ u^I-\Pi u\rangle_h. \tag{2.26}$$

From (2.21) and the definition of u^I, we have

$$|\langle w,\ u^I-\Pi u\rangle_h|\leqslant\|w\|_{0,\Omega}\cdot\|u^I-\Pi u\|_{0,\Omega^h}$$
$$=\|w\|_{0,\Omega}\Big(\sum_{\tau'\in\mathcal{T}_h'}\|u^I-\Pi u\|^2_{0,\tau'}\Big)^{1/2}\leqslant Ch^2\|w\|_{0,\Omega}\cdot\|u\|_{2,\Omega}. \tag{2.27}$$

Applying (2.15), we have

$$\langle w,\ \Pi u-u^h\rangle_h=\langle w,\ \Pi(u-\psi)-(u-\psi)\rangle_h+\langle w,\ u-\psi\rangle_h+\langle w,\ \Pi\psi-u^h\rangle_h$$
$$=\langle w,\ \Pi(u-\psi)-(u-\psi)\rangle_h+\langle w,\ \Pi\psi-u^h\rangle_h. \tag{2.28}$$

Due to (2.9), then

$$\langle w,\ \Pi(u-\psi)-(u-\psi)\rangle_h\leqslant\|w\|_{0,\Omega}\cdot\|\Pi(u-\psi)-(u-\psi)\|_{0,\Omega^h}$$
$$\leqslant Ch^2\|w\|_{0,\Omega}\cdot|u-\psi|_{2,\Omega}. \tag{2.29}$$

The second term on the right side of (2.28) can be written as

$$\langle w,\ \Pi\psi-u^h\rangle_h=\langle w,\ \psi^I-u^h\rangle_h+\langle w,\ \Pi\psi-\psi^I\rangle_h. \tag{2.30}$$

Since $w\geqslant0$, a.e. in Ω, and $\psi^I-u^h\leqslant0$ at the nodes on Ω^h, and ψ^I-u^h is piecewise linear, so $\psi^I-u^h\leqslant0$ on Ω^h. Thus $\langle w,\ \psi^I-u^h\rangle_h\leqslant0$, from which and (2.21) we have

$$\langle w,\ \Pi\psi-u^h\rangle_h\leqslant\langle w,\ \Pi\psi-\psi^I\rangle_h\leqslant\|w\|_{0,\Omega}\cdot\|\Pi\psi-\psi^I\|_{0,\Omega^h}$$
$$\leqslant Ch^2\|w\|_{0,\Omega}\cdot\|\psi\|_{2,\Omega}. \tag{2.31}$$

From (2.26)—(2.31), it can be seen that

$$\int_{\Omega^h}(-\Delta u+u-f)(u^I-u^h)dx\leqslant Ch^2\|w\|_{0,\Omega}\{\|u\|_{2,\Omega}+\|\psi\|_{2,\Omega}\}. \tag{2.32}$$

From (2.25), (2.32) and Lemma 2, we have

$$a_h(u,\ u^I-u^h)-\langle f,\ u^I-u^h\rangle_h\leqslant Ch^2+Ch\|u^I-u^h\|_h, \tag{2.33}$$

where C is a constant dependent only on $f\in L^2(\Omega)$, ψ, $u\in H^2(\Omega)$.

If $\|u^I-u^h\|_h\geqslant h$, then from (2.33),

$$\frac{a_h(u,\ u^I-u^h)-\langle f,\ u^I-u^h\rangle_h}{\|u^I-u^h\|_h}\leqslant Ch, \tag{2.34}$$

from which and (2.16), (2.24), we can see

$$\|u-u^h\|_h\leqslant Ch.$$

If $\|u^I-u^h\|_h\leqslant h$, then from (2.24) and the triangle inequality,

$$\|u-u^h\|_h\leqslant\|u-u^I\|_h+\|u^I-u^h\|_h\leqslant Ch.$$

Thus the proof is completed.

Remark 1. It should be noted that for a triangular subdivision the number of midpoints of all edges is nearly 3 times the number of all vertices. So the result of Theorem 2 is of interest only in its theoretic respect.

§ 3. Wilson's Element

In this section, we will consider Wilson's element approximation of problem (2.1) on a rectangular domain Ω in R^2.

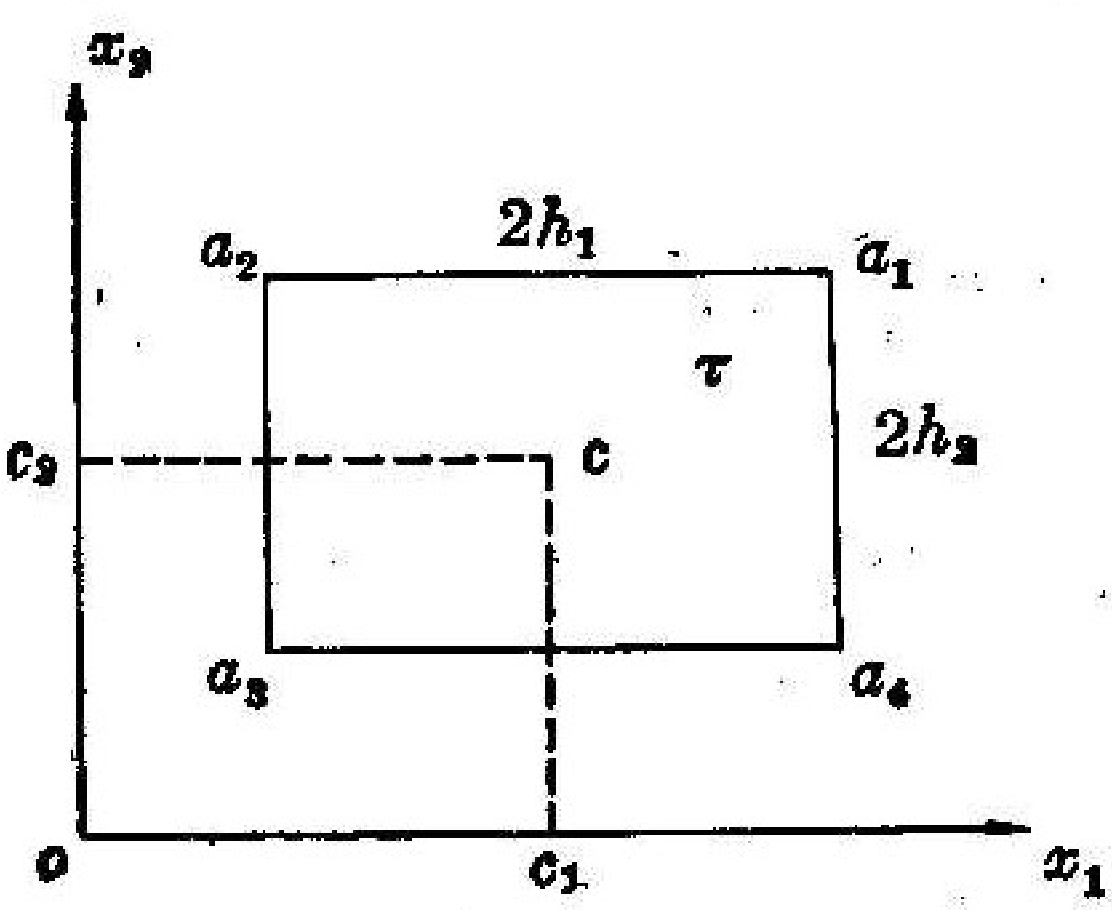

Fig. 3

Let $\tau\in\mathscr{T}_h$ be a rectangle, whose vertices will be denoted by $a_i\,(1\leqslant i\leqslant 4)$ (Fig. 3). By the same notations as in [5, 10, 11], Wilson's element $(\tau, p_\tau, \Sigma_\tau)$, $\forall\tau\in\mathscr{T}_h$, is defined as

$$P_\tau=P_2(\tau),$$

$$\Sigma_\tau=\Big\{p(a_i),\ 1\leqslant i\leqslant 4;\quad (h_i^2/h_1h_2)\int_\tau \partial_{ii}p\,dx,\ 1\leqslant i\leqslant 2\Big\}.$$

Then for any given $p\in P_\tau$, we have

$$p(x)=\sum_{i=1}^{4}p(a_i)p_i(x)+\sum_{i=1}^{2}\varphi_i(p)\cdot q_i(x), \tag{3.1}$$

where

$$\begin{cases} p_1(x)=\frac{1}{4}\Big(1+\frac{x_1-c_1}{h_1}\Big)\Big(1+\frac{x_2-c_2}{h_2}\Big),\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ p_4(x)=\frac{1}{4}\Big(1+\frac{x_1-c_1}{h_1}\Big)\Big(1-\frac{x_2-c_2}{h_2}\Big); \end{cases} \tag{3.2}$$

$$\begin{cases} q_i(x)=\frac{1}{8}\Big[\Big(\frac{x_i-c_i}{h_i}\Big)^2-1\Big], & 1\leqslant i\leqslant 2,\\ \varphi_i(p)=\frac{h_i^2}{h_1h_2}\int_\tau \partial_{ii}p\,dx, & 1\leqslant i\leqslant 2, \end{cases} \tag{3.3}$$

$$c=\frac{1}{4}\sum_{i=1}^{4}a_i. \tag{3.4}$$

Let the interpolation operator $\Pi_\tau: H^2(\tau)\to p_\tau$ be defined as follows: for each given $v\in H^2(\tau)$,

$$\Pi_\tau v=\sum_{i=1}^{4}v(a_i)p_i(x)+\sum_{i=1}^{2}\varphi_i(v)q_i(x). \tag{3.5}$$

Let X_h be a piecewise polynomial space consisting of above Wilson's rectangular elements $(\tau, p_\tau, \Sigma_\tau)$ $\forall\tau\in\mathscr{T}_h$, with norm $\|\cdot\|_h=\sum_{\tau\in\mathscr{T}_h}\|\cdot\|_{1,\tau}^2)^{1/2}$. Since $X_h\not\subset H^1(\Omega)\cap C^0(\bar\Omega)$, Wilson's element is nonconforming. We consider approximation of the obstacle problem as follows:

$$\begin{cases} \text{find } u_h\in K_h, \text{ such that}\\ a_h(u_h, v_h-u_h)\geqslant\langle f, v_h-u_h\rangle_h \quad \forall v_h\in K_h, \end{cases} \tag{3.6}$$

where

$$K_h=\{v_h\in X_h:\ v_h(a_i)\geqslant\psi(a_i),\ 1\leqslant i\leqslant 4,\ \varphi_i(v_h)\leqslant\varphi_i(\psi),\ 1\leqslant i\leqslant 2, \\ \forall\tau\in\mathscr{T}_h;\ v_h(Q)=g(Q)\ \forall\text{ boundary nodes } Q\in\partial\Omega\}. \tag{3.7}$$

We also assume that the subdivision $\mathscr{T}_h$ is regular and the inverse assumption holds, i.e., there exist σ, ν=const. >0, such that

$$\bar{h}_\tau \leqslant \sigma \underline{h}_\tau, \ h \leqslant \nu \bar{h}_\tau \quad \forall \tau \in \mathscr{T}_h, \ h>0, \tag{3.8}$$

where $\bar{h}_\tau$ and $\underline{h}_\tau$ are the lengths of longest and shortest edges of τ respectively and $h=\max\limits_{\tau\in\mathscr{T}_h} \bar{h}_\tau$.

In order to get the error estimate $\|u-u_h\|_h$, we need

Lemma 3. *Let another interpolation operator $\widetilde{\Pi}_\tau: H^2(\tau)\to P_\tau$ be defined as follows: for each given $v\in H^2(\tau)$,*

$$\widetilde{\Pi}_\tau v=\sum_{i=1}^4 v(a_i)p_i(x)+\sum_{i=1}^4 \Phi_i(v, \psi)\cdot q_i(x), \tag{3.9}$$

where

$$\Phi_i(v, \psi)=\min(\varphi_i(v), \varphi_i(\psi)). \tag{3.10}$$

Then

$$\varphi_i(\widetilde{\Pi}_\tau v)=\Phi_i(v, \psi)\leqslant\varphi_i(\psi), \quad \widetilde{\Pi}_\tau v(a_i)=v(a_i). \tag{3.11}$$

And if $v\in K$, $\widetilde{\Pi}v|_\tau=\widetilde{\Pi}_\tau v$, then

$$\widetilde{\Pi}v\in K_h. \tag{3.12}$$

The proof is easy, so we omit it.

It should be noted that Lemmas 1 (for $l=2, 3$) and 2 are also true for Wilson's element (c.f. [5] and [12]).

From Theorem 1 and Lemmas 1—3 for Wilson's element, we derive the following error estimate:

Theorem 3. *Assume that $f\in L^2(\Omega)$, ψ and $g\in H^2(\Omega)$, u and u_h are the solutions of* (2.1) *and* (3.6) *respectively, and that hypothesis* (3.8) *holds for subdivision $\mathscr{T}_h$. Then there exists a constant C independent of h, such that*

$$\|u-u_h\|_h\leqslant Ch(\|w\|_{0,\Omega}+|u|_{2,\Omega}+|\psi|_{2,\Omega}), \tag{3.13}$$

where

$$w=-\Delta u+u-f\in L^2(\Omega). \tag{3.14}$$

Proof. From Theorem 1 and Lemma 3 we can see that

$$\|u-u_h\|_h<C\left\{\|u-\widetilde{\Pi}u\|_h+\frac{a_h(u, \widetilde{\Pi}u-u_h)-\langle f, \widetilde{\Pi}u-u_h\rangle_h}{\|\widetilde{\Pi}u-u_h\|_h}\right\}. \tag{3.15}$$

Let us first estimate

$$\|u-\widetilde{\Pi}u\|_h=(\sum_{\tau\in\mathscr{T}_h}\|u-\widetilde{\Pi}_\tau u\|_{1,\tau}^2)^{1/2}. \tag{3.16}$$

Due to Lemma 1 and the triangle inequality,

$$\begin{aligned}\|u-\widetilde{\Pi}_\tau u\|_{1,\tau}&\leqslant\|u-\Pi_\tau u\|_{1,\tau}+\|\Pi_\tau u-\widetilde{\Pi}_\tau u\|_{1,\tau}\\&\leqslant Ch|u|_{2,\tau}+\|\Pi_\tau u-\widetilde{\Pi}_\tau u\|_{1,\tau}.\end{aligned} \tag{3.17}$$

And from (3.5), (3.9), we have

$$\begin{aligned}\|\Pi_\tau u-\widetilde{\Pi}_\tau u\|_{1,\tau}&\leqslant\sum_{i=1}^2|\varphi_i(u)-\Phi_i(u, \psi)|\cdot\|q_i\|_{1,\tau}\\&\leqslant\sum_{i=1}^2(h_i^2/h_1h_2)\left|\int_\tau\partial_{ii}(u-\psi)dx\right|\cdot\|q_i\|_{1,\tau}\\&\leqslant Ch|u-\psi|_{2,\tau}\cdot\sum_{i=1}^2\|q_i\|_{1,\tau}.\end{aligned} \tag{3.18}$$

It is easy to verify that

$$\begin{cases}\|q_i\|_{0,\tau}=(h_1h_2/30)^{1/2},\\ |q_i|_{1,\tau}=(h_1h_2/12h_i^2)^{1/2}, \quad i=1,2.\end{cases} \tag{3.19}$$

Thus

$$\sum_{i=1}^{2}\|q_i\|_{1,\tau}\leqslant C(\sigma) \quad \forall\tau\in\mathscr{T}_h. \tag{3.20}$$

From (3.18) and (3.20), we have

$$\|\Pi_\tau u-\tilde{\Pi}_\tau u\|_{1,\tau}\leqslant Ch(|u|_{2,\tau}+|\psi|_{2,\tau}), \tag{3.21}$$

from which and (3.17), it can be seen that

$$\|u-\tilde{\Pi}_\tau u\|_{1,\tau}\leqslant Ch(|u|_{2,\tau}+|\psi|_{2,\tau}). \tag{3.22}$$

Thus we have, taking (3.16) into account,

$$\|u-\tilde{\Pi}u\|_h< Ch(|u|_{2,\Omega}+|\psi|_{2,\Omega}). \tag{3.23}$$

We now estimate the second term on the right side of (3.15) by use of Green's formula:

$$\begin{aligned}&a_h(u,\tilde{\Pi}u-u_h)-\langle f,\tilde{\Pi}u-u_h\rangle\\ &=\sum_{\tau\in\mathscr{T}_h}\int_\tau[\nabla u\cdot\nabla(\tilde{\Pi}u-u_h)+u(\tilde{\Pi}u-u_h)]dx-\int_\Omega f\cdot(\tilde{\Pi}u-u_h)dx\\ &=\int_\Omega(-\Delta u+u-f)(\tilde{\Pi}u-u_h)dx+\sum_{\tau\in\mathscr{T}_h}\int_{\partial\tau}\partial_{\nu_\tau}u\cdot(\tilde{\Pi}u-u_h)d\gamma.\end{aligned} \tag{3.24}$$

Let $w=-\Delta u+u-f\in L^2(\Omega)$. Then the first term on the right side of (3.24) is

$$\begin{aligned}&\int_\Omega(-\Delta u+u-f)(\tilde{\Pi}u-u_h)dx\\ &=\langle w,\tilde{\Pi}u-u_h\rangle=\langle w,\tilde{\Pi}u-\Pi u\rangle+\langle w,\Pi u-u_h\rangle.\end{aligned} \tag{3.25}$$

From (3.5), (3.9) and (3.19), we can see that

$$\begin{aligned}\|\tilde{\Pi}_\tau u-\Pi_\tau u\|_{0,\tau}&\leqslant\sum_{i=1}^{2}|\varphi_i(u)-\Phi_i(u,\psi)|\cdot\|q_i\|_{0,\tau}\\ &\leqslant Ch^2|u-\psi|_{2,\tau}, \quad \forall\tau\in\mathscr{T}_h,\end{aligned}$$

from which

$$\|\tilde{\Pi}u-\Pi u\|_{0,\Omega}=\Big(\sum_{\tau\in\mathscr{T}_h}\|\tilde{\Pi}_\tau u-\Pi_\tau u\|_{0,\tau}^2\Big)^{1/2}\leqslant Ch^2|u-\psi|_{2,\Omega}. \tag{3.26}$$

Thus

$$|\langle w,\tilde{\Pi}u-\Pi u\rangle|\leqslant\|w\|_{0,\Omega}\cdot\|\tilde{\Pi}u-\Pi u\|_{0,\Omega}\leqslant Ch^2\|w\|_{0,\Omega}\cdot|u-\psi|_{2,\Omega}. \tag{3.27}$$

The second term on the right side of (3.25) can be written as

$$\langle w,\Pi u-u_h\rangle=\langle w,\Pi(u-\psi)-(u-\psi)\rangle+\langle w,u-\psi\rangle+\langle w,\Pi\psi-u_h\rangle. \tag{3.28}$$

Since (2.15), $\langle w,u-\psi\rangle=0$. Due to $u_h\in K_h$, $\Pi\psi\leqslant u_h$, and (2.15) $w\geqslant0$ a.e. in Ω, so $\langle w,\Pi\psi-u_h\rangle\leqslant0$. Thus from (3.28) and Lemma 1 we have

$$\langle w,\Pi u-u_h\rangle\leqslant\langle w,\Pi(u-\psi)-(u-\psi)\rangle\leqslant Ch^2\|w\|_{0,\Omega}\cdot|u-\psi|_{2,\Omega}. \tag{3.29}$$

From (3.27), (3.29) and (3.25) we can see

$$\int_\Omega(-\Delta u+u-f)(\tilde{\Pi}u-u_h)dx\leqslant Ch^2\|w\|_{0,\Omega}\cdot|u-\psi|_{2,\Omega}, \tag{3.30}$$

from which and (3.24), by Lemma 2, we have

$$a_h(u, \tilde{\Pi}u-u_h)-\langle f, \tilde{\Pi}u-u_h\rangle \leqslant Ch^2\|w\|_{0,\Omega}\cdot|u-\psi|_{2,\Omega}+Ch|u|_{2,\Omega}\cdot\|\tilde{\Pi}u-u_h\|_h. \tag{3.31}$$

If $\|\tilde{\Pi}u-u_h\|_h\geqslant h$, then from (3.31), we have

$$\frac{a_h(u, \tilde{\Pi}u-u_h)-\langle f, \tilde{\Pi}u-u_h\rangle}{\|\tilde{\Pi}u-u_h\|_h}\leqslant ch(\|w\|_{0,\Omega}+|u|_{2,\Omega}+|\psi|_{2,\Omega}),$$

from which and (3.15), (3.22), comes

$$\|u-u_h\|_h\leqslant ch(\|w\|_{0,\Omega}+|u|_{2,\Omega}+|\psi|_{2,\Omega}).$$

If $\|\tilde{\Pi}u-u_h\|_h\leqslant h$, then by use of the triangle inequality and (3.23) it can be seen immediately that

$$\|u-u_h\|_h\leqslant\|u-\tilde{\Pi}u\|_h+\|\tilde{\Pi}u-u_h\|_h\leqslant ch(|u|_{2,\Omega}+|\psi|_{2,\Omega}).$$

Remark 2. For Wilson's finite element approximation to the obstacle problem we obtain, on the one hand, the convergence rate of order $O(h)$, which is optimal due to the recent result in [13]; on the other hand, a number of numerical experiments show that the approximations of Wilson's element are better than those of the bilinear element for second order problems[14,15]. So we feel that Wilson's element is preferable in the approximation to the obstacle problem on a rectangular domain.

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