HIGHLY ACCURATE NUMERICAL SOLUTIONS OF ELLIPTIC BOUNDARY VALUE PROBLEMS ON GENERAL REGIONS*

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Abstract

We prove that Lin Qun and Lu Tao's splitting extrapolation method and correction method can be effectively applied to raise the accuracy of the numerical solution of elliptic boundary value problems on general regions, i.e., to obtain approximate solutions with fourth- or fifth-order precision in the maximum norm.

§ 1. Introduction

The solution of the linear elliptic boundary value problem

$$\begin{cases} Lu(x)\equiv\sum_{j=1}^{N}\left[a_j(x)\dfrac{\partial^2 u(x)}{\partial x_j^2}+b_j(x)\dfrac{\partial u}{\partial x_j}\right]+d(x)u(x)=f(x), & x\in\Omega,\\ u(x)=g(x), & x\in\partial\Omega \end{cases} \tag{1}$$

by the finite difference method has a long history, see, e.g., [1] and rferences therein. It is well-known that there are many methods, based on the asymptotic expansion, for acceleration of convergence, but the one for problem (1) was obtained by Böhmer[1] only recently.

In this paper, we first give a simpler proof of Böhmer's result. Then we explain how to obtain solutions with fourth- or fifth-order precision by the splitting extrapolation method[7]. Finally, based on the idea in [8], we formulate the correction analogue for (1) and prove that the approximate solution has fourth-order precision.

The splitting extrapolation method[7] can save much computational work and storage in comparison with the usual extrapolation along all variables, for it is essentially equal to the procedure in which the one-dimensional extrapolation is done N times, where N is the dimension of problem (1). Moreover, it is appropriate for parallel computers. The correction method has the advantage that to obtain a more accurate solution, one does not need to solve the original discrete problem on a smaller mesh, but to solve another discrete problem on the original mesh, which is easier.

§ 2. Formulation of Difference Analogue

Let us consider the numerical solution for (1) in which Ω is an arbitrary

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bounded and connected region in the N-dimensional Euclidean space R^N. To define the analogue, we first introduce the following notations

e_j, j-th unit vector in the j-th co-ordinate direction,

$h>0$, the step, $\mathbb{N}=\{1, 2, \cdots, N\}$,

$\mathbb{Z}=\{0, \pm1, \pm2, \cdots\}$,

$N_h=\Omega\cap\{x\in R^N: x_j=hn_j, n_j\in\mathbb{Z}, \forall j\in\mathbb{N}\}$,

$\Omega_h=\{x\in N_h: x\pm he_j\in\Omega, \forall j\in\mathbb{N}\}$,

$\Omega'_h=N_h-\Omega_h$.

For every $x\in\Omega'_h$, by notations we know that there exist a non-empty subset $I=I^+\cup I^-$ of $\mathbb{N}$ such that $x+he_j\bar{\in}\Omega$, $\forall j\in I^+$, $x-he_j\bar{\in}\Omega$, $\forall j\in I^-$ and $x\pm he_j\in\Omega$, $\forall j\in \mathbb{N}-I$. We assume that $\partial\Omega$ is smooth enough and h is so small that we have $x-(v-1)he_j\in\Omega$, $v=2, \cdots, k$, $\forall j\in I^+$ and $x+(v-1)he_j\in\Omega$, $v=2, \cdots, k$, $\forall j\in I^-$. Moreover, for any $j\in I^\pm$, there is exactly one intersection point $x\pm(1-s_j^\pm)he_j$, $0\leqslant s_j^\pm<1$, of the line segment from x to $x\pm he_j$ with $\partial\Omega$. Taking $x\in\Omega'_h$, we let $\partial\Omega_h$ be the set consisting of the intersection points.

Now, we define an operator L_h: $F_h\to F_h$, where $F_h=\{u_h: N_h\cup\partial\Omega_h\to R\}$. For every $x\in\Omega_h$, let

$$L_hu_h(x)=\frac{1}{h^2}\sum_{j=1}^N a_j(x)[u_h(x+he_j)-2u_h(x)+u_h(x-he_j)]$$
$$+\frac{1}{2h}\sum_{j=1}^N b_j(x)[u_h(x+he_j)-u_h(x-he_j)]+d(x)u_h(x). \tag{2.1}$$

For every $x\in\partial\Omega_h$, $L_hu_h(x)=u_h(x)$. Finally, for every $x\in\Omega'_h$, we still use (2.1) to define $L_hu_h(x)$. To this end, noting that $x\pm he_j\bar{\in}\Omega$, $\forall j\in I^\pm$, we replace $u_h(x\pm he_j)$ in (2.1) by

$$u_h(x\pm he_j)=\alpha_{j,-1}^\pm u_h(x\pm(1-s_j^\pm)he_j)+\sum_{v=0}^{k-1}\alpha_{j,v}^\pm u_h(x\mp vhe_j), \tag{2.2}$$

where

$$\alpha_{j,-1}^\pm=k!\Big/\prod_{l=1}^k(l-s_j^\pm),\quad \alpha_{j,v}^\pm=(-1)^{v+1}s_j^\pm/(v+1-s_j^\pm), \tag{2.3}$$

which is meaningful from the assumptions. Thus, L_h is well defined and we obtain the discretization problem corresponding to the continuous problem (1), i.e., the difference analogue

$$\begin{cases} L_hu_h(x)=f(x), & x\in\Omega_h\cup\Omega'_h,\\ u_h(x)=g(x), & x\in\partial\Omega_h.\end{cases} \tag{2.4}$$

It should be pointed out that the discrete problem (2.4) is essentially the same as that in [1].

§ 3. A Priori Estimate for the Discrete Problem

In this section we rewrite the results of Bramble-Hubbard[3,4] in a more convenient form for our use.

We express the operator L_h as

$$L_h u_h(x) = \sum_{y \in N_h \cup \partial\Omega_h} \sigma(x, y) u_h(y). \quad (3.1)$$

Let S_h be any non-empty subset of $N_h \cup \partial\Omega_h$ and v_h: $S_h \to R$. Write

$$\|v_h\|_{S_h} = \max_{x \in S_h} |v_h(x)|, \quad \|v_h\| = \|v_h\|_{N_h \cup \partial\Omega_h}. \quad (3.2)$$

Now let us consider the necessary conditions for applying Bramble-Hubbard's results on the operator L_h

$$\sigma(x, x) < 0, \quad \forall x \in \Omega_h \cup \Omega'_h, \quad (3.3)$$

$$\sigma(x, y) \geqslant 0, \quad \forall y \neq x, \ \forall x \in \Omega_h, \ y \in N_h, \quad (3.4)$$

$$|\sigma(x, x)| \geqslant \sum_{y \neq x} |\sigma(x, y)|, \quad \forall x \in \Omega_h, \ y \in N_h, \quad (3.5)$$

$\exists \delta$, $0 < \delta < 1$, such that

$$\delta\, \sigma(x, x) \geqslant \sum_{y \in N_h - x} |\sigma(x, y)|, \quad \forall x \in \Omega'_h. \quad (3.6)$$

Here the validities of (3.5) and (3.6) clearly imply, when h is small enough, that

$$h^2 |\sigma(x, x)| \geqslant \varepsilon_0 > 0, \quad \forall x \in \Omega_h \cup \Omega'_h,$$

where ε_0 is independent of h. For this we may write those results in [4, Theorem 2.1] as

Lemma 1. *Let the conditions (3.3)—(3.6) are valid when h is small enough, then if $\|v_h\|_{\partial\Omega_h} = 0$ and h is small enough, any $v_h \in F_h$ satisfies the key estimation*

$$\|v_h\| \leqslant C(\|L_h v_h\|_{\Omega_h} + h^2 \|L_h v_h\|_{\Omega'_h}), \quad (3.7)$$

where C denotes some constant independent of h. Such constants will always be denoted by C, though not necessarily the same at each occurrence.

§ 4. Asymptotic Expansion of the Discretization Error

Lemma 2. *Let s satisfy $0 \leqslant s < 1$, then the linear equation*

$$\begin{cases} \alpha_{-1} + \alpha_0 + \alpha_1 + \cdots + \alpha_{k-1} = 1, \\ s\alpha_{-1} + \alpha_0 + 2\alpha_1 + \cdots + k\alpha_{k-1} = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ s^k \alpha_{-1} + \alpha_0 + 2^k \alpha_1 + \cdots + k^k \alpha_{k-1} = 0 \end{cases} \quad (4.1)$$

have exactly one solution

$$\alpha_{-1} = k! \Big/ \prod_{l=1}^{k} (l - s), \quad \alpha_v = (-1)^{v+1} \binom{k}{v+1} s \Big/ (v + 1 - s), \quad 0 \leqslant v \leqslant k - 1$$

and

$$\psi(k, s) \geqslant \frac{1}{3}, \quad 0 \leqslant k \leqslant 4, \ \forall s \in [0, 1),$$

$$\psi\left(k, \frac{1}{2}\right) \leqslant -1, \quad k \geqslant 5,$$

where $\psi(k, s) = 1 + |\alpha_0| - \sum_{v=1}^{k-1} |\alpha_v|$ ($\psi(0, s)$ and $\psi(1, s)$ are naturally regarded as 1 and $1 + |\alpha_0|$, respectively).

Proof. By means of calculation of Vandermode's determinant and Cramer's rule, it is obvious that (4.1) has exactly one solution and the solution has the given expression.

For the rest part of Lemma 2, we may suppose that $k\geqslant 2$. From

$$\begin{aligned}\psi(k,s)&=1+\frac{s}{1-s}\binom{k}{1}-\sum_{v=1}^{k-1}\binom{k-1}{v}\frac{k}{v+1-s}+\sum_{v=1}^{k-1}\binom{k-1}{v}\frac{k}{v+1}\\&=1-k\int_0^1 dt+k\int_0^1 t^{-s}\,dt-k\int_0^1 t^{-s}\Big[\sum_{v=1}^{k-1}\binom{k-1}{v}t^v\Big]dt\\&\quad+k\int_0^1\Big[\sum_{v=1}^{k-1}\binom{k-1}{v}t^v\Big]dt\\&=1-k\int_0^1(t^{-s}-1)[(1+t)^{k-1}-2]\,dt\end{aligned}$$

we may consider k as a continuous variable. It is easy to see that $\psi'_k(k,s)\leqslant 0$, that is, $\psi(k,s)$ is a monotone decreasing function on k, so we only have to prove $\psi(4,s)\geqslant 1/3$, $\forall s\in[0,1)$ and $\psi\left(5,\frac{1}{2}\right)\leqslant -1$. In fact, we have from (4.1) that

$$\psi(k,s)=1-\alpha_0-\sum_{\substack{v=1\\2\nmid v}}^{k-1}\alpha_v+\sum_{\substack{v=1\\2\nmid v}}^{k-1}\alpha_v=\alpha_{-1}+2\sum_{\substack{v=1\\2\mid v}}^{k-1}\alpha_v=2-2\sum_{\substack{v=1\\2\mid v}}^{k-1}|\alpha_v|;$$

therefore

$$\begin{aligned}\psi(4,s)&=4!\Big/\prod_{l=1}^{4}(l-s)-2\cdot4\cdot\frac{s}{3-s}\\&\geqslant\frac{1}{24}[24-8s(1-s)(2-s)(4-s)]\\&\geqslant\frac{1}{3}[3-s(1-s)\cdot2\cdot4]\geqslant\frac{1}{3}.\end{aligned}$$

We also have

$$\frac{1}{2}\prod_{l=1}^{5}\left(l-\frac{1}{2}\right)\psi\left(5,\frac{1}{2}\right)\leqslant 60-10\cdot\frac{1}{2}\prod_{l=1}^{5}\left(l-\frac{1}{2}\right)\leqslant -40.$$

The proof of Lemma 2 is completed.

Throughout this paper we assume that a_j, b_j, d, f, $g\in C(\bar{\Omega})$, $\min_{j=1}^{N}\min_{x\in\Omega}a_j(x)=a>0$, $d\leqslant 0$, and there exists exactly one solution for problem (1) and u, the unique solution, can be extended to Ω^* such that $u\in C^{q+\alpha}(\Omega^*)$, where Ω^* is some open region containing the set $\Omega\cup\{x+te_j\colon x\in N_h,\ |t|\leqslant h,\ j\in\mathbb{N}\}$, $q\geqslant 2$, $0<\alpha<1$. These assumptions are guaranteed by [6, Theorem 6.19 and Lemma 3.7] when

$$a_j,\ b_j,\ d,\ f\in C^{q-2+\alpha}(\bar{\Omega}),\quad g\in C^{q+\alpha}(\partial\Omega)\quad\text{and}\quad\partial\Omega\in C^{q+\alpha}.$$

In the end, to estimate the truncation error by Taylor's formula, we always suppose that the line segment from x to y is inside the region Ω^*.

Under the above hypotheses we have

Theorem 1. *Let the fixed k satisfy $0\leqslant k\leqslant 4$, then there exists h_0, $h_0>0$, such that, if $h<h_0$, the discrete problem (2.4) has a unique solution u_h and the discretization error $\varepsilon_h(x)=u_h(x)-u(x)$ admits the estimation*

$$\|\varepsilon_h\| \leqslant Ch^{\min(2,k+1,q-2+\alpha)}. \tag{4.2}$$

Proof. Using Taylor's formula in every direction, we obtain, for any $x \in \Omega_h$,

$$L_h v_h(x) = Lu(x) + \sum_{j=1}^{N} a_j(x) \sum_{v=1}^{\left[\frac{q-2}{2}\right]} \frac{2h^{2v}}{(2v+2)!} D_j^{2v+2} u(x)$$

$$+ \sum_{j=1}^{N} b_j(x) \sum_{v=1}^{\left[\frac{q-1}{2}\right]} \frac{h^{2v}}{(2v+1)!} D_j^{2v+1} u(x) + O(h^{q-2+\alpha}) \tag{4.3}$$

in which the terms containing h^m should be considered as $O(h^{q-2+\alpha})$ when $m \geqslant q-2+\alpha$. For every $x \in \Omega_h'$, because of Lemma 2 and the fact $u \in C^{q+\alpha}(\Omega^*)$, we deduce, for any $j \in I^{\pm}$,

$$u(x \pm he_j) = \alpha_{j,-1}^{\pm} u(x \pm (1-s_j^{\pm}) he_j) + \sum_{v=1}^{k-1} \alpha_{j,v}^{\pm} u(x \mp vhe_j) + O(h^{k+1}). \tag{4.4}$$

Hence (4.3) is also valid for every $x \in \Omega_h'$, but for this case the 0 terms should be $O(h^{\min(k-1,q-2+\alpha)})$. If the conditions of Lemma 1 are valid, these clearly imply Theorem 1. It is easy to check conditions (3.3)—(3.5) when $h < a/b$, where $b = \max_{j=1}^{N} \max_{x \in \bar{\Omega}} |b_j(x)|$ (may be zero). So in the following we only have to prove that condition (3.6) is valid.

Let $x \in \Omega_h'$ and $h < a/b$. Since $a_j(x) \pm \frac{h}{2} b_j(x) \geqslant 0$, $\alpha_{j,0}^{\pm} \leqslant 0$ and $\alpha_{j,1}^{\pm} \geqslant 0$, by definition we have

$$h^2\sigma(x, x) = -\sum_{j \in \mathbf{N}-I} [2a_j(x)] + h^2 d(x) - \sum_{j \in I^+} \left\{ 2a_j(x) + \left[a_j(x) + \frac{h}{2} b_j(x) \right] \alpha_{j,0}^{+} \right\}$$

$$- \sum_{j \in I^-} \left\{ 2a_j(x) + \left[a_j(x) - \frac{h}{2} b_j(x) \right] \alpha_{j,0}^{-} \right\} \tag{4.5}$$

and

$$h^2 \sum_{y \in N_h - x} |\sigma(x, y)| = \sum_{j \in \mathbf{N}-I} [2a_j(x)]$$

$$+ \sum_{j \in I^+} \left\{ \left[a_j(x) - \frac{h}{2} b_j(x) \right] + \left[a_j(x) + \frac{h}{2} b_j(x) \right] \sum_{v=1}^{k-1} |\alpha_{j,v}^{+}| \right\}$$

$$+ \sum_{j \in I^-} \left\{ \left[a_j(x) + \frac{h}{2} b_j(x) \right] + \left[a_j(x) - \frac{h}{2} b_j(x) \right] \sum_{v=1}^{k-1} |\alpha_{j,v}^{-}| \right\}. \tag{4.6}$$

For $k=0$ or 1, it is trivial to prove (4.6). For $k \geqslant 2$, combination of (4.5) and (4.6) leads to

$$\delta \left\{ 2a_j(x) + \left[a_j(x) \pm \frac{h}{2} b_j(x) \right] |\alpha_{j,0}^{\pm}| \right\}$$

$$\geqslant a_j(x) \mp \frac{h}{2} b_j(x) + \left[a_j(x) \pm \frac{h}{2} b_j(x) \right] \sum_{v=1}^{k-1} |\alpha_{j,v}^{\pm}|. \tag{4.7}$$

To prove (4.7), noting $\sum_{v=1}^{k-1} |\alpha_{j,v}^{\pm}| \leqslant \sum_{v=1}^{k-1} \binom{k}{v+1} \leqslant 2^k$ and the definitions of a and b, we only have to prove

$$2+|\alpha_{j,0}^{\pm}|-hb|\alpha_{j,0}^{\pm}|/a>1+\sum_{v=1}^{k-1}|\alpha_{j,v}^{\pm}|+h(1+2^k)b/a. \tag{4.8}$$

Taking $h_0>0$ such that $h_0(1+2^k+2^{k+1})b<\frac{1}{12}a$, then when $h<h_0$ and $|\alpha_{j,0}^{\pm}|>2^{k+1}$, the left side of $(4.8)\geqslant 2+\frac{1}{2}|\alpha_{j,0}^{\pm}|\geqslant 2+2^k$, but the right side$\leqslant 1+2^k+\frac{1}{12}$; when $h<h_0$ and $|\alpha_{j,0}^{\pm}|\leqslant 2^{k+1}$, by Lemma 2 and the choice of h_0, (4.8) is also valid. This completes the proof of Theorem 1.

Theorem 1 asserts the convergence of the approximate solution. Further, under the same hypotheses, we have

Theorem 2. *Let* $0\leqslant k\leqslant 4$, *then in the maximum norm* $\|\cdot\|=\|\cdot\|_{N_h\cup\partial\Omega_h}$ *the asymptotic expansion*

$$u(x)=u_h(x)+h^2e_2(x)+h^4e_4(x)+O(h^{\min(k+1,q-2+\alpha)}) \tag{4.9}$$

holds for $x\in N_h\cup\partial\Omega_h$, *where* $e_{2v}\in C^{q-2v+\alpha}(\overline{\Omega})$ $(v=1,2)$ *are functions independent of* h.

Proof. For $k=0$ or 1, (4.9) is naturally regarded as

$$u=u_h+O(h^{\min(k+1,q-2+\alpha)}),$$

which is the assertion of Theorem 1. For $k\geqslant 2$, we may suppose $q\geqslant 4$. By [6] and the fact $u\in C^{q+\alpha}(\Omega^*)$ the equation

$$Le_2(x)=\sum_{j=1}^{N}\Big[\frac{2}{4!}a_jD_j^4u+\frac{1}{3!}b_jD_j^3u\Big]=f_2,\quad \Omega,$$

$$e_2(x)=0,\quad \partial\Omega$$

has a unique solution $e_2\in C^{q-2+\alpha}(\overline{\Omega})$ and it can be extended. Let $\varepsilon_2=u-u_h-h^2e_2$. From (4.3), (4.4) we have

$$L_h\varepsilon_2(x)=L_hu-Lu-h^2f_2-h^2(L_he_2-Le_2)=O(h^{\min(4,q-2+\alpha)}),\quad x\in\Omega_h,$$

$$L_h\varepsilon_2(x)=L_hu-Lu-h^2f_2-h^2(L_he_2-Le_2)+O(h^{k-1})=O(h^{\min(k-1,q-2+\alpha)}),\quad x\in\Omega_h'.$$

Therefore, by applying Lemma 1 to ε_2, (4.9) is valid when $2\leqslant k\leqslant 3$. Finally, for $k=4$, we may suppose $q\geqslant 6$. Proceeding as before, it is easy to find the desired e_4 to be the solution of the following equation

$$Le_4=\sum_{j=1}^{N}\Big[2a_j\Big(\frac{1}{6!}D_j^6u+\frac{1}{4!}D_j^4e_2\Big)+b_j\Big(\frac{1}{5!}D_j^5u+\frac{1}{3!}D_j^3u\Big),\quad \Omega,$$

$$e_4=0,\quad \partial\Omega$$

which implies that the proof of Lemma 2 is completed.

Remark 1. When $k\geqslant 5$, we see from above proofs that the conditions of Bramble-Hubbard's results are no longer valid. In this sence, Böhmer's result ([1, Theorem 2]) is best possible, that is to say, Peryra-Proskurowski-Widlund's conjecture ([9, p. 9]) is perhaps not true.

§ 5. Fourth- and Fifth-order Formulations of Splitting Extrapolation

For simplicity we used uniform mesh formerly, but if we let h_j be the step of the j-th direction, $h=\max_{j=1}^{N} h_j$ and $L_{h_1,\cdots,h_N}$ denote the corresponding discrete operator ($L_{h_1,\cdots,h_N}=L_h$ when $h_1=\cdots=h_N$), then we still have similar results. In fact, we again have Lemma 1 from the proofs in [4]. Thus the proofs of Lemma 2 and Theorem 2 imply

Theorem 2*. *Let $0\leqslant k\leqslant 4$, then there exists $h_0>0$ such that, if $h<h_0$, the discrete equation (2.4) in which L_h replaced by $L_{h_1,\cdots,h_N}$ has unique solution $u_{h_1,\cdots,h_N}$ and the asymptotic expansion*

$$u=u_{h_1,\cdots,h_N}+\sum_{j=1}^{N} h_j^2 e_{2,j}+\sum_{j=1}^{N} h_j^4 e_{4,j}+O(h^{\min(k+1,q-2+\alpha)})$$

holds, where $e_{2v,j}\in C^{q-2v+\alpha}(\bar{\Omega})$ $(j=1,\cdots,N;\ v=1,2)$ are solutions of

$$Le_{2,j}=\frac{2}{4!}a_j D_j^4 u+\frac{1}{3!}b_j D_j^3 u,\quad \Omega,$$

$$e_{2,j}=0,\quad \partial\Omega$$

and

$$Le_{4,j}=2a_j\left(\frac{1}{6!}D_j^6 u+\frac{1}{4!}D_j^4 e_{2,j}\right)+b_j\left(\frac{1}{5!}D_j^5 u+\frac{1}{3!}D_j^3 e_{2,j}\right),\quad \Omega,$$

$$e_{4,j}=0,\quad \partial\Omega$$

respectively.

Theorem 2* is the basis of extrapolation. By this and the splitting extrapolation method[7] we obtain

$$\frac{1}{3}\left\{4\sum_{j=1}^{N} u_{h_1,\cdots,h_j/2,\cdots,h_N}-(4N-3)u_{h_1,\cdots,h_N}\right\}=u+O(h^{\min(4,k+1,q-2+\alpha)}) \tag{5.1}$$

which gives an approximate solution of (1) with fourth-order precision when $3\leqslant k\leqslant 4$ and $q\geqslant 6$. Further, when $k=4$ and $q\geqslant 7$, both

$$\frac{1}{45}\left\{64\sum_{i=1}^{N} u_{h_1,\cdots,h_i/4,\cdots,h_N}-20\sum_{i=1}^{N} u_{h_1,\cdots,h_i/2,\cdots,h_N}+(45-44N)u_{h_1,\cdots,h_N}\right\}=u+O(h^5) \tag{5.2}$$

and

$$\frac{1}{120}\left\{243\sum_{i=1}^{N} u_{h_1,\cdots,h_i/3,\cdots,h_N}-128\sum_{i=1}^{N} u_{h_1,\cdots,h_i/2,\cdots,h_N}+(120-115N)u_{h_1,\cdots,h_N}\right\}=u+O(h^5) \tag{5.3}$$

give the approximate solutions with fifth-order precision.

§ 6. Fourth-order Correction Analogue

Based on the idea in [8], we have for problem (1) the correction analogue

$$\begin{cases} L_h u_h^*=f+h^2\sum_{j=1}^{N}\left[\frac{2}{4!}a_j d_j^4 u_h+\frac{1}{3!}b_j d_j^2\delta_j u_h\right], & \Omega_h,\\ L_h u_h^*=f, & \Omega_h',\\ u_h^*=g, & \partial\Omega_h, \end{cases} \tag{6.1}$$

where the discrete difference operators d_j^2, d_j^4 and δ_j are defined as

$$d_j^2 Z_h=h^{-2}[Z_h(x+he_j)-2Z_h(x)+Z_h(x-he_j)],$$

$$d_j^4 = d_j^2 \cdot d_j^2,$$
$$\delta_j Z_h = (2h)^{-1}[Z_h(x+he_j) - Z_h(x-he_j)]$$

in which Z_h: $\{x \in R^N: x_j = n_j h, n_j \in \mathbb{Z}, \forall j \in \mathbb{N}\} \to R$ is any mesh function. By above definitions, $d_j^4 u_h(x)$ and $d_j^2 \delta_j u_h(x)$ both use the value of u_h at the points $x \pm 2he_j$. If some of these points are out of $\bar{\Omega}$, at which the values of u_h may be replaced by the right side of (2.2), then (6.1) is well defined. We have

Theorem 3. *Let* $0 \leqslant k \leqslant 4$ *and* $q \geqslant \max(4, 3+[N/2])$ *then, under the basic hypotheses made before, there exists* $h_0 > 0$ *such that, if* $h < h_0$, (6.1) *has unique solution* u_h^* *and* u_h^* *satisfies*

$$u_h^* = u + O(h^4) \tag{6.2}$$

on the mesh set $N_h \cup \partial\Omega_h$ *in the maximum norm.*

Proof. Taking into account the corresponding (4.3) when $x \in \Omega_h'$, we deduce

$$L_h(u - u_h^*) = O(h^2), \quad \Omega_h'. \tag{6.3}$$

So by Lemma 1 we only have to prove

$$L_h(u - u_h^*) = O(h^4), \quad \Omega_h. \tag{6.4}$$

Further, noting (6.1) and (4.3), we only have to prove

$$D_j^4 u - d_j^4 u_h = O(h^2), \quad \Omega_h \tag{6.5}$$

and

$$D_j^3 u - d_j^2 \delta_j u_h = O(h^2), \quad \Omega_h. \tag{6.6}$$

These are the known results in [11, Theorem 4.1] or [5]. The proof is thus completed.

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