ON THE STRUCTURE OF BÉZIER NETS*[1)]

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Abstract

The distance between two neighbouring multivariate Bézier nets $G_n^m\hat{f}$ and $G_n^{m+1}\hat{f}$ is proved to be $O(m^{-2})$ in this paper. As a consequence, the sequence of Bézier nets is uniformly convergent with the optimal approximation order $O(m^{-1})$. Furthermore, the structures of Bézier nets are explored by investigating how the piecewise linear surface $G_n^m\hat{f}$ tends to the Bézier surface of C^∞.

§ 1. Introduction

It is well known that the Bézier surfaces have been established as a mathematical basis of many CAD systems. "Bernstein–Bézier approximations have recently become very popular and no fewer than 1/4 of the titles at this symposium contain Bézier's name"[1]. The Bézier nets associated with Bézier surface are a very useful tool in exploring the Bézier surface. Many properties of Bézier nets have been explored, such as the limit of Bézier nets, the variation diminishing properties, the convexity preservation. The approximation order of Bézier nets in the univariate case has been shown in [5] to be $O(1/m)$. [5] also shows the relationship between the convergence of Bézier nets and the approximation of Bernstein–Bézier polynomials.

In this paper we are concerned with the Bézier nets and Bézier surface on a triangle in R^n for the bivariate and multivariate cases. Starting from the point of view that the Bézier nets are obtained by successive piecewise linear interpolation, we prove that the distance between the neighbouring Bézier nets $G_n^m\hat{f}$ and $G_n^{m+1}\hat{f}$ is $O(1/m^2)$. As an immediate consequence, the sequence of Bézier nets is uniformly convergent with the approximation order $O(1/m)$. In searching for the representation of the limit of Bézier nets, we show how the piecewise liner surface $G_n^m\hat{f}$ tends to the Bézier surface of C^∞. Therefore the structures of Bézier nets are explored more clearly.

Now, we introduce some notations used in this paper.

The domain of the Bézier surface is a triangle T with three vertices $T_i=(\hat{x}_i, \hat{y}_i)$, $i=1, 2, 3$.

Every point P in T is identified with its barycentric coordinates, i.e.,

$$P=(x, y)=uT_1+vT_2+wT_3,$$
$$u\geqslant 0,\ v\geqslant 0,\ w\geqslant 0,$$
$$u+v+w=1.$$

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Let f be a function defined on T, and $f\left(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}\right)$ be denoted by $f_{i,j,k}$ for $i+j+k=n$.

$E_k (k=1, 2, 3)$ are shift operators, defined by

$$E_1 f_{i,j,k} = f_{i+1,j,k},$$
$$E_2 f_{i,j,k} = f_{i,j+1,k},$$
$$E_3 f_{i,j,k} = f_{i,j,k+1},$$

and $E_{-k} = (E_k)^{-1}$, $k=1, 2, 3$.

Denote by $\Delta_{k,s}^{r,t} (1 \leqslant k, s, r, t \leqslant 3)$ the partial difference operator of order 2, i.e.,

$$\Delta_{k,s}^{r,t} = (E_r - E_t)(E_{-k} - E_{-s}).$$

G_n^m is a degree raising operator of order m, defined by

$$G_n^1 f_{i,j,k} = \frac{1}{n+1}(iE_{-1} + jE_{-2} + kE_{-3}) f_{i,j,k}$$

for $i+j+k=n+1$ and

$$G_n^m = G_{n+m-1}^1 G_n^{m-1}.$$

Denote by $G_n^m \hat{f}$ the m-th Bézier net for $P = \left(\frac{i}{n+m}, \frac{j}{n+m}, \frac{k}{n+m}\right) \in T$:

$$G_n^m \hat{f}(P) = G_n^m f_{i,j,k}.$$

$B_n(f)$ is a Bernstein polynomial of degree n, i.e.,

$$B_n(f) = \sum_{i+j+k=n} f_{i,j,k} B_{i,j,k}^n (u, v, w)$$

where $\left\{B_{i,j,k}^n = \frac{n!}{i!j!k!} u^i v^j w^k\right\}$ are Bernstein polynomial basis functions.

Throughout this paper we use the maximum norm, i.e.,

$$\|f\| = \max_{P \in T} |f(P)|.$$

§ 2. Main Results

In this section we will present our results on the structure of Bézier nets over a triangle. In order to get an estimate of the distance between $G_n^m \hat{f}$ and $G_n^{m+1} \hat{f}$, we first prove the following identity concerning G_n^m and $\Delta_{k,s}^{r,t}$.

Lemma 1. $G_n^1 \Delta_{k,s}^{r,t} - \Delta_{k,s}^{r,t} G_n^1 = \frac{1}{n+1}(E_{-k} + E_{-s}) \Delta_{k,s}^{r,t}$,

$$G_{n+m-1}^1 \Delta_{k,s}^{r,t} G_n^{m-1} - \Delta_{k,s}^{r,t} G_n^m = \frac{1}{n+m}(E_{-k} + E_{-s}) \Delta_{k,s}^{r,t} G_n^{m-1}.$$

Proof. It is easy to verify by the definition.

Lemma 2. *For* $1 \leqslant k, s, r, t \leqslant 3$, $i_s \geqslant 1$, $i_k \geqslant 1$,

$$|\Delta_{k,s}^{r,t} G_n^m f_{i_1,i_2,i_3}| \leqslant \frac{n(n-1)}{(n+m)(n+m-1)} \max_{\substack{j_1+j_2+j_3=n \\ j_s \geqslant 1, j_k \geqslant 1}} |\Delta_{k,s}^{r,t} f_{j_1,j_2,j_3}|.$$

Proof. Without loss of generality, we let $t=k=1$, $r=s=2$. By Lemma 1, we get

$$
\begin{aligned}
|\Delta^{2,1}_{1,2}G^m_n f_{i_1,i_2,i_3}| &= \frac{1}{n+m}\left|\{(i_1-1)E_{-1}+(i_2-1)E_{-2}+i_3E_{-3}\}\Delta^{2,1}_{1,2}G^{m-1}_n f_{i_1,i_2,i_3}\right| \\
&\leqslant \frac{n+m-2}{n+m}\max_{\substack{j_1+j_2+j_3=n+m-1\\ j_1\geq 1,\, j_2\geq 1}}|\Delta^{2,1}_{1,2}G^{m-1}_n f_{j_1,j_2,j_3}| \\
&\leqslant \frac{n(n-1)}{(n+m)(n+m-1)}\max_{\substack{j_1+j_2+j_3=n\\ j_1\geq 1,\, j_2\geq 1}}|\Delta^{2,1}_{1,2}f_{j_1,j_2,j_3}|.
\end{aligned}
$$

Remark. Lemma 2 shows that the partial divided difference of order 2 of the m-th Bézier net can be controlled by that of the initial Bézier net $G^0_n\hat{f}$, i.e.,

$$
\max_{\substack{i_1+i_2+i_3=n+m\\ i_k\geq 1,\, i_s\geq 1}}\left|\frac{\Delta^{r,t}_{k,s}G^m_n f_{i_1,i_2,i_3}}{(n+m)^{-2}}\right| \leqslant \max_{\substack{j_1+j_2+j_3=n\\ j_k\geq 1,\, j_s\geq 1}}\left|\frac{\Delta^{r,t}_{k,s}f_{j_1,j_2,j_3}}{n^{-2}}\right|
$$

Now we can get the estimate of $\|G^m_n\hat{f}-G^{m+1}_n\hat{f}\|$.

Theorem 1.

$$
\|G^{m+1}_n\hat{f}-G^m_n\hat{f}\| \leqslant \frac{1}{3}\,\frac{n(n-1)}{(n+m+1)(n+m-1)}M,
$$

where

$$
M := \max_{\substack{1\leq k<s\leq 3\\ 1\leq r<t\leq 3}}\ \max_{\substack{i_1+i_2+i_3=n\\ i_k\geq 1,\, i_s\geq 1}}|\Delta^{r,t}_{k,s}f_{i_1,i_2,i_3}|.
$$

Proof. It is easy to know that the partition lines of the m-th Bézier net and the $(m+1)$-th Bézier net divide T into a partition of polygons. Since a linear function on a polygon gains its maximum at some vertices, it is sufficient to estimate $\|G^m_n\hat{f}-G^{m+1}_n\hat{f}\|$ at the crossing points of the partition lines of m-th and $(m+1)$-th Bézier nets.

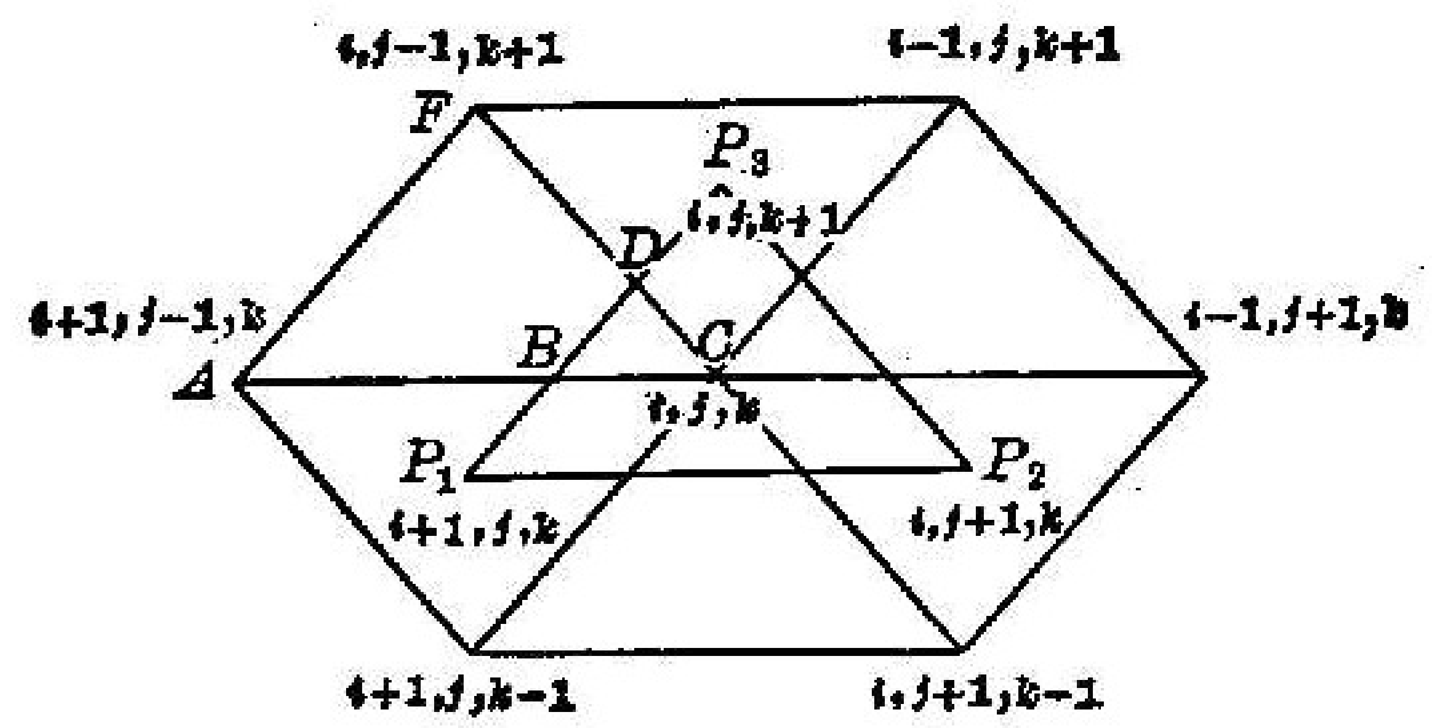

Fig. 1

From Fig. 1 we know that

$$
\begin{aligned}
B &= \frac{j}{n+m+1}A+\frac{i+k+1}{n+m+1}, \\
C &= \frac{i+j}{n+m}P_1+\frac{k}{n+m}P_3, \\
D &= \frac{i}{n+m}P_1+\frac{j+k}{n+m}P_3 \\
&= \frac{j}{n+m}F+\frac{i+k+1}{n+m+1}C,
\end{aligned}
$$

$$
C = \frac{i}{n+m}P_1+\frac{j}{n+m}P_2+\frac{k}{n+m}P_3.
$$

Therefore it follows that

$$
\begin{aligned}
G^m_n\hat{f}(B) &= \frac{j}{n+m+1}G^m_n f_{i+1,j-1,k}+\frac{i+k+1}{n+m+1}G^m_n f_{i,j,k}, \\
G^{m+1}_n\hat{f}(B) &= \frac{i+j}{n+m}G^{m+1}_n\hat{f}(P_1)+\frac{k}{n+m}G^{m+1}_n\hat{f}(P_3) \\
&= \frac{i+j}{(n+m)(n+m+1)}\{(i+1)I+jE_1E_{-2}+kE_1E_{-3}\}G^m_n f_{i,j,k} \\
&\quad +\frac{k}{(n+m+1)(n+m)}\{iE_3E_{-1}+jE_3E_{-2}+(k+1)I\}G^m_n f_{i,j,k}
\end{aligned}
$$

and

$$G_n^{m+1}\hat{f}(B)-G_n^m\hat{f}(B)=\left\{\frac{jk}{(n+m)(n+m+1)}\Delta_{2,3}^{3,1}+\frac{ik}{(n+m)(n+m+1)}\Delta_{1,3}^{3,1}\right\}G_n^m f_{i,j,k}.$$

Similarly, we may get

$$G_n^{m+1}\hat{f}(D)-G_n^m\hat{f}(D)=\left\{\frac{ij}{(n+m)(n+m+1)}\Delta_{1,2}^{3,1}+\frac{ik}{(n+m)(n+m+1)}\Delta_{1,3}^{3,1}\right\}G_n^m f_{i,j,k},$$

$$G_n^{m+1}\hat{f}(C)-G_n^m\hat{f}(C)=\left\{\frac{ij}{(n+m)(n+m+1)}\Delta_{2,1}^{1,2}+\frac{jk}{(n+m)(n+m+1)}\Delta_{3,2}^{2,3}+\frac{ki}{(n+m)(n+m+1)}\Delta_{3,1}^{1,3}\right\}G_n^m f_{i,j,k}.$$

By Lemma 2 we get

$$|G_n^{m+1}\hat{f}(B)-G_n^m\hat{f}(B)|\leqslant\frac{1}{3}\,\frac{n(n-1)}{(n+m-1)(n+m+1)}M,$$

$$|G_n^{m+1}\hat{f}(C)-G_n^m\hat{f}(C)|\leqslant\frac{1}{3}\,\frac{n(n-1)}{(n+m-1)(n+m+1)}M,$$

and

$$|G_n^{m+1}\hat{f}(D)-G_n^m\hat{f}(D)|\leqslant\frac{1}{3}\,\frac{n(n-1)}{(n+m-1)(n+m+1)}M.$$

Noticing that $G_n^{m+1}\hat{f}$ is the piecewise interpolation to $G_n^m\hat{f}$ and the barycentric coodinate is symmetric, we have

$$\|G_n^m\hat{f}-G_n^{m+1}\hat{f}\|\leqslant\frac{1}{3}\,\frac{n(n-1)}{(n+m-1)(n+m+1)}M.$$

Remark. From the definition of Bézier nets, we know that the m-th Bézier net $G_n^m\hat{f}$ is obtained by the linear interpolation to $G_n^{m-1}\hat{f}$. If the Bézier net $G_n^{m-1}\hat{f}$ were of $C^2(T)$, we could immediately conclude that

$$\|G_n^m\hat{f}-G_n^{m-1}\hat{f}\|=O(1/m^2).$$

But $G_n^{m-1}\hat{f}$ in fact is of $C^0(T)$ not of $C^2(T)$. Therefore the above estimate is not obvious. A proper selection of knots plays a very important role here.

From the result of Theorem 1 we may now derive the following conclusion.

Corollary 1. The sequence of Bézier nets $G_n^m\hat{f}$ is uniformly convergent on triangle T.

Denote $U_n(P)=\lim\limits_{m\to\infty}G_n^m\hat{f}(P)$, $P\in T$. By Corollary 1, $U_n(P)$ is continuous on T. In the rest of this section we would like to show that $U_n(P)$ in effect is the Bernstein-Bézier polynomial $B_n(f)$.

By the definition we easily get the following identity.

Lemma 3. *For $1\leqslant r<s\leqslant 3$ and $i+j+k=n+m-1$, following identity holds*

$$(n+m)(E_r-E_s)G_n^m f_{i,j,k}=nG_{n-1}^m(E_r-E_s)f_{i,j,k}.$$

Put $g_{i,j,k}=n(E_2-E_1)f_{i,j,k}$, $i+j+k=n-1$. The associated Bézier net $G_{n-1}^m\hat{g}$ is uniformly convergent on T by Lemma 3 and Corollary 1.

We denote

$$V_n(P)=\lim_{m\to\infty}G_{n-1}^m\hat{g}(P),\quad P\in T.$$

We will prove that $V_n(P)$ is the directional derivative of

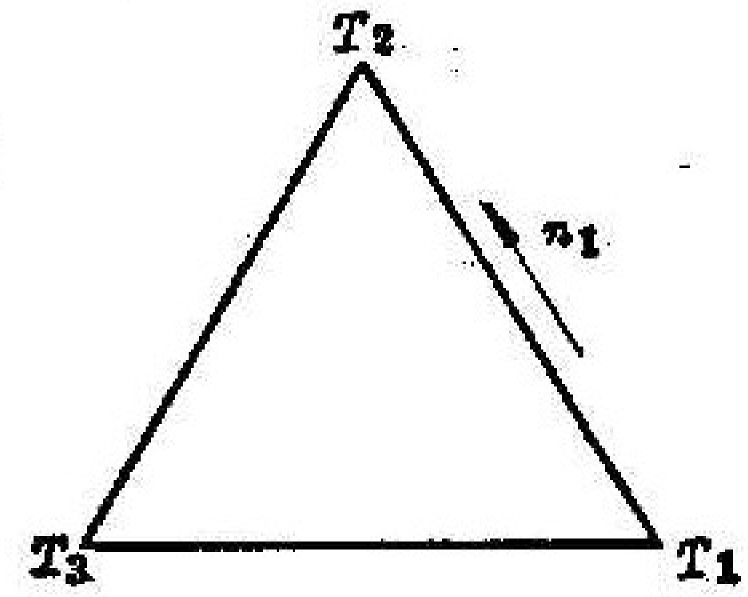

Fig. 2

$U_n(P)$ in the direction n_1 (up to a constant factor).

Lemma 4. *For any* $(x, y)=(u, v, w)\in T$ *and* $(x_0, y_0)=(u+v, 0, w)\in T$, $U_n(P)$ *and* $V_n(P)$ *are related by*

$$\int_{(x_0,y_0)}^{(x,y)} V_n(x, y)dx=(\hat{x}_2-\hat{x}_1)(U_n(x, y)-U_n(x_0, y_0)),$$

where the integration is carried out on the straight line connecting (x, y) *and* (x_0, y_0).

Proof. For $(x, y)=(u, v, w)\in T$, we can find

$$P_{j_m}^m=\left(\frac{i_m}{n+m}, \frac{j_m}{n+m}, \frac{k_m}{n+m}\right)\in T$$

such that

$$P_{j_m}^m\to(u, v, w), \text{ as } m\to\infty.$$

Therefore

$$P_0^m=\left(\frac{i_m+j_m}{n+m}, 0, \frac{k_m}{n+m}\right)\to(u+v, 0, w), \text{ as } m\to\infty.$$

After primary calculation we get

$$\int_{P_0^m}^{P_{j_m}^m} G_{n-1}^m\hat{g}\,dx=\frac{n+m}{n+m-1}(\hat{x}_2-\hat{x}_1)\left\{\frac{1}{2}(G_n^m f_{i_m-1,j_m+1,k_m}+G_n^m f_{i_m,j_m,k_m})\right.$$
$$\left.-\frac{1}{2}(G_n^m f_{i_m+j_m-1,1,k_m}+G_n^m f_{i_m+j_m,0,k_m})\right\},$$

Since $G_{n-1}^m\hat{g}$ converges to $V_n(P)$ uniformly on T and $G_n^m f$ converges to $U_n(P)$ uniformly on T, letting m go to infinity we get

$$\int_{(x_0,y_0)}^{(x,y)} V_n(x, y)d_x=(\hat{x}_2-\hat{x}_1)(U_n(x, y)-U_n(x_0, y_0)).$$

As a result of Lemma 4, we get the following conclusion.

Theorem 2. *For* $r\geqslant 0$, $s\geqslant 0$, $r+s\leqslant n$, *denote*

$$g_{i,j,k}=\frac{n!}{(n-r-s)!}(E_1-E_3)^r(E_2-E_3)^s f_{i,j,k}, \quad i+j+k=n-r-s.$$

Then

$$\lim_{m\to\infty} G_{n-r-s}^m\hat{g}=\frac{\partial^{r+s}U_n(u, v, 1-u-v)}{\partial u^r\partial v^s}.$$

Now we can prove the Farin Theorem and estimate the convergence rate of Bézier nets.

Theorem 3. *Bézier net* $G_n^m\hat{f}$ *converges to* $B_n(f)$ *uniformly on* T *with the rate*

$$\|B_n(f)-G_n^m\hat{f}\|\leqslant\frac{1}{3}\,\frac{n(n-1)}{n+m-1}M$$

where M *is the same as that defined in* Theorem 1. *Furthermore, the approximation order* $O(1/m)$ *is optimal.*

Proof. By Theorem 2 we know that $U_n(P)=\lim_{m\to\infty}G_n^m f$ is a polynomial of degree n because linear functions are invariants of the Bézier net and

$$\left.\frac{\partial^{r+s}U_n(u, v, 1-u-v)}{\partial u^r\partial v^s}\right|_{T_i}=\left.\frac{\partial^{r+s}B_n(u, v, 1-u-v)}{\partial u^r\partial v^s}\right|_{T_i}$$

for $r+s\leqslant n$, $i=1, 2, 3$.

Hence the two polynomials $U_n(P)$ and $B_n(f)$ are identical. It implies that $G_n^m\hat{f}$ converges to $B_n(f)$ uniformly on T. Therefore

$$\|B_n(f)-G_n^m\hat{f}\|\leqslant\sum_{r=m}^{\infty}\|G_n^{r+1}\hat{f}-G_n^r\hat{f}\|.$$

Apply Theorem 1 to $\|G_n^{r+1}\hat{f}-G_n^r\hat{f}\|$ and we get the estimate

$$\|B_n(f)-G_n^m\hat{f}\|\leqslant\frac{1}{3}\,\frac{n(n-1)}{n+m-1}\,M.$$

Consequently, Bézier nets $G_n^m\hat{f}$ converge to $B_n(f)$ with approximation order $O(1/m)$. We will prove that the order $O(1/m)$ is optimal because the approximation order of a Bézier net cannot be higher than that of a Bernstein-Bézier polynomial $B_n(f)$.

Denote by $N_{i,j,k}^m(u, v, w)$ the piecewise linear function with knots $\left(\frac{i}{n+m}, \frac{j}{n+m}, \frac{k}{n+m}\right)\in T$ and

$$N_{i,j,k}^m(A_{r,s,t})=\delta_{i,r}\cdot\delta_{j,s}\cdot\delta_{k,t}$$

for $A_{r,s,t}=\left(\frac{r}{n+m}, \frac{s}{n+m}, \frac{t}{n+m}\right)\in T$. Then,

$$G_n^m\hat{f}=\sum_{i+j+k=n+m}G_n^m f_{i,j,k}N_{i,j,k}^m(u, v, w).$$

We denote $P_{n+m}X=\sum_{i+j+k=n+m}X(A_{i,j,k})N_{i,j,k}^m(u, v, w)$ for $X\in C(T)$. Let $X=u^2=B_2(f)$. If

$$\|X-G_2^m\hat{f}\|=o(1/m),$$

then

$$\max_{i+j+k=n+2}|X(A_{i,j,k})-G_2^m\hat{f}(A_{i,j,k})|=\|G_2^m\hat{f}-P_{2+m}X\|$$
$$\leqslant\|G_2^m\hat{f}-X\|+\|X-P_{2+m}X\|=o(1/m).$$

Therefore it follows that

$$\|B_{2+m}(f)-X\|=\|\Sigma(X(A_{i,j,k})-G_2^m\hat{f}(A_{i,j,k}))B_{i,j,k}^{2+m}\|$$
$$\leqslant\max_{i+j+k=2+m}|X(A_{i,j,k})-G_n^m\hat{f}(A_{i,j,k})|=o(1/m).$$

But it is well known that

$$|B_{2+m}(X)-X|=\frac{1}{4}\,\frac{1}{m+2}.$$

This contradiction shows that the appoximation order $O(1/m)$ of the Bézier net is optimal.

§ 3. Results of the Multivariate case

For convenience, we introduce some notations.

The domain of the Bézier surface is a proper simplex T_m in R^{m-1} for $m\geqslant3$, i.e.,

$$T_m=\left\{\sum_{r=1}^{m}u_rA_r\,\middle|\,u_r\geqslant0\text{ and }\sum_{r=1}^{m}u_r=1\right\}$$

and

$$Vol_{m-1}\,T_m>0.$$

For any point $P\in T_m$, we identify it with its barycentric coordinate $U=(u_1, \cdots, u_m)$.

For $i\in Z_+^m$, $i=(i_1, \cdots, i_m)$, we simply denote

$$f_i:=f\left(\frac{i_1}{|i|}, \cdots, \frac{i_m}{|i|}\right).$$

$e_r(r=1, 2, \cdots, m)$ is the r-th row vector of the unit metrix.

E_r $(r=1, 2, \cdots, m)$ is the r-th shift operator defined by

$$E_r f_i=f_{i+e_r},$$

and $E_{-r}=(E_r)^{-1}$ for $r=1, 2, \cdots, m$.

Denote

$$\Delta_{k,s}^{r,t}=(E_r-E_t)(E_{-k}-E_{-s})$$

The Bézier nets are denoted by $G_n^p\hat{f}$ for $p=1, 2, \cdots$, and the multivariate Bernstein-Bézier polynomial is denoted by $B_n(f)$, i.e.,

$$B_n(f; U)=\sum_{|i|=n}\frac{n!}{i!}U^i f_i.$$

Then we can generalize the results of the bivariate case in section 2 to the multivariate case.

Theorem 4.

$$\|G_n^{p+1}\hat{f}-G_n^p\hat{f}\|\leqslant\frac{(m-1)n(n-1)}{2m(n+p+1)(n+p-1)}M_n,$$

where

$$M_n=\max_{\substack{1\leqslant k<s\leqslant m\\ 1\leqslant r<t\leqslant m}}\ \max_{\substack{|i|=n\\ i_k\geqslant 1, i_s\geqslant 1}}|\Delta_{k,s}^{r,t}f_i|.$$

Theorem 5. *The Bézier nets approximate the associated Bézier surface with the rate*

$$\|B_n(f)-G_n^p\hat{f}\|\leqslant\frac{1}{2}\ \frac{m-1}{m}\ \frac{n(n-1)}{n+p-1}M_n,$$

where M is as defined in Theorem 4. *And the approximation order $O(1/p)$ is optimal.*

Corollary 2. For $f\in C^2(T)$, the Bernstein-Bézier polynomials $B_n(f)$ approximate f with the rate

$$\|B_n(f)-f\|\leqslant\frac{1}{2}\ \frac{m-1}{m}\ nM_n+O(1/n^2),$$

where M_n is the same as that defined in Theorem 4.

The proofs of above theorems are similar to those of bivariate case. For the p-th B-net $G_n^p f$, the domain T_m is devided into a partition of subsimplices by many simplices of dimension $m-2$. Since a linear function defined on a simplex gaints its maximum and minimum at some vertices, analogue to the proof of Theorem 1, it is sufficient to estimate $\|G_n^p\hat{f}-G_n^{p+1}\hat{f}\|$ at all crossing points of all partition simplices of $G_n^p\hat{f}$ and $G_n^{p+1}\hat{f}$. In addition, all lemmas presented in this paper have their generalization for multivariate case. Therefore we can prove Theorem 4, Theorem 5 and Theorem 6 in the similar way. The details of the proof are omitted for brief.