ON NUMERICAL METHODS FOR ROBUST POLE ASSIGNMENT IN CONTROL SYSTEM DESIGN (II)[*1)]

SUN JI-GUANG (孙继广)
(*Computing Center, Academia Sinica, Beijing, China*)

Abstract

L. R. Fletcher[1] has pointed out that how to best exploit any freedom of choice in regard to the eigenvalues of a closed-loop system is an unsolved problem for robust pole assignment in control system design. This paper suggests a numerical method to solve this problem. Numerical results show that the method is feasible.

§ 1. Introduction

Throughout this paper we shall use the same notational convention as in [7].

The following robust assignment problem has been investigated (Ref. [1], [2], [3], [7]):

Problem RPA. Given a real $n\times n$ matrix A, a real full rank $n\times m$ matrix $B(m<n)$ and a set $\mathscr{L}$ of n complex numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$, closed under complex conjugation, find a real $m\times n$ matrix F and a non-singular $n\times n$ matrix X satisfying

$$(A+BF)X=X\Lambda, \tag{1.1}$$

where $\Lambda=\mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$, such that the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of $A+BF$ are as insensitive to perturbations in the matrix $A+BF$ as possible.

This paper investigates an unsolved problem proposed by L. R. Fletcher[1].

L. R. Fletcher has pointed out that in practice the eigenvalue spectrum $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ "is usually required to be contained in some region of the complex plane rather than to be precisely some given complex numbers. Numerical experimentation indicates that some eigenvalue spectra are much more sensitive than others to perturbations in A, B and F so that choosing Λ to minimize this sensitivity is an important practical issue about which virtually nothing is known." ([1, p. 169]) Hence, the problem of how to best exploit any freedom of choice in regard to the eigenvalue spectrum is well worth investigating.

The unsolved problem may be formulated as follows:

Problem RPA 1. Given a real $n\times n$ matrix A, a real full rank $n\times m$ matrix $B(m<n)$, p segments $\mathscr{L}_1, \mathscr{L}_2, \cdots, \mathscr{L}_p$ lying in the real axis $\mathbb{R}$ and regions $\mathscr{D}_1, \mathscr{D}_2, \cdots, \mathscr{D}_q$ of the complex plane $\mathbb{C}$, where

$$\mathscr{L}_j=\{\xi\in\mathbb{R}: \lambda_{j,1}\leqslant\xi\leqslant\lambda_{j,2}\}, \quad j=1, 2, \cdots, p, \tag{1.2}$$

$$\mathscr{D}_j=\{z=\xi+i\eta\in\mathbb{C}: \mu_{j,1}\leqslant\xi\leqslant\mu_{j,2},\ \nu_{j,1}\leqslant\eta\leqslant\nu_{j,2}\}, \quad j=1, 2, \cdots, q \tag{1.3}$$

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and $p+2q=n$, find n numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$, a nonsingular $n\times n$ matrix X and a real $m\times n$ matrix F satisfying the equation (1.1) and

$$\lambda_j \in \mathscr{L}_j, \quad j=1, 2, \cdots, p, \tag{1.4}$$

$$\lambda_{p+j} \in \mathscr{D}_j, \ \lambda_{p+q+j} = \bar{\lambda}_{p+j}, \quad j=1, 2, \cdots, q, \tag{1.5}$$

such that the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of $A+BF$ are as insensitive to perturbations in the matrix $A+BF$ as possible.

Clearly, Problem RPA and Problem RPA1 are both inverse algebraic eigenvalue problems.

The aim of this paper is to suggest a numerical method for solving Problem RPA1. For simplicity we consider only the case where the pair (A, B) is controllable ([1]) and $p=n$. Moreover, for convenience we write

$$\lambda_j = \lambda_j(t_j) = \lambda_{j,1} + (\lambda_{j,2} - \lambda_{j,1})\sin^2 t_j, \quad t_j \in \mathbb{R} \ \forall j. \tag{1.6}$$

The idea and technique described in this paper may be used to solve Problem RPA1 in the case of $p<n$.

The procedure of the numerical method for solving Problem RPA1 consist of two basic steps ([2], [7]):

Step A—X. Compute the decomposition

$$B = (U_0^{(B)}, U_1^{(B)})\begin{pmatrix} Z \\ 0 \end{pmatrix}, \tag{1.7}$$

where $(U_0^{(B)}, U_1^{(B)})$ is a real orthogonal matrix and Z nonsingular;

Construct orthogonal bases comprised by the columns of matrices $S_j(t_j)$ and $\hat{S}_j(t_j)$ for the space $\mathscr{S}_j(t_j) \equiv \mathscr{N}(U_1^{(B)T}(A-\lambda_j(t_j)I))$ and its complement, $\hat{\mathscr{S}}_j(t_j)$ for $\lambda_j(t_j) \in \mathscr{L}_j$, $j=1, 2, \cdots, n$;

Select vectors $x_j = S_j(t_j)w_j \in \mathscr{S}_j(t_j)$, $j=1, 2, \cdots, n$ such that $X=(x_1, x_2, \cdots, x_n)$ is well-conditioned.

Step F. Find the matrix $M=A+BF$ by solving $MX=X\Lambda$ and compute F explicitly from

$$F = Z^{-1}U_0^{(B)T}(M-A). \tag{1.8}$$

Obviously, to find a well-conditioned matrix X is the key of the above mentioned procedure. In the next section we reduce the problem for finding a well-conditioned matrix X to an unconstrained optimization problem. In Section 3 we deduce a formula of the gradient vector for the objective function described in Section 2, and in Section 4 we use the DFP algorithm to solve the unconstrained optimization problem. Numerical results are given in Section 5.

§ 2. An Optimization Problem

Let

$$X=(x_1, x_2, \cdots, x_n), \quad Y=X^{-T}=(y_1, y_2, \cdots, y_n), \tag{2.1}$$

$$c_j = \|x_j\|_2 \|y_j\|_2 \geqslant 1, \quad j=1, 2, \cdots, n \tag{2.2}$$

and

$$c=(c_1, c_2, \cdots, c_n)^T, \tag{2.3}$$

where

$$x_j = x_j(w_j, t_j) = S_j(t_j)w_j, \quad w_j \in \mathbb{R}^{m_j}, \ w_j \neq 0, \ t_j \in \mathbb{R} \tag{2.4}$$

and

$$S_j(t_j)^T S_j(t_j) = I^{(m_j)}, \quad j=1, 2, \cdots, n. \tag{2.5}$$

It is known[7] that the quantity

$$\nu_c(D) \equiv \|Dc\|_2 / \|D\|_F \tag{2.6}$$

is a reasonable measure of the conditioning of the eigenproblem (1.1), where

$$D = \mathrm{diag}(d_1, d_2, \cdots, d_n), \quad d_j > 0 \ \forall j. \tag{2.7}$$

Hence, if we can determine $w_1, w_2, \cdots, w_n$ and $t_1, t_2, \cdots, t_n$ such that the measure $\nu_c(D)$ takes its minimum, then the corresponding X is well-conditioned.

Let

$$w = (w_1^T, w_2^T, \cdots, w_n^T)^T, \quad w_j = (w_{1j}, \cdots, w_{m_j,j})^T \in \mathbb{R}^{m_j} \ \forall j, \tag{2.8}$$

$$t = (t_1, t_2, \cdots, t_n)^T, \quad t_j \in \mathbb{R} \ \forall j \tag{2.9}$$

and $N = m_1 + m_2 + \cdots + m_n$. Moreover, let

$$X_j = (x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n), \tag{2.10}$$

$$\hat{w}_j = (w_1^T, \cdots, w_{j-1}^T, w_{j+1}^T, \cdots, w_n^T)^T \tag{2.11}$$

and

$$\hat{t}_j = (t_1, \cdots, t_{j-1}, t_{j+1}, \cdots, t_n)^T, \quad j=1, 2, \cdots, n. \tag{2.12}$$

Using singular value decomposition we obtain

$$X_j = (U_j, u_j)\begin{pmatrix} \Sigma_j \\ 0 \end{pmatrix} V_j^T, \quad u_j \in \mathbb{R}^n, \ \Sigma_j = \mathrm{diag}(\sigma_{j,1}, \cdots, \sigma_{j,n-1}), \tag{2.13}$$

where (U_j, u_j) and V_j are orthogonal matrices, and

$$X_j = X_j(\hat{w}_j, \hat{t}_j), \quad U_j = U_j(\hat{w}_j, \hat{t}_j), \quad u_j = u_j(\hat{w}_j, \hat{t}_j),$$
$$\Sigma_j = \Sigma_j(\hat{w}_j, \hat{t}_j), \quad V_j = V_j(\hat{w}_j, \hat{t}_j).$$

By [7] ([7, § 3]) we have

$$[\nu_c(D)]^2 = \sum_{j=1}^{n} \frac{\delta_j w_j^T w_j}{[u_j(\hat{w}_j, \hat{t}_j)^T S_j(t_j) w_j]^2} \equiv f(w, t), \tag{2.14}$$

where

$$\delta_j = \frac{d_j^2}{\sum_{j=1}^{n} d_j^2} \ \forall j, \quad \sum_{j=1}^{n} \delta_j = 1. \tag{2.15}$$

Hence, the problem for finding a well-conditioned matrix X may be reduced to the following unconstrained optimization problem:

$$\text{minimize } f(w, t), \tag{2.16}$$

where $f(w, t)$, w and t are defined by (2.14), (2.8) and (2.9), respectively.

§ 3. Formula of the Gradient Vector

Let

$$\frac{\partial f(w,t)}{\partial w_l} = \left(\frac{\partial f(w,t)}{\partial w_{1l}}, \frac{\partial f(w,t)}{\partial w_{2l}}, \cdots, \frac{\partial f(w,t)}{\partial w_{m_l,l}}\right)^T, \quad l=1, 2, \cdots, n \tag{3.1}$$

and

$$\frac{\partial f(w,t)}{\partial t}=\left(\frac{\partial f(w,t)}{\partial t_1}, \frac{\partial f(w,t)}{\partial t_2}, \cdots, \frac{\partial f(w,t)}{\partial t_n}\right)^T, \tag{3.2}$$

where $f(w, t)$, w and t are defined by (2.14), (2.8) and (2.9), respectively. Then the gradient vector $\nabla f(w, t)$ may be represented by

$$\nabla f(w,t)=\left(\left(\frac{\partial f(w,t)}{\partial w_1}\right)^T, \left(\frac{\partial f(w,t)}{\partial w_2}\right)^T, \cdots, \left(\frac{\partial f(w,t)}{\partial w_n}\right)^T, \left(\frac{\partial f(w,t)}{\partial t}\right)^T\right)^T. \tag{3.3}$$

In this section we deduce a formula of the gradient vector $\nabla f(w, t)$.

3.1. Expressions of $\dfrac{\partial f(w,t)}{\partial w_l}$, $l=1, 2, \cdots, n$.

By Theorem 3.1 of [7] we know that

$$\frac{\partial f(w,t)}{\partial w_l}=2\delta_l\left[w_l-\frac{w_l^T w_l S_l(t_l)^T u_l(\hat{w}_l, \hat{t}_l)}{u_l(\hat{w}_l, \hat{t}_l)^T S_l(t_l) w_l}\right]\Big/[u_l(\hat{w}_l, \hat{t}_l)^T S_l(t_l) w_l]^2$$
$$+\sum_{\substack{j=1\\ j\neq l}}^{n}\frac{\delta_j w_j^T w_j}{u_j(\hat{w}_j, \hat{t}_j)^T S_j(t_j) w_j} Y_{j,l}(w,t) S_l(t_l)^T Z_j(\hat{w}_j, \hat{t}_j) S_j(t_j) w_j /$$
$$[u_j(\hat{w}_j, \hat{t}_j)^T S_j(t_j) w_j]^2, \quad l=1, 2, \cdots, n, \tag{3.1.1}$$

where

$$Y_{j,l}(w,t)=u_j(\hat{w}_j, \hat{t}_j)^T S_l(t_l) w_l \cdot I^{(m_l)}+S_l(t_l)^T u_j(\hat{w}_j, \hat{t}_j) w_l^T \tag{3.1.2}$$

and

$$Z_j(\hat{w}_j, \hat{t}_j)=U_j(\hat{w}_j, \hat{t}_j)\Sigma_j(\hat{w}_j, \hat{t}_j)^{-2}U_j(\hat{w}_j, \hat{t}_j)^T, \tag{3.1.3}$$

among which $S_j(t_j)$, $U_j(\hat{w}_j, \hat{t}_j)$, $u_j(\hat{w}_j, \hat{t}_j)$ and $\Sigma_j(\hat{w}_j, \hat{t}_j)$ have been defined in the above two sections.

3.2. Expression of $\dfrac{\partial f(w,t)}{\partial t}$.

Now we consider $\partial f(w, t)/\partial t$ defined by (3.2).

3.2.1. First observe that the matrices $S_j(t_j)$ and $\hat{S}_j(t_j)$ described in Step A—X may be obtained by singular value decomposition of $(A^T-\lambda_j(t_j)I)U_1^{(B)}$ ([2]):

$$(A^T-\lambda_j(t_j)I)U_1^{(B)}=(\hat{S}_j(t_j), S_j(t_j))\begin{pmatrix}\Gamma_j(t_j)\\ 0\end{pmatrix}T_j(t_j)^T, \tag{3.2.1}$$

where $(\hat{S}_j(t_j), S_j(t_j))$ and $T_j(t_j)$ are orthogonal matrices, $S_j(t_j)\in\mathbb{R}^{n\times m_j}$, and $\Gamma_j(t_j)\in\mathbb{R}^{(n-m_j)\times(n-m_j)}$ is a diagonal matrix with positive diagonal elements (Note: Since the pair (A, B) is controllable, the matrix $(A^T-\lambda_j(t_j)I)U_1^{(B)}$ is necessarily of full-rank (see [2, p. 1134]).

Let

$$H_j(t_j)=(A^T-\lambda_j(t_j)I)U_1^{(B)}U_1^{(B)T}(A-\lambda_j(t_j)I). \tag{3.2.2}$$

From (3.2.1) we have

$$H_j(t_j)=(\hat{S}_j(t_j), S_j(t_j))\begin{pmatrix}\Gamma_j(t_j)^2 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\hat{S}_j(t_j)^T\\ S_j(t_j)^T\end{pmatrix}.$$

The following theorem shows that if for an arbitrarily fixed point $t_j^*\in\mathbb{R}$ we have a decomposition

$$H_j(t_j) = (\hat{S}_j^*, S_j^*)\begin{pmatrix} \Gamma_j^{*2} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \hat{S}_j^{*T} \\ S_j^{*T} \end{pmatrix},$$

where $(\hat{S}_j^*, S_j^*) \in \mathbb{R}^{n\times n}$ is an orthogonal matrix with $S_j^* \in \mathbb{R}^{n\times m_j}$, and $\Gamma_j^* \in \mathbb{R}^{(n-m_j)\times(n-m_j)}$ is a diagonal matrix with positive elements, then there exist some neighbourhood $\mathscr{B}_{t_j^*}^{(j)} \subset \mathbb{R}$ of t_j^*, $(\hat{S}_j(t_j), S_j(t_j)) \in \mathbb{R}^{n\times n}$ with $S_j(t_j) \in \mathbb{R}^{n\times m_j}$, and a full-rank matrix $\Gamma_j(t_j) \in \mathbb{R}^{(n-m_j)\times(n-m_j)}$, such that $\hat{S}_j(t_j)$, $S_j(t_j)$ and $\Gamma_j(t_j)$ are real matrix-valued analytic functions of $t_j \in \mathscr{B}_{t_j^*}^{(j)}$ satisfying

$$(\hat{S}_j(t_j), S_j(t_j))^T(\hat{S}_j(t_j), S_j(t_j)) = I^{(n)}, \quad \forall t_j \in \mathscr{B}_{t_j^*}^{(j)}, \tag{3.2.3}$$

$$(\hat{S}_j(t_j^*), S_j(t_j^*)) = (\hat{S}_j^*, S_j^*), \quad \Gamma_j(t_j^*) = \Gamma_j^* \tag{3.2.4}$$

and

$$H_j(t_j) = (\hat{S}_j(t_j), S_j(t_j))\begin{pmatrix} \Gamma_j(t_j)^2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \hat{S}_j(t_j)^T \\ S_j(t_j)^T \end{pmatrix}, \quad \forall t_j \in \mathscr{B}_{t_j^*}^{(j)}. \tag{3.2.5}$$

Moreover, according to the following theorem we can obtain an expression of $\left(\frac{dS_j(t_j)}{dt_j}\right)_{t_j=t_j^*}$.

Theorem 3.1. *Let $H(\tau) \in \mathbb{R}^{n\times n}$ be a real matrix-valued analytic function of the real variable τ in some neighbourhood $\mathscr{B}(0)$ of the origin, $H(\tau)^T = H(\tau)$ and*

$$\operatorname{rank}(H(\tau)) = n - m, \quad \forall \tau \in \mathscr{B}(0).$$

Suppose that there is an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ such that

$$Q = (Q_1, Q_2), \quad H(0) = Q\begin{pmatrix} H_1 & 0 \\ 0 & 0 \end{pmatrix}Q^T, \tag{3.2.6}$$

where

$$Q_2 \in \mathbb{R}^{n\times m}, \quad H_1 \in \mathbb{R}^{(n-m)\times(n-m)}, \quad \operatorname{rank}(H_1) = n - m. \tag{3.2.7}$$

Then

1) *there exist some neighbourhood $\mathscr{B}_0 \subset \mathscr{B}(0)$ of the origin, $Q(\tau) = (Q_1(\tau), Q_2(\tau)) \in \mathbb{R}^{n\times n}$ with $Q_2(\tau) \in \mathbb{R}^{n\times m}$, and a full-rank matrix $H_1(\tau) \in \mathbb{R}^{(n-m)\times(n-m)}$, such that $Q_1(\tau)$, $Q_2(\tau)$ and $H_1(\tau)$ are real matrix-valued analytic functions of $\tau \in \mathscr{B}_0$ satisfying*

$$Q(\tau)^T Q(\tau) = I^{(n)}, \quad \forall \tau \in \mathscr{B}_0, \tag{3.2.8}$$

$$Q(0) = Q, \quad H_1(0) = H_1 \tag{3.2.9}$$

and

$$H(\tau) = Q(\tau)\begin{pmatrix} H_1(\tau) & 0 \\ 0 & 0 \end{pmatrix}Q(\tau)^T, \quad \forall \tau \in \mathscr{B}_0; \tag{3.2.10}$$

2) *we have the following expression of $\left(\frac{dQ_2(\tau)}{d\tau}\right)_{\tau=0}$:*

$$\left(\frac{dQ_2(\tau)}{d\tau}\right)_{\tau=0} = -Q_1 H_1^{-1} Q_1^T \left(\frac{dH(\tau)}{d\tau}\right)_{\tau=0} Q_2. \tag{3.2.11}$$

Proof. 1) Let

$$\tilde{H}(\tau) = Q^T H(\tau) Q = \begin{pmatrix} \tilde{H}_{11}(\tau) & \tilde{H}_{21}(\tau)^T \\ \tilde{H}_{21}(\tau) & \tilde{H}_{22}(\tau) \end{pmatrix}, \quad \tilde{H}_{22}(\tau) \in \mathbb{R}^{m\times m}. \tag{3.2.12}$$

It is obvious that

$$\widetilde{H}_{11}(0)=H_1,\quad \widetilde{H}_{21}(0)=0,\quad \widetilde{H}_{22}(0)=0. \tag{3.2.13}$$

We introduce a matrix-valued function

$$\begin{cases} G(Y,\tau)=\widetilde{H}_{21}(\tau)^T+\widetilde{H}_{11}(\tau)Y-Y\widetilde{H}_{22}(\tau)-Y\widetilde{H}_{21}(\tau)Y, \\ Y=(\eta_{ij})\in\mathbb{R}^{(n-m)\times m},\quad \tau\in\mathscr{B}(0). \end{cases} \tag{3.2.14}$$

Clearly, the function $G(Y,\tau)=(g_{ij}(Y,\tau))$ is analytic for $Y\in\mathbb{R}^{(n-m)\times m}$ and $\tau\in\mathscr{B}(0)$, and it satisfies

$$g_{ij}(0,0)=0,\quad i=1,2,\cdots,n-m,\ j=1,2,\cdots,m$$

and

$$\left(\det\frac{\partial(g_{11},\cdots,g_{1m},g_{21},\cdots,g_{2m},\cdots,g_{n-m,1},\cdots,g_{n-m,m})}{\partial(\eta_{11},\cdots,\eta_{1m},\eta_{21},\cdots,\eta_{2m},\cdots,\eta_{n-m,1},\cdots,\eta_{n-m,m})}\right)_{Y=0,\ \tau=0}=\det(I^{(m)}\otimes H_1)\neq0,$$

where $\otimes$ denotes the Kronecker product (see [5, p. 8—9]). Hence by the implicit function theorem ([6, § 1]) the equation

$$G(Y,\tau)=0 \tag{3.2.15}$$

has a unique real analytic solution $Y=Y(\tau)$ in some neighbourhood $\mathscr{B}_1\subset\mathscr{B}(0)$ of the origin, and $Y(0)=0$.

Combining (3.2.12), (3.2.14) and (3.2.15) we can verify that the real matrix-valued analytic function $Y(\tau)$ satisfies

$$\begin{pmatrix} K_1(\tau) & 0 \\ 0 & K_2(\tau)\end{pmatrix}\begin{pmatrix} I & -Y(\tau) \\ Y(\tau)^T & I\end{pmatrix}H(\tau)\begin{pmatrix} I & Y(\tau) \\ -Y(\tau)^T & I\end{pmatrix}\begin{pmatrix} K_1(\tau) & 0 \\ 0 & K_2(\tau)\end{pmatrix}$$
$$=\begin{pmatrix} H_1(\tau) & 0 \\ 0 & H_2(\tau)\end{pmatrix},\quad \forall\tau\in\mathscr{B}_1, \tag{3.2.16}$$

among which

$$K_1(\tau)=(I+Y(\tau)Y(\tau)^T)^{-\frac{1}{2}},\quad K_2(\tau)=(I+Y(\tau)^TY(\tau))^{-\frac{1}{2}}, \tag{3.2.17}$$

$$H_1(\tau)=K_1(\tau)\widetilde{H}_1(\tau)K_1(\tau),\quad H_2(\tau)=K_2(\tau)\widetilde{H}_2(\tau)K_2(\tau), \tag{3.2.18}$$

$$\widetilde{H}_1(\tau)=\widetilde{H}_{11}(\tau)-Y(\tau)\widetilde{H}_{21}(\tau)-\widetilde{H}_{21}(\tau)^TY(\tau)^T+Y(\tau)\widetilde{H}_{22}(\tau)Y(\tau)^T \tag{3.2.19}$$

and

$$\widetilde{H}_2(\tau)=\widetilde{H}_{22}(\tau)+\widetilde{H}_{21}(\tau)Y(\tau)+Y(\tau)^T\widetilde{H}_{21}(\tau)^T+Y(\tau)^T\widetilde{H}_{11}(\tau)Y(\tau). \tag{3.2.20}$$

Obviously, the real matrix-valued functions $H_1(\tau)$ and $H_2(\tau)$ are analytic in $\mathscr{B}_1$. Since $\widetilde{H}_{11}(0)=H_1$, $\operatorname{rank}(H_1)=n-m$ and $\operatorname{rank}(H(\tau))=n-m$ $\forall\tau\in\mathscr{B}(0)$, from (3.2.17)—(3.2.20) it follows that there exists some neighbourhood $\mathscr{B}_0\subset\mathscr{B}_1$ of the origin such that

$$\operatorname{rank}(H_1(\tau))=n-m,\quad H_2(\tau)=0,\quad \forall\tau\in\mathscr{B}_0.$$

Therefore if we set

$$Q_1(\tau)=Q\begin{pmatrix} I \\ -Y(\tau)^T\end{pmatrix}K_1(\tau),\ Q_2(\tau)=Q\begin{pmatrix} Y(\tau) \\ I\end{pmatrix}K_2(\tau),\ Q(\tau)=(Q_1(\tau),Q_2(\tau)), \tag{3.2.21}$$

then $Q(\tau)$ is a matrix-valued analytic function of $\tau\in\mathscr{B}_0$, and $Q(\tau)$, $H_1(\tau)$ and $H(\tau)$ satisfy the relations (3.2.8)—(3.2.10).

2) From (3.2.16) we see that

$$H(\tau)Q_2(\tau)=0, \quad \forall \tau \in \mathscr{B}_0, \tag{3.2.22}$$

in which the matrix $Q_2(\tau)$ is defined by (3.2.21). Utilizing the real analyticity of $Y(\tau)$ in $\mathscr{B}_0$, from (3.2.22) we get

$$\left(\frac{dH(\tau)}{d\tau}\right)_{\tau=0} Q_2(0)+H(0)\left(\frac{dQ_2(\tau)}{d\tau}\right)_{\tau=0}=0.$$

Combining with (3.2.6), (3.2.9) and

$$\left(\frac{dQ_2(\tau)}{d\tau}\right)_{\tau=0}=Q\begin{pmatrix}\left(\frac{dY(\tau)}{d\tau}\right)_{\tau=0}\\ 0\end{pmatrix}=Q_1\left(\frac{dY(\tau)}{d\tau}\right)_{\tau=0}, \tag{3.2.23}$$

we obtain

$$\left(\frac{dH(\tau)}{d\tau}\right)_{\tau=0} Q_2+Q\begin{pmatrix}H_1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\left(\frac{dY(\tau)}{d\tau}\right)_{\tau=0}\\ 0\end{pmatrix}=0,$$

and thus

$$\left(\frac{dY(\tau)}{d\tau}\right)_{\tau=0}=-H_1^{-1}Q_1^T\left(\frac{dH(\tau)}{d\tau}\right)_{\tau=0} Q_2. \tag{3.2.24}$$

Substituting (3.2.24) into (3.2.23) we deduce (3.2.11). ∎

By Theorem 3.1 the real matrix-valued analytic function $S_j(t_j)$ defined by (3.2.3)—(3.2.5) has derivative

$$\left(\frac{dS_j(t_j)}{dt_j}\right)_{t_j=t_j^*}=-\hat{S}_j^*\Gamma_j^{*-2}\hat{S}_j^{*T}\left(\frac{dH_j(t_j)}{dt_j}\right)_{t_j=t_j^*} S_j^*. \tag{3.2.25}$$

But from (1.6) and (3.2.2),

$$\left(\frac{dH_j(t_j)}{dt_j}\right)_{t_j=t_j^*}=-(\lambda_{j,2}-\lambda_{j,1})\sin 2t_j^*[U_1^{(B)}U_1^{(B)T}(A-\lambda_j^*I)+(A^T-\lambda_j^*I)U_1^{(B)}U_1^{(B)T}], \tag{3.2.26}$$

where

$$\lambda_j^*=\lambda_{j,1}+(\lambda_{j,2}-\lambda_{j,1})\sin^2 t_j^*. \tag{3.2.27}$$

Hence, substituting (3.2.26) and (3.2.27) into (3.2.25) we obtain

$$\left(\frac{dS_j(t_j)}{dt_j}\right)_{t_j=t_j^*}=(\lambda_{j,2}-\lambda_{j,1})\sin 2t_j^*\hat{S}_j^*\Gamma_j^{*-2}\hat{S}_j^{*T}[U_1^{(B)}U_1^{(B)T}(A-\lambda_j^*I)+(A^T-\lambda_j^*I)U_1^{(B)}U_1^{(B)T}]S_j^*. \tag{3.2.28}$$

3.2.2. Let

$$A_j(\hat{w}_j, \hat{t}_j^*)=X_j(\hat{w}_j, \hat{t}_j^*)X_j(\hat{w}_j, \hat{t}_j^*)^T, \tag{3.2.29}$$

where $\hat{w}_j$ is defined by (2.11), $\hat{t}_j^*$ and $X_j(\hat{w}_j, \hat{t}_j^*)$ are defined by (2.12), (2.10) and (2.14), $t^*=(t_1^*, t_2^*, \cdots, t_n^*)^T$ is an arbitrarily fixed point of $\mathbb{R}^n$. From (2.10)—(2.12) and (2.4) we have

$$A_j(\hat{w}_j, \hat{t}_j^*)=\sum_{\substack{k=1\\ k\neq j}}^{n} S_k^* w_k w_k^T S_k^{*T}, \tag{3.2.30}$$

where $S_1^*, S_2^*, \cdots, S_n^*$ are defined by (3.2.1), i.e., by

$$(A^T-\lambda_j(t_j^*)I)U_1^{(B)}=(\hat{S}_j^*,\ S_j^*)\begin{pmatrix}\Gamma_j^*\\ 0\end{pmatrix}T_j^{*T}, \tag{3.2.31}$$

among which $(\hat{S}_j^*,\ S_j^*)$ and T_j^* are orthogonal matrices, $S_j^*\in\mathbb{R}^{n\times m_j}$, and $\Gamma_j^*\in\mathbb{R}^{(n-m_j)\times(n-m_j)}$ is a diagonal matrix with positive elements.

Utilizing (2.13) and the same argument used in the proof of Theorem 3.1 (or § 2 of [6]) there exists some neighbourhood $\mathscr{D}^{(j)}_{\hat{w}_j^*,\hat{t}_j^*}$ of $(\hat{w}_j^{*T},\ \hat{t}_j^{*T})^T$ in the product vector space $\mathbb{R}^{N-m_j}\times\mathbb{R}^{n-1}$, such that

$$A_j(\hat{w}_j,\ \hat{t}_j)=(U_j(\hat{w}_j,\ \hat{t}_j),\ u_j(\hat{w}_j,\ \hat{t}_j))\begin{pmatrix}\Sigma_j(\hat{w}_j,\ \hat{t}_j)^2 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}U_j(\hat{w}_j,\ \hat{t}_j)^T\\ u_j(\hat{w}_j,\ \hat{t}_j)^T\end{pmatrix}, \tag{3.2.32}$$

where $u_j(\hat{w}_j,\ \hat{t}_j)$ is a vector-valued analytic function of $(\hat{w}_j^T,\ \hat{t}_j^T)^T\in\mathscr{D}^{(j)}_{\hat{w}_j^*,\hat{t}_j^*}$, and $\Sigma_j(\hat{w}_j,\ \hat{t}_j)\in\mathbb{R}^{(n-1)\times(n-1)}$ is diagonal matrix. Assume that the matrix $X_j(\hat{w}_j,\ \hat{t}_j)$ is of full column rank. Then the matrix $\Sigma_j(\hat{w}_j,\ \hat{t}_j)$ is necessarily non-singular, and $u_j(\hat{w}_j,\ \hat{t}_j)$ is a unit eigenvector of $A_j(\hat{w}_j,\ \hat{t}_j)$ corresponding to the simple eigenvalue zero. Thus, by Theorem 2.4 of [6] we get a expression of the partial derivative of $u_j(\hat{w}_j,\ \hat{t}_j)$ with respect to t_l at $t=t^*$:

$$\left(\frac{\partial u_j(\hat{w}_j,\ \hat{t}_j)}{\partial t_l}\right)_{t=t^*}=-U_j(\hat{w}_j,\ \hat{t}_j^*)\Sigma_j(\hat{w}_j,\ \hat{t}_j^*)^{-2}U_j(\hat{w}_j,\ \hat{t}_j^*)^T\left(\frac{\partial A_j(\hat{w}_j,\ \hat{t}_j)}{\partial t_l}\right)_{t=t^*}u_j(\hat{w}_j,\ \hat{t}_j^*),\quad l\neq j. \tag{3.2.33}$$

Observe that by Theorem 3.1 one can define real matrix-valued analytic functions $S_k(t_k)$ for $k=1,\ 2,\ \cdots,\ j-1,\ j+1,\ \cdots,\ n$. Therefore, for any point $w\in\mathbb{R}^N$ and any point t in some neighbourhood $\mathscr{B}_{t^*}\subset\mathbb{R}^n$ of t^*, we have

$$A_j(\hat{w}_j,\ \hat{t}_j)=\sum_{\substack{k=1\\k\neq j}}^{n}S_k(t_k)w_kw_k^TS_k(t_k)^T, \tag{3.2.34}$$

and thus

$$\left(\frac{\partial A_j(\hat{w}_j,\ \hat{t}_j)}{\partial t_l}\right)_{t=t^*}=\left(\frac{dS_l(t_l)}{dt_l}\right)_{t_l=t_l^*}w_lw_l^TS_l^{*T}+S_l^*w_lw_l\left(\frac{dS_l(t_l)}{dt_l}\right)^T_{t_l=t_l^*}, \tag{3.2.35}$$

where the derivative $\left(\frac{dS_l(t_l)}{dt_l}\right)_{t_l=t_l^*}$ is given by (3.2.28). Substituting (3.2.35) and (3.2.28) into (3.2.33) we obtain the expression of $\left(\frac{\partial u_j(\hat{w}_j,\ \hat{t}_j)}{\partial t_l}\right)_{t=t^*}$.

3.2.3. Now we give the expressions of $\frac{\partial f(w,\ t)}{\partial t_l}$ for $l=1,\ 2,\ \cdots,\ n$.

First of all, we define $S_j(t_j)$ and $u_j(\hat{w}_j,\ \hat{t}_j)$ for $j=1,\ 2,\ \cdots,\ n$ contained in (2.14) as follows: for arbitrarily fixed points $w^*\in\mathbb{R}^N$ and $t^*=(t_1^*,\ t_2^*,\ \cdots,\ t_n^*)^T\in\mathbb{R}^n$, by Theorem 3.1 we define $S_j(t_j)$ for $j=1,\ 2,\ \cdots,\ n$, which are real matrix-value analytic functions in some neighbourhood $\mathscr{B}^{(j)}_{t_j^*}\subset\mathbb{R}$ of t_j^*; at the same time, according to subsection 3.2.2 we define $u_j(\hat{w}_j,\ \hat{t}_j)$ for $j=1,\ 2,\ \cdots,\ n$, which are real vector-valued analytic functions in some neighbourhood $\mathscr{D}^{(j)}_{\hat{w}_j^*,\hat{t}_j^*}$ of $(\hat{w}_j^{*T},\ \hat{t}_j^{*T})^T\in\mathbb{R}^{N-m_j}\times\mathbb{R}^{n-1}$ for $j=1,\ 2,\ \cdots,\ n$. respectively.

From (2.14) we get

$$\left(\frac{\partial f(w,\ t)}{\partial t_l}\right)_{t=t^*}=\left[\frac{\partial}{\partial t_l}\left(\sum_{j=1}^{n}\frac{\delta_jw_j^Tw_j}{[u_j(\hat{w}_j,\ \hat{t}_j)^TS_j(t_j)w_j]^2}\right)\right]_{t=t^*}$$

$$=-2\sum_{\substack{j=1\\j\neq l}}^{n}\frac{\delta_j w_j^T w_j}{[u_j(\hat{w}_j,\hat{t}_j^*)^T S_j^* w_j]^3}\left(\frac{\partial u_j(\hat{w}_j,\hat{t}_j)}{\partial t_l}\right)^T_{t=t^*} S_j^* w_j$$

$$-\frac{2\delta_l w_l^T w_l}{[u_l(\hat{w}_l,\hat{t}_l^*)^T S_l^* w_l]^3} u_l(\hat{w}_l,\hat{t}_l^*)^T\left(\frac{dS_l(t_l)}{dt_l}\right)_{t_l=t_l^*} w_l, \tag{3.2.36}$$

among which $\left(\frac{\partial u_j(\hat{w}_j,\hat{t}_j)}{\partial t_l}\right)_{t=t^*}$ and $\left(\frac{dS_l(t_l)}{dt_l}\right)_{t_l=t_l^*}$ are given by (3.2.33)—(3.2.35) and (3.2.38), respectively.

Combining Subsections 3.1—3.2 we obtain a formula of the gradient vector $\nabla f(w,t)$, in which the expressions of $\frac{\partial f(w,t)}{\partial w_1}, \cdots, \frac{\partial f(w,t)}{\partial w_n}$ are given by (3.1.1)—(3.1.3), and $\frac{\partial f(w,t)}{\partial t_1}, \cdots, \frac{\partial f(w,t)}{\partial t_n}$ are given by (3.2.36), (3.2.33)—(3.2.35) and (3.2.28).

3.3. Application

In the next section we shall use the DFP algorithm to solve the problem (2.16). Let $(w^{(k)^T}, t^{(k)^T})^T$ be the k-th iterative point of the DFP iterative process. Assume that

$$(A^T-\lambda_j^{(k)}I)U_1^{(B)}=(\hat{S}_j^{(k)}, S_j^{(k)})\begin{pmatrix}\Gamma_j^{(k)}\\0\end{pmatrix}T_j^{(k)^T},\quad j=1,2,\cdots,n \tag{3.3.1}$$

and

$$X_j(\hat{w}_j^{(k)},\hat{t}_j^{(k)})=(U_j^{(k)},u_j^{(k)})\begin{pmatrix}\Sigma_j^{(k)}\\0\end{pmatrix}V_j^{(k)^T},\quad u_j^{(k)}\in\mathbb{R}^n,\quad j=1,2,\cdots,n \tag{3.3.2}$$

are singular value decomposition (see (3.2.1) and (2.13)), where

$$\lambda_j^{(k)}=\lambda_{j,1}+(\lambda_{j,2}-\lambda_{j,1})\sin^2 t^{(k)} \tag{3.3.3}$$

and

$$X_j(\hat{w}_j^{(k)},\hat{t}_j^{(k)})=(S_1^{(k)}w_1^{(k)},\cdots,S_{j-1}^{(k)}w_{j-1}^{(k)},S_{j+1}^{(k)}w_{j+1}^{(k)},\cdots,S_n^{(k)}w_n^{(k)}). \tag{3.3.4}$$

Then by Subsections 3.1—3.2 we have

$$\nabla f(w^{(k)},t^{(k)})=\left(\left(\frac{\partial f(w,t)}{\partial w_1}\right)^T,\cdots,\left(\frac{\partial f(w,t)}{\partial w_n}\right)^T,\frac{\partial f(w,t)}{\partial t_1},\cdots,\frac{\partial f(w,t)}{\partial t_n}\right)^T_{\substack{w=w^{(k)}\\t=t^{(k)}}}, \tag{3.3.5}$$

where

$$\left(\frac{\partial f(w,t)}{\partial w_l}\right)_{\substack{w=w^{(k)}\\t=t^{(k)}}}=2\delta_l\left(w_l^{(k)}-\frac{w_l^{(k)^T}w_l^{(k)}S_l^{(k)^T}u_l^{(k)}}{u_l^{(k)^T}S_l^{(k)}w_l^{(k)}}\right)\Big/(u_l^{(k)^T}S_l^{(k)}w_l^{(k)})^2$$

$$+2\sum_{\substack{j=1\\j\neq l}}^{n}\frac{\delta_j w_j^{(k)^T}w_j^{(k)}}{u_j^{(k)^T}S_j^{(k)}w_j^{(k)}}Y_{j,l}^{(k)}S_l^{(k)^T}Z_j^{(k)}S_j^{(k)}w_j^{(k)}/(u_j^{(k)^T}S_j^{(k)}w_j^{(k)})^2,$$

$$l=1,2,\cdots,n \tag{3.3.6}$$

with

$$Y_{j,l}^{(k)}=u_j^{(k)^T}S_l^{(k)}w_l^{(k)}I+S_l^{(k)^T}u_j^{(k)}w_l^{(k)^T} \tag{3.3.7}$$

and

$$Z_j^{(k)}=U_j^{(k)}\Sigma_j^{(k)-2}U_j^{(k)^T}, \tag{3.3.8}$$

$$\left(\frac{\partial f(w,t)}{\partial t_l}\right)_{\substack{w=w^{(k)}\\ t=t^{(k)}}}$$
$$=-2\sum_{\substack{j=1\\ j\neq l}}^{n}\frac{\delta_j w_j^{(k)T}w_j^{(k)}}{u_j^{(k)T}S_j^{(k)}w_j^{(k)}}\left(\frac{\partial u_j(\hat{w}_j,\hat{t}_j)}{\partial t_l}\right)^T_{\substack{w=w^{(k)}\\ t=t^{(k)}}} S_j^{(k)}w_j^{(k)}\Big/(u_j^{(k)T}S_j^{(k)}w_j^{(k)})^2$$
$$-\frac{2\delta_l w_l^{(k)T}w_l^{(k)}}{u_l^{(k)T}S_l^{(k)}w_l^{(k)}}u_l^{(k)T}\left(\frac{dS_l(t_l)}{dt_l}\right)_{t_l=t_l^{(k)}} w_l^{(k)}\Big/(u_l^{(k)T}S_l^{(k)}w_l^{(k)})^2,\quad l=1,2,\cdots,n \tag{3.3.9}$$

with

$$\left(\frac{\partial u_j(\hat{w}_j,\hat{t}_j)}{\partial t_l}\right)_{\substack{w=w^{(k)}\\ t=t^{(k)}}}=-Z_j^{(k)}\left[\left(\frac{dS_l(t_l)}{dt_l}\right)_{t_l=t_l^{(k)}} w_l^{(k)}w_l^{(k)T}S_l^{(k)T}\right.$$
$$\left.+S_l^{(k)}w_l^{(k)}w_l^{(k)T}\left(\frac{dS_l(t_l)}{dt_l}\right)^T_{t_l=t_l^{(k)}}\right]u_j^{(k)} \tag{3.3.10}$$

and

$$\left(\frac{dS_l(t_l)}{dt_l}\right)_{t_l=t_l^{(k)}}=(\lambda_{l,2}-\lambda_{l,1})\sin 2t_l^{(k)}\cdot\hat{S}_l^{(k)}\Gamma_l^{(k)-2}\hat{S}_l^{(k)T}(U_1^{(B)}U_1^{(B)T}A$$
$$+A^TU_1^{(B)}U_1^{(B)T}-2\lambda_l^{(k)}U_1^{(B)}U_1^{(B)T})S_l^{(k)}. \tag{3.3.11}$$

§ 4. An Algorithm

The Davidon–Fletcher–Powell method ([4, p. 194]) is now applied to solving the optimization problem (2.16):

Initialization Step. Let $\varepsilon>0$ be the termination scalar. Choose an initial vector $w^{(0)}=(w_1^{(0)T},\cdots,w_n^{(0)T})^T\in\mathbb{R}^N$ with $w_j^{(0)}\in\mathbb{R}^{m_j}\ \forall j$, $t^{(0)}=(t_1^{(0)},\cdots,t_n^{(0)})^T\in\mathbb{R}^n$, and an $(N+n)\times(N+n)$ positive definite matrix H_0 (e.g., $H_0=I^{(N+n)}$).

Main Step.

(1) Let $k:=0$.

(2) Compute $g_k=\nabla f(w^{(k)},t^{(k)})$, $p_k=-H_kg_k$.

(3) Determine $\begin{pmatrix} w^{(k+1)}\\ t^{(k+1)}\end{pmatrix}=\begin{pmatrix} w^{(k)}\\ t^{(k)}\end{pmatrix}+\lambda_kp_k$ by means of an approximate minimization

$$f(w^{(k)}+\lambda_kp_k',\ t^{(k)}+\lambda_kp_k'')\approx\min_{\lambda>0}f(w^{(k)}+\lambda p_k',\ t^{(k)}+\lambda p_k''),$$

where

$$p_k=\begin{pmatrix} p_k'\\ p_k''\end{pmatrix},\quad p_k'\in\mathbb{R}^N,\ p_k''\in\mathbb{R}^n.$$

(4) Set $\Delta w^{(k)}=w^{(k+1)}-w^{(k)}$ and $\Delta t^{(k)}=t^{(k+1)}-t^{(k)}$. If $\left\|\begin{pmatrix}\Delta w^{(k)}\\ \Delta t^{(k)}\end{pmatrix}\right\|_2\leqslant\varepsilon$ then $\begin{pmatrix} w^{(k+1)}\\ t^{(k+1)}\end{pmatrix}$ is an approximate optimal solution; if $\left\|\begin{pmatrix}\Delta w^{(k)}\\ \Delta t^{(k)}\end{pmatrix}\right\|_2>\varepsilon$, go to Step (5).

(5) Compute $g_{k+1}=\nabla f(w^{(k+1)},t^{(k+1)})$, $g_{k+1}-g_k=h_k$,

$$H_{k+1}=H_k+\frac{\begin{pmatrix}\Delta w^{(k)}\\ \Delta t^{(k)}\end{pmatrix}(\Delta w^{(k)T},\ \Delta t^{(k)T})}{(\Delta w^{(k)T},\ \Delta t^{(k)T})h_k}-\frac{H_kh_kh_k^TH_k^T}{h_k^TH_kh_k}$$

and

$$p_{k+1} = -H_{k+1} g_{k+1};$$

replace k by $k+1$, and go to Step (3).

Remark 4.1. If for some k and j the matrix $X_j(\hat{w}_j^{(k)}, \hat{t}_j^{(k)})$ or the matrix $(A^T - \lambda_j^{(k)} I) U_1^{(B)}$ has very small singular value, then we must choose another initial vector $(w^{(0)^T}, t^{(0)^T})^T$ anew.

Remark 4.2. At Step (2) we compute the gradient vector by the formulas (3.3.1)—(3.3.11).

Remark 4.3. At Step (3) the line search techniques based on curve fitting procedures, such as cubic fit and quadratic fit ([4, p. 142]), are feasible in practice.

§ 5. Numerical Results

Test Example. $n=3$, $m=2$ (see [3], [7]).

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \tag{5.1}$$

$$\begin{gathered} [\lambda_{1,1}, \lambda_{1,2}] = [-0.3, -0.1], \quad [\lambda_{2,1}, \lambda_{2,2}] = [-0.5, -0.1], \\ [\lambda_{3,1}, \lambda_{3,2}] = [-12, -8]. \end{gathered} \tag{5.2}$$

It is easy to verify that the matrix B has a decomposition (1.7) with

$$U_0^{(B)} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} \\ 0 & \frac{\sqrt{6}}{3} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \end{pmatrix}, \quad U_1^{(B)} = \begin{pmatrix} -\frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{pmatrix}, \quad Z = \begin{pmatrix} \sqrt{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{6}}{2} \end{pmatrix}.$$

We have calculated this example on a L-340 Computer in single precision with the numerical method described in this paper.

We choose $s = 15^{-4}$ as the termination scalar and $\delta_1 = \delta_2 = \delta_3 = 1/3$ (the factors δ_1, δ_2 and δ_3 are defined by (2.15)). With the initial parameters $t_1^{(0)} = \frac{\pi}{4}$, $t_2^{(0)} = \frac{\pi}{3}$, $t_3^{(0)} = \frac{\pi}{4}$ and the initial vectors $w_1^{(0)} = (1, 0)^T$, $w_2^{(0)} = (0, 1)^T$ and $w_3^{(0)} = \frac{1}{\sqrt{2}}(1, 1)^T$, after five iterations (see Main Step described in § 4) we obtain a convergent solution

$$t_1^{(5)} = 0.72508, \quad t_2^{(5)} = 1.28949, \quad t_3^{(5)} = -0.02338$$

and

$$w_1^{(5)} = \begin{pmatrix} 1.04644 \\ -0.06712 \end{pmatrix}, \quad w_2^{(5)} = \begin{pmatrix} -0.18302 \\ 1.03705 \end{pmatrix}, \quad w_3^{(5)} = \begin{pmatrix} 1.77673 \\ -0.52682 \end{pmatrix}.$$

The corresponding approximate solution $\{\lambda_1, \lambda_2, \lambda_3; X; F\}$ of Problem RPA1 is

$$\lambda_1 = \lambda_1(t_1^{(5)}) = -0.21204, \quad \lambda_2 = \lambda_2(t_2^{(5)}) = -0.13083, \quad \lambda_3 = \lambda_3(t_3^{(5)}) = -11.9978,$$

$$X = (x_1, x_2, x_3) = \begin{pmatrix} 0.90643 & -0.52293 & -1.71900 \\ 0.50038 & 0.12963 & 0.63122 \\ 0.16595 & 0.90483 & 0.28453 \end{pmatrix},$$

$$F=\begin{pmatrix} 1.32390 & -3.95266 & 1.21351 \\ -4.02153 & 7.97770 & -4.43657 \end{pmatrix}$$

with $\|x_1\|_2=1.04859$, $\|x_2\|_2=1.05308$, $\|x_3\|_2=1.85320$ and $\|F\|_2=10.8431$. The condition numbers of the eigenvalues λ_1, λ_2, λ_3 are

$$c_1=1.31167,\quad c_2=1.33810,\quad c_3=1.64963,$$

respectively, and for $c=(c_1, c_2, c_3)^T$ we have $\|c\|_2=2.49645$.

It is worthwhile to point out that for the matrices A and B given by (5.1), if $\lambda_1=-0.2$, $\lambda_2=-0.2$ and $\lambda_3=-10$ are assigned as the eigenvalues of $A+BF$, then using Method (I) described in [7] with $\delta_1=\delta_2=\delta_3=1/3$, $\varepsilon=10^{-8}$ and $w_1^{(0)}=(1, 0)^T$, $w_2^{(0)}=(0, 1)^T$, $w_3^{(0)}=\frac{1}{\sqrt{2}}(1, 1)^T$, after five iterations we obtain an approximate solution $\{X; F\}$ of Problem RPA:

$$X=(x_1, x_2, x_3)=\begin{pmatrix} 0.93979 & -0.54052 & -1.06046 \\ 0.49838 & 0.13725 & 0.33460 \\ 0.12104 & 0.92488 & 0.20796 \end{pmatrix},$$

$$F=\begin{pmatrix} 2.61121 & -6.56373 & 2.39685 \\ -3.52182 & 6.90222 & -4.03693 \end{pmatrix}$$

with $\|x_1\|_2=1.07063$, $\|x_2\|_2=1.08$, $\|x_3\|_2=1.13127$ and $\|F\|_2=11.4455$. The condition numbers of the eigenvalues λ_1, λ_2, λ_3 are

$$c_1=1.39486,\quad c_2=1.39449,\quad c_3=1.78934,$$

respectively, and for $c=(c_1, c_2, c_3)^T$ we have $\|c\|_2=2.66308$.

Remark 5.1. For the matrices A, B and the segments $[\lambda_{1,1}, \lambda_{1,2}]$, $[\lambda_{2,1}, \lambda_{2,2}]$, $[\lambda_{3,1}, \lambda_{3,2}]$ given by (5.1) and (5.2), with several different sets of initial vectors $w_1^{(0)}$, $w_2^{(0)}$, $w_3^{(0)}$ and initial parameters $t_1^{(0)}$, $t_2^{(0)}$, $t_3^{(0)}$, we have obtained different approximate solutions $\{\lambda_1, \lambda_2, \lambda_3; X; F\}$ of Problem RPA1 by using the numerical method suggested in this paper, but the corresponding vectors $c=(c_1, c_2, c_3)^T$ defined by (2.1)—(2.3) have about the same norm $\|c\|_2=2.5$.

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