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AN INTERIOR POINT METHOD FOR LINEAR PROGRAMMING*[1)]

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Abstract

In this paper we present an interior point method which solves a linear programming problem by using an affine transformation. We prove under certain assumptions that the algorithm converges to an optimal solution even if the dual problem is degenerate as long as the primal is bounded, or to a ray direction if the optimal value of the objective function is unbounded.

§1. Introduction

This paper presents an interior point method for solving a linear programming problem (LP), and proves the convergence of the algorithm under certain assumptions. The algorithm was previously mentioned in [7], in which the convergence was proven under the assumptions that the primal and dual problems are both nondegenerate, and that the problem is bounded. In fact, the algorithm performs well in more general cases, but under an additional assumption that the LP problem satisfies the condition F, which will be defined in Section 2. The condition F implies that if the feasible region of the LP problem is bounded, then the primal problem is nondegenerate and that if the feasible region is unbounded, then the primal problem is nondegenerate, and there are at least $m+1$ nonzero components in the vector of any ray direction.

In this paper we show, under the assumption that the LP problem satisfies the above condition F, that the algorithm converges to an optimal solution for a bounded primal problem, even if the dual problem is degenerate, and to an extreme point if the dual problem is nondegenerate. We also prove that in the case of the unboundedness of the LP problem the algorithm converges to a ray direction, along which the minimum value is unbounded.

§2. Algorithm

This section describes the interior point method for solving the LP problem by use of affine transformations.

We consider the following standard form of the linear programming problem:

$$\text{minimize } z=c^T x, \tag{2.1}$$

$$\text{subject to } Ax=b, \tag{2.2}$$

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$$x \geqslant 0, \tag{2.3}$$

where A is an $m \times n$ real matrix with rank m, and $m<n$. b and c are real vectors in R^m and R^n, respectively, and x is a real variable in R^n.

Let $S=[x: Ax=b,\ x\geqslant 0]$ denote the feasible region of the LP problem, then the LP problem is feasible if S is nonempty.

Definition 2.1. *The LP problem satisfies the condition F if S is nonemptg, and for any $x\in S$, the matrix AD^2A^T is of full rank, where D is a diagonal matrix containing components of x.*

From the definition, it is easy to see that if the feasible region S is bounded, then the condition F is equivalent to nondegeneracy of the primal, and if S is unbounded, then for any ray direction ν of S, the matrix $AD_\nu^2A^T$ is of full rank, where D_ν is a diagonal matrix containing components of ν.

Suppose that the LP problem satisfies the condition F, and that $\bar{x}$ is a strictly interior feasible point. Then an affine transformation and its inverse can be defined as follows:

$$x' = D^{-1}x, \tag{2.4}$$

$$x = Dx'. \tag{2.5}$$

With the above transformation the original LP problem is transformed into the following linear programming problem in x'-space:

$$\text{minimize } z = c^TDx', \tag{2.6}$$

$$\text{subject to } ADx' = b, \tag{2.7}$$

$$x' \geqslant 0, \tag{2.8}$$

where D is a diagonal matrix containing components of $\bar{x}$.

Obviously, from the definition of D the point $\bar{x}$ in x-space is mapped into the point $e^T=(1, 1, \cdots, 1)$ in x'-space. Hence, in order to get the maximum rate of decrease of the objective function, a large step away from the point e to a new point in x'-space is taken along the negative of the projective gradient direction. The new point is transformed back into the x-space, and an iterative point is obtained.

Given a strictly interior point $x^{(0)}$, the algorithm described above, which creates a sequence of points $x^{(0)}, x^{(1)}, \cdots$, is defined more formally as follows.

Algorithm A.

$k: =0$. Given $x^{(0)}$, a strictly interior feasible point.

(1) Define

$$D_k = \operatorname{diag}(x_1^{(k)}, x_2^{(k)}, \cdots, x_n^{(k)}), \tag{2.9}$$

$$A_k = AD_k. \tag{2.10}$$

(2) Compute the vector $c_p^{(k)}$ and its norm $\|c_p^{(k)}\|_2$ by

$$c_p^{(k)} = [I - A_k^T(A_kA_k^T)^{-1}A_k]D_kc \tag{2.11}$$

and

$$\|c_p^{(k)}\|_2 = \sqrt{(c_p^{(k)})^T(c_p^{(k)})}. \tag{2.12}$$

(3) Normalize $c_p^{(k)}$,

$$p^{(k)}=\frac{c_p^{(k)}}{\|c_p^{(p)}\|_2}. \tag{2.13}$$

go to the next step.

(4) Determine the largest step which mag be taken and generate a new point

$$\lambda_k=\frac{1}{q_k}, \tag{2.14}$$

where

$$q_k=\max[p_i^{(k)}]$$

and let

$$x^{(k+1)}=x^{(k)}-\alpha\lambda_k D_k p(k), \tag{2.15}$$

where $\alpha\in(0, 1)$.

k: $k+1$, and go to step (1).

It is clear that the bulk of the computational effort in each iteration is from step (2), which ensures the feasibility of the new point.

§ 3. Stopping Rule in the Case of Unboundedness

This section discusses the behavior of the vector $p^{(k)}$ defined by (2.13) under the assumption that the sequence $x^{(k)}$ generated by Algorithm A is unbounded.

As is well known, the dual problem plays a very important role in the solution of the LP problem. If the minimum value of the LP problem is unbounded, then its dual problem is infeasible. This is also true in Algorithm A. That is, if the sequence $x^{(k)}$ generated by Algorithm A is unbounded, then the dual problem is infeasible. To show this, we define a vector y, which is an approximation to the dual vector with respect to an interior point $\bar{x}$:

$$y=(AD^2A^T)^{-1}AD^2c. \tag{3.1}$$

It follows from the assumptions that $(AD^2A^T)^{-1}$ exists for all feasible points x, and from this it can be shown that it is continuous in x, and so is y. The major results of this section are as follows.

Theorem 3.1. *Suppose that the LP problem satisfies the condition F. Then the sequence $x^{(k)}$ generated by Algorithm A is unbounded if and only if*

$$\lim_{k\to\infty} p^{(k)}\leqslant 0, \tag{3.2}$$

where $p^{(k)}$ is defined by (2.13).

Theorem 3.2. *Suppose that the LP problem satisfies the condition F. Assume that the sequence $x^{(k)}$ generated by Algorithm A is unbounded, and there are l unbounded components in $x^{(k)}$. Then for each unbounded component $x_i^{(k)}$,*

$$\lim_{k\to\infty} p_i^{(k)}=\frac{-1}{\sqrt{l}} \tag{3.3}$$

holds.

In order to prove these theorems several lemmas are introduced.

Lemma 3.3. *Suppose that the LP problem satisfies the condition F. Then the sequence $y^{(k)}$ is bounded, where $y^{(k)}$ is defined by* (3.1) *with respect to $x^{(k)}$.*

Proof. If the sequence $x^{(k)}$ generated by Algorithm A is bounded, then the conclusion is trivial. Otherwise, from the assumption that the LP problem satisfies the condition F, it follows that for all k, the matrix $AD_k^2A^T$ is of full rank, where D_k is a diagonal matrix with respect to $x^{(k)}$.

Let

$$M_k=\sqrt{(x^{(k)})^T(x^{(k)})} \tag{3.4}$$

and

$$\bar{x}^{(k)}=\frac{x^{(k)}}{M_k}, \tag{3.5}$$

$$\bar{D}_k=\frac{1}{M_k}D_k. \tag{3.6}$$

Thus, for all k, the matrix

$$\frac{1}{M_k^2}\cdot(AD_k^2A^T)=A\left(\frac{1}{M_k}D_k\right)^2A^T=A\bar{D}_k^2A^T$$

is of full rank.

By definitions (3.1) and (3.6), it is clear that

$$y^{(k)}=(AD_k^2A^T)^{-1}AD_k^2c=(A\bar{D}_k^2A^T)^{-1}A\bar{D}_k^2c. \tag{3.7}$$

Let

$$E_k=\bar{D}_kA^T(A\bar{D}_k^2A^T)^{-1}A\bar{D}_k.$$

Since the matrix E_k is symmetric and idempotent, it is a projection matrix. Hence, for all k

$$\|\bar{D}_kA^Ty^{(k)}\|_2=\|\bar{D}_kA^T(A\bar{D}_k^2A^T)^{-1}A\bar{D}_k^2c\|_2$$
$$=\|E_k\bar{D}_kc\|_2\leqslant\|\bar{D}_kc\|_2\leqslant\|c\|_2$$

which implies that the sequence $y^{(k)}$ is bounded.

This completes the proof of the lemma.

Lemma 3.4. *Suppose that the LP problem satisfies the condition F. If the sequence $x^{(k)}$ generated by Algorithm A is unbounded, and there is a subset $N_1\subset N$, such that for all k, each component $x_i^{(k)}$ $(i\in N_1)$ is bounded, then*

$$\lim_{k\to\infty}p_i^{(k)}=0,\quad i\in N_1, \tag{3.8}$$

where $p^{(k)}$ is defined by (2.13).

Proof. It follows from the assumptions that there exists a large $M>0$, such that for all k

$$x_i^{(k)}<M,\quad i\in N_1 \tag{3.9}$$

holds. Suppose that M_k, $\bar{x}^{(k)}$, and $\bar{D}_k$ are defined by (3.4), (3.5) and (3.6), respectively. Thus

$$x^{(k)}=M_k\bar{x}^{(k)} \tag{3.10}$$

and

$$\lim_{k\to\infty}M_k=+\infty. \tag{3.11}$$

By (3.5) and (3.7), we have

$$c_p^{(k)}=D_k[c-A^T(AD_k^2A^T)^{-1}AD_k^2c]=D_k[c-A^Ty^{(k)}]=D_kr^{(k)} \tag{3.12}$$

and $r^{(k)}\neq 0$ for all k since the sequence $x^{(k)}$ tends to infinity.

It follows from Lemma 3.3 that $y^{(k)}$ is bounded. So, there is some large $\bar{M}$ such that for all k

$$\|r^{(k)}\|_2<\bar{M}. \tag{3.13}$$

From (3.6), it is clear that

$$D_k=M_k\bar{D}_k \tag{3.14}$$

By substituting (3.14) into (3.12), we obtain

$$c_p^{(k)}=M_k\bar{D}_k r^{(k)}. \tag{3.15}$$

By definition (2.13), it follows that

$$p^{(k)}=\frac{\bar{D}_k r^{(k)}}{\|\bar{D}_k r^{(k)}\|_2}. \tag{3.16}$$

Then, from (3.16), (3.13) and (3.11), it is straightforward to show that for a given sufficiently small $s>0$, there is an integer k_0, such that for all $k>k_0$

$$\left(M\frac{\bar{M}}{M_k}\right)<s.$$

This implies that

$$\lim_{k\to\infty}p_i^{(k)}=0,\quad i\in N_1$$

which proves the lemma.

Lemma 3.5. *With the same assumptions as in Theorem 3.2, there exists a subsequence $x^{(k_l)}$, such that for each unbounded component $x_i^{(k_l)}$,*

$$\lim_{k_l\to\infty}p_i^{(k_l)}=\frac{-1}{\sqrt{l}} \tag{3.17}$$

holds.

Proof. The proof follows from the assumption that for all k, matrix $AD_k^2A^T$ is of full rank, where D_k is defined by (2.9).

Suppose that M_k, $\bar{x}^{(k)}$ and $\bar{D}_{(k)}$ are defined by (3.4), (3.5) and (3.6), respectively. It is clear that $\bar{x}^{(k)}$ is a bounded sequence, so that there are a subsequence $\bar{x}^{(k_l)}$ and an $\bar{x}$ such that

$$\lim_{k_l\to\infty}\bar{x}^{(k_l)}=\bar{x}. \tag{3.18}$$

Since y is a continuous function of x, it is straightforward to show that there are also a vector $\bar{y}$ and a subsequence $y^{(k_l)}$ corresponding to $\bar{x}^{(k_l)}$ such that

$$\lim_{k_l\to\infty}y^{(k_l)}=\bar{y} \tag{3.19}$$

which implies that there are also a vector $\bar{r}\neq0$ and a subsequence $r^{(k_l)}$ with respect to $y^{(k_l)}$ such that

$$\lim_{k_l\to\infty}r^{(k_l)}=\bar{r}. \tag{3.20}$$

By definition (2.13),

$$\lim_{k_l\to\infty}p^{(k_l)}=\bar{p}=\frac{\bar{D}\bar{r}}{\|\bar{D}\bar{r}\|_2}, \tag{3.21}$$

where $\bar{D}$ is a diagonal matrix with respect to $\bar{x}$.

It follows from (3.18) and (3.21) that there exists a vector v such that

$$v=\lim_{k_l\to\infty}-\frac{\bar{D}_{k_l}P^{(k_l)}}{\|\bar{D}_{k_l}P^{(k_l)}\|_2}=\frac{-\bar{D}\bar{P}}{\|\bar{D}\bar{P}\|_2} \tag{3.22}$$

along which the subsequence $x^{(k_l)}$ tends to infinity. Therefore, there are a sufficiently small $s>0$ and an integer $\bar{k}_l>0$ such that for all $k_l>\bar{k}_l$

$$\left|\frac{x^{(k_l)}-x^{(\bar{k}_l)}}{\|x^{(k_l)}-x^{(\bar{k}_l)}\|_2}-\bar{x}\right|\leqslant s.$$

Then

$$\lim_{k_l\to\infty}\frac{x^{(k_l)}-x^{(\bar{k}_l)}}{\|x^{(k_l)}-x^{(\bar{k}_l)}\|_2}=\bar{x}. \tag{3.23}$$

By (3.18), it is also easy to show that

$$\lim_{k_l\to\infty}\frac{-\bar{D}_{k_l}P^{(k_l)}}{\|\bar{D}_{k_l}P^{(k_l)}\|_2}=\bar{x}. \tag{3.24}$$

It follows from (3.22), (3.23) and (3.24) that

$$v=-\frac{\bar{D}P}{\|\bar{D}\bar{P}\|_2}=\bar{x}.$$

Hence, if $\bar{x}_{i_t}\neq 0$, then

$$\bar{p}_{i_t}=-\|\bar{D}\bar{p}\|_2,\quad t=1,2,\cdots,l$$

which implies that

$$\bar{p}_i=\lim_{k_l\to\infty}p_i^{(k_l)}=-\frac{1}{\sqrt{l}}.$$

This completes the proof of the lemma.

Now we prove Theorems 3.1 and 3.2.

The proof of Theorem 3.2 is straightforward. From the conclusion of Lemma 3.5 it follows that there exists a subsequence $x^{(k_l)}$ such that

$$\lim_{k_l\to\infty}p^{(k_l)}=\bar{p}=\frac{\bar{D}\bar{r}}{\|\bar{D}\bar{r}\|_2}.$$

Hence, for a given sufficiently small $s>0$, there is an integer $\bar{k}_l$ such that for all $k_l>\bar{k}_l$

$$\left|-\frac{1}{\sqrt{l}}-p_{i_t}^{(k_l)}\right|<s,\quad t=1,2,\cdots,l \tag{3.25}$$

and

$$|p_i^{(k_l)}|<s,\quad i\in N_1, \tag{3.26}$$

where N_1 is described in Lemma 3.4.

Now for each $x^{(k_l+1)}$ defined by (2.15), by Algorithm A it is easy to show that for all $k_l>\bar{k}_l$

$$x_{i_t}^{(k_l+1)}=(1-\alpha\lambda_{k_l}p^{(k_l)})x_{i_t}^{(k_l)},\quad t=1,2,\cdots,l, \tag{3.27}$$

which implies that there is a sufficiently small $s_1>0$, such that for all $k_l>\bar{k}_l$

$$\|\bar{x}^{(k_l+1)}-\bar{x}^{(k_l)}\|_2<s_1,$$

where $\bar{x}^{(k_l+1)}$ and $\bar{x}^{(k_l)}$ are defined by (3.5). Since y is a continuous function of x, there exists a sufficiently small $s_2>0$, such that for all $k_l>\bar{k}_l$

$$\|r^{(k_l+1)}-r^{(k_l)}\|_2<s_2. \tag{3.28}$$

It follows from (3.27), (3.28), (3.12) and definition (2.13) that there is a sufficiently small $s_3>0$, such that for all $k_l>\bar{k}_l$

$$\|p^{(k_l+1)}-p^{(k_l)}\|_2<s_3.$$

Hence, it is straightforward to show that

$$\lim_{k\to\infty} p_{i_t}^{(k)}=\bar{p}_{i_t}=-\frac{1}{\sqrt{l}},\quad t=1, 2, \cdots, l$$

which proves the theorem.

The proof of Theorem 3.1 is as follows:

The necessary condition is a straightforward result of Theorem 3.2 and Lemma 3.4.

It is also easy to show that the sufficient condition is true. By the assumptions that $s>0$ is a sufficiently small constant, that $p^{(k)}$ is a unit vector, and that

$$p_i^{(k)}<s$$

there are l negative components in $p^{(k)}$ and the absolute value of every negative component is greater than s. Thus, it follows from formula (2.15) and Lemma 3.4 that the sequence $x^{(k)}$ is unbounded as k tends to infinity.

This completes the proof of the theorem.

The following corollary is a direct result of Theorem 3.2.

Corollary 3.6. With the same assumptions as in Theorem 3.1, the sequence $x^{(k)}$ generated by Algorithm A is unbounded if and only if

$$\lim_{k\to\infty}\|c_p^{(k)}\|_2\to\infty,$$

where $\|c_p^{(k)}\|_2$ is defined by (2.12).

§ 4. Convergence

We will show that under the assumption that the LP problem satisfies the condition F, the algorithm converges to an optimal solution if the minimum value of the LP problem is bounded, or generates a ray direction if the optimal value of the objective function is unbounded.

Theorem 4.1. *Suppose that the LP problem satisfies the condition F. If the minimum value of the LP problem is bounded, then the sequence $x^{(k)}$ generated by Algorithm A converges to an optimal solution.*

Theorem 4.2. *Suppose that the LP problem satisfies the condition F. If the minimum value of the LP problem is unbounded, then there exists a vector $p\leqslant 0$, such that the sequence $p^{(k)}$ defined by* (3.10) *converges to p.*

Now we introduce several lemmas.

Lemma 4.3. *Suppose that the LP problem satisfies the condition F. Then the sequence $x^{(k)}$ generated by Algorithm A is bounded if and only if*

$$\lim_{k\to\infty}\|c_p^{(k)}\|_2=0, \tag{4.1}$$

where $\|c_p^{(k)}\|_2$ is defined by (2.12).

Proof. In order to prove necessity, let

$$z^{(k)}=c^T x^{(k)}. \tag{4.2}$$

Then, by the formula (2.15),

$$z^{(k+1)}=c^Tx^{(k+1)}=z^{(k)}-\alpha\lambda_k\|c_p^{(k)}\|_2 \tag{4.3}$$

and hence $z^{(k)}$ corresponding to $x^{(k)}$ is a monotonically decreasing sequence with a lower bound. It follows from (4.3) and the boundedness of the sequence $x^{(k)}$ that there exists a $\bar{z}$ such that

$$\lim_{k\to\infty} z^{(k)}=\bar{z}. \tag{4.4}$$

From (4.3), it is easy to see that

$$z^{(k)}=z^{(0)}-\alpha\sum_{0}^{k-1}\lambda_l\|c_p^{(l)}\|_2. \tag{4.5}$$

It follows from (4.4) and (4.5) that

$$\lim_{k\to\infty}\|c_p^{(k)}\|_2=0.$$

It should be clear that sufficiency is also satisfied. Otherwise, by Corollary 3.6, if the sequence $x^{(k)}$ is unbounded, then

$$\lim_{k\to\infty}\|c_p^{(k)}\|_2=\infty$$

and this contradicts the assumption that

$$\lim_{k\to\infty}\|c_p^{(p)}\|_2=0.$$

Hence the lemma is proven.

Lemma 4.4. *With the same assumptions as in Lemma 4.3, if the minimum value of the LP problem is bounded, then the sequence $x^{(k)}$ generated by Algorithm A is bounded.*

Proof. If the feasible region of the LP problem is bounded, then the conclusion of the lemma is trivial.

Assume that the feasible region of the LP problem is unbounded, and let

$$V=[v\colon Av=0,\ v\geqslant 0]. \tag{4.6}$$

Then, V is nonempty, and for any $v\in V$, $v\neq 0$,

$$c^Tv\geqslant 0 \tag{4.7}$$

holds since the minimum value of the LP problem is bounded.

Now suppose that the conclusion of the lemma is not true. Then the sequence $x^{(k)}$ generated by Algorithm A is unbounded. It follows from Theorem 3.2 and Lemma 3.4 that there is a vector p such that

$$\lim_{k\to\infty}p^{(k)}=p. \tag{4.8}$$

Let

$$\bar{v}^{(k)}=-\frac{D_kp^{(k)}}{\|x^{(k)}\|_2}.$$

Then

$$A\bar{v}^{(k)}=-AD_k\frac{p^{(k)}}{\|x^{(k)}\|_2}=0$$

and for all k

$$c^T\bar{v}^{(k)}=-\frac{c^T D_{kp}^{(k)}}{\|x^{(k)}\|_2}=-\frac{\|c_p^{(k)}\|_2}{\|x^{(k)}\|_2}<0. \tag{4.9}$$

The formula can be written in the form

$$c^T\bar{v}^{(k)}=-c^T\bar{D}_{kp}(k), \tag{4.10}$$

where $\bar{D}_k$ is defined by (3.6). Thus, the sequence $\bar{x}^{(k)}$ is bounded. So there are a subsequence $\bar{x}^{(k_l)}$ and a vector $\bar{x}$ such that

$$\lim_{k_l\to\infty}\bar{x}^{(k_l)}=\bar{x}. \tag{4.11}$$

It follows from (4.9) and (4.11) that there are a vector $\bar{v}$ and a subsequence $\bar{v}^{(k_l)}$ such that

$$\lim_{k_l\to\infty}\bar{v}^{(k_l)}=\bar{v}$$

and

$$\bar{v}=-\bar{D}p\geqslant 0,$$
$$c^T\bar{v}=-c^T\bar{D}p\leqslant 0$$

which contradicts (4.7). So the conclusion of the lemma is true.

This completes the proof of the lemma.

Now we state the proof of Theorems 4.1 and 4.2.

The proof of Theorem 4.1 follows from the assumptions and results of Lemma 4.4, namely that the sequence $x^{(k)}$ generated by Algorithm A is bounded. Hence, there are a feasible solution $\bar{x}$ and a subsequence $x^{(k_l)}$ such that

$$\lim_{k_l\to\infty}x^{(k_l)}=\bar{x} \tag{4.12}$$

which implies that there are a vector $\bar{r}$ and a subsequence $r^{(k_l)}$ with respect to $x^{(k_l)}$ such that

$$\lim_{k_l\to\infty}r^{(k_l)}=\bar{r}. \tag{4.13}$$

Thus, by Lemma 4.3 and (3.14),

$$\lim_{k\to\infty}\|c_p^{(k)}\|_2=0 \tag{4.14}$$

and

$$\bar{x}_i\bar{r}_i=0. \tag{4.15}$$

Let

$$J=[i:\ \bar{x}_i=0,\ i\in N]. \tag{4.16}$$

Then there exists an integer k_0 such that for all $k_l>k_0$

$$(c_p^{(k_l)})_i>0,\quad i\in J. \tag{4.17}$$

Otherwise, given a sufficiently small $\delta>0$, there is an integer $\bar{k}>0$ such that for all $k_l>\bar{k}$

$$\|x^{(k_l)}-\bar{x}\|_2<\delta \tag{4.18}$$

and

$$(c_p^{(k_l)})_i\leqslant 0,\quad i\in J. \tag{4.19}$$

Hence, for all $k_l>\bar{k}$

$$p_i^{(k_l)}<0,\quad i\in J. \tag{4.20}$$

It follows from (2.15) and (4.27) that for all $k>\bar{k}$

$$x_i^{(k_1)} \geqslant x_i^{(\bar{k})} > 0, \quad i \in J$$

which implies that the subscript i is not in J. This contradicts (4.23), so that (4.24) holds.

By (4.22) and (4.24), it follows that if $\bar{x}_i>0$, then $\bar{r}_i=0$; or that if $\bar{x}_i=0$, then $\bar{r}_i>0$. Thus, $\bar{x}$ is an optimal solution.

From (2.15) and since $c_p^{(k)}$ is a continuous function of x, $\bar{x}$ is the only accumulation point of the sequence $x^{(k)}$. This completes the proof of the theorem.

Corollary 4.5. With the same assumptions as in Theorem 4.1, if the dual problem is nondegenerate, then the sequence $x^{(k)}$ generated by Algorithm A converges to a unique optimal solution.

The proof of Theorem 4.2 follows from the assumption that the dual cf the LP problem is infeasible since the minimum value of the LP problem is unbounded.

Now we assert that the sequence $x^{(k)}$ generated by Algorithm A is unbounded. If this is not true, then $x^{(k)}$ is a bounded sequence

$$\lim_{k\to\infty} x^{(k)} = \bar{x}$$

which implies that the dual problem of the LP problem is feasible. This contradicts the fact that the dual problem is infeasible. Hence, the assertion is true. It follows from Theorems 3.1 and 3.2 that there is a vector $p\leqslant 0$ such that the sequence $p^{(k)}$ converges to p. This completes the proof of the theorem.

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