THE MULTIGRID METHOD WITH CORRECTION PROCEDURE*

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§ 1. Introduction

The general form of the MGE method for solving the boundary value problem of elliptic partial differential equations suggested in [1] suits both the finite difference scheme and the finite element scheme resulting from elliptic differential equations. In order to decrease the number of multigrid iterations on each level, Cai et al. suggested a revised MGE method by using auxiliary grids in [2—3]. For the special equation $-\Delta u=f(x, u)$, we combine the correction procedure with multigrid method so that the interpolation level number is 1. The computational work needed by this method is less than any revised MGE method[2-3].

§ 2. The Multigrid Method with Correction Procedure

For simplicity, we consider the model problem

$$\begin{cases} \Delta u=f(x, u), & \text{in } \Omega, \\ u=g, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a one-, two- or three-dimensional domain, and $\partial\Omega$ is the boundary of the domain Ω. Suppose that Ω consists of some squares in the two-dimensional case or of some cubes in the three-dimensional case, and that the solution u is smooth enough and

$$f_1(u)=f'_u(x, u)\geqslant 0. \tag{2}$$

$\Omega_k\subset\Omega$ $(k=0, 1, \cdots, 1)$ are uniform discretized grids of the domain, whose width is h_k, and

$$\Omega_k\subset\Omega_{k+1}, \quad h_k=\xi h_{k+1}.$$

The ratio of step size ξ is usually 2.

Let Δ_k be the 5-point approximation of the Laplace operator Δ in the two-dimensional case and the 7-point approximation in the three-dimensional case on the grid Ω_k as usual.

Let Δ_k^* be the 5-point approximation defined by

$$\Delta_k^* u(x_1, x_2)=\left(\sum u(x_1\pm h_k, x_2\pm h_k)-4u(x_1, x_2)\right)/2h_k^2, \tag{3}$$

$$\sum u(x_1\pm h_k, x_2\pm h_k)=u(x_1+h_k, x_2-h_k)+u(x_1+h_k, x_2+h_k) +u(x_1-h_k, x_2-h_k)+u(x_1-h_k, x_2+h_k)$$

in the two-dimensional case and the 9-point approximation defined by

* Received January 2, 1986.

$$\Delta_k^* u(x_1, x_2, x_3) = (\sum u(x_1 \pm h_k, x_2 \pm h_k, x_3 \pm h_k) - 8u(x_1, x_2, x_3))/4h_k^2 \tag{4}$$

in the three-dimensional case.

Consider the finite difference solution u_k defined by

$$\begin{cases} \Delta_k u_k = f(x, u_k), & x \in \Omega_k, \\ u_k = g - h_k^2 f(x, g)/12, & x \in \partial\Omega_k \end{cases} \tag{5}$$

and a correction solution φ_k defined by the linearized finite difference equation

$$\begin{cases} (\Delta_k^* - f_1(u_k))\varphi_k = f(x, u_k) - f_1(u_k)u_k + h_k^2 f_1(u_k) f(x, u_k)/4, & x \in \Omega_k, \\ \varphi_k = g - h_k^2 f(x, g)/12, & x \in \partial\Omega_k. \end{cases} \tag{6}$$

The finite difference equations (5) and (6) can be denoted by the abstract equations

$$L_k u_k = F_k, \tag{7}$$

$$L_k^* \varphi_k = F_k^*, \tag{8}$$

where L_k, L_k^* are discretized matrices and u_k, φ_k, F_k and F_k^* are grid functions.

In [4], Lin and Lu have proved

$$\frac{2}{3} u_k + \frac{1}{3} \varphi_k + \frac{1}{12} h_k^2 f(x, u_k) = u + O(h_k^4), \quad x \in \overline{\Omega}_k \tag{9}$$

under reasonable conditions. It is obvious that if the following condition

$$\bar{u}_k = u_k + O(h_k^4), \quad \bar{\varphi}_k = \varphi_k + O(h_k^4), \text{ in } \overline{\Omega}_k \tag{10}$$

is valid, then one has also

$$\frac{2}{3} \bar{u}_k + \frac{1}{3} \bar{\varphi}_k + \frac{1}{12} h_k^2 f(x, \bar{u}_k) = u + O(h_k^4), \quad \text{in } \overline{\Omega}_k. \tag{11}$$

In order to avoid solving equations (7) and (8) directly, we wish to find the approximations of u_k and φ_k indirectly by using the solutions u_{k-1} and φ_{k-1} at the level $k-1$. We can prove the following proposition.

Proposition 1. Let $h_{k-1} = 2h_k$ $(k=1, \cdots, l)$; then

$$\frac{3}{4} u_{k-1} + \frac{1}{4} \varphi_{k-1} + \frac{1}{16} h_{k-1}^2 f(x, u_{k-1}) = u_k + O(h_{k-1}^4), \quad \text{on } \overline{\Omega}_k. \tag{12}$$

Proof. Let u_k^* be the solution of the equation

$$\begin{cases} (\Delta_k^* - f_1(u_k))u_k^* = f(x, u_k) - f_1(u_k)u_k, & x \in \overline{\Omega}_k, \\ u_k^* = g - \frac{1}{12} h_k^2 f(x, g), & x \in \partial\Omega_k. \end{cases}$$

Then one may obtain

$$u_{k-1} - u + \frac{1}{12} h_{k-1}^2 v_1 = O(h_{k-1}^4), \quad \text{on } \overline{\Omega}_{k-1}, \tag{13}$$

$$u_{k-1}^* - u + \frac{1}{12} h_{k-1}^2 v_2 = O(h_{k-1}^4), \quad \text{on } \overline{\Omega}_{k-1} \tag{14}$$

from the proof of Proposition 1 in [4]. Hence

$$\frac{3}{4} u_{k-1} + \frac{1}{4} u_{k-1}^* - u_k + \frac{1}{16} h_{k-1}^2 \left(\frac{2}{3} v_1 + \frac{1}{3} v_2\right) = O(h_{k-1}^4). \tag{15}$$

Set

$$w = \frac{2}{3} v_1 + \frac{1}{3} v_2.$$

Then we have

$$w=f(x,\ u_{k-1})+z_{k-1}+O(h_{k-1}^2), \tag{16}$$

where z_{k-1} is the solution of the following equation

$$\begin{cases}(\Delta_{k-1}^*-f_1(u_{k-1}))z_{k-1}=f_1(u_{k-1})f(x,\ u_{k-1}), & \text{on } \Omega_{k-1},\\ z_{k-1}=0, & \text{on } \partial\Omega_{k-1}.\end{cases} \tag{17}$$

Then (12) results from (16) and

$$\varphi_{k-1}=u_{k-1}^*+\frac{1}{4}h_{k-1}^2z_{k-1}.$$

Proposition 2. Let $\xi=2$; then

$$\frac{1}{2}u_{k-1}+\frac{1}{2}\varphi_{k-1}+\frac{1}{16}h_{k-1}^2(2f(x,\ u_{k-1})-f(x,\ u_k))=\varphi_k+O(h_{k-1}^4),\ \text{on } \bar{\Omega}_{k-1}. \tag{18}$$

Proof. One may obtain

$$\frac{1}{2}u_{k-1}+\frac{1}{2}u_{k-1}^*-u_k^*+\frac{1}{16}h_{k-1}^2w=O(h_{k-1}^4),\quad \text{on } \Omega_{k-1} \tag{19}$$

from (13) and (14). Since

$$w=f(x,\ u_{k-1})+z_{k-1}+O(h_{k-1}^2)=f(x,\ u_k)+z_k+O(h_{k-1}^2)$$

hence

$$w=2f(x,\ u_{k-1})+2z_{k-1}-f(x,\ u_k)-z_k+O(h_{k-1}^2). \tag{20}$$

So (18) results from (20) and

$$\varphi_{k-1}=u_{k-1}^*+\frac{1}{4}\bar{z}_{k-1}h_{k-1}^2,\quad \varphi_k=u_k^*+h_{k-1}^2z_k/16. \tag{21}$$

It is proved that the approximation of high accuracy of u_k and φ_k results from u_{k-1} and φ_{k-1} on the grid $\bar{\Omega}_k$ by using the interpolation operator. Let

$$\mathrm{INT}(\cdot,\ k):\mathscr{U}_{k-1}\to\mathscr{U}_k \tag{22}$$

be a suitable interpolation operator. In general, assume that INT is an affine operator:

$$\mathrm{INT}(\cdot,\ k)=\Pi_{k-1}^k\cdot+w_k, \tag{23}$$

where $w_k\in\mathscr{U}_k$ is a fixed grid function, and

$$\Pi_{k-1}^k:\mathscr{U}_{k-1}\to\mathscr{U}_k \tag{24}$$

is a linear operator. The interpolation error is assumed to be of order p:

$$\|\mathrm{INT}(u_k,\ k)-u_k\|\leqslant Kh_k^p,\quad k=1,\ \cdots,\ l, \tag{25}$$

where K and p are independent of k.

By using the above results, we may construct the following algorithm:

Algorithm 1.

compute $\bar{u}_0=u_0,\ \bar{\varphi}_0=\varphi_0$

for $k=1$ (1) l *do*

begin

$$\tilde{u}_k:=\mathrm{INT}\left(\frac{3}{4}\bar{u}_{k-1}+\frac{1}{4}\bar{\varphi}_{k-1}+\frac{1}{16}h_{k-1}^2f(x,\ \bar{u}_{k-1}),\ k\right)$$

$$\bar{u}_k:=\mathrm{MGI}^r(\tilde{u}_k,\ k,\ L_k,\ F_k)$$

$$\tilde{\varphi}_k:=\mathrm{INT}\left(\frac{1}{2}\bar{u}_{k-1}+\frac{1}{2}\bar{\varphi}_{k-1}+\frac{1}{16}h_{k-1}^2(2f(x,\ \bar{u}_{k-1})-f(x,\ u_k)),\ k\right)$$

$$\bar{\varphi}_k := \mathrm{MGI}^r(\tilde{\varphi}_k, k, L_k^*, F_k^*)$$

end

$$\bar{u}_1 := \frac{2}{3}\bar{u}_1 + \frac{1}{3}\bar{\varphi}_1 + \frac{1}{12}h_1^2 f(x, \bar{u}_1)$$

end;

In the above algorithm, operators

$$\mathrm{MGI}^r(\cdot, k\ L_k, F_k): \mathscr{U}_k \to \mathscr{U}_k,$$
$$\mathrm{MGI}^r(\cdot, k, L_k^*, F_k^*): \mathscr{U}_k \to \mathscr{U}_k$$

are used for a suitable procedure consisting of r iteration steps of a suitable iterative multigrid method for (7) and (8) (using grids $\Omega_0, \Omega_1, \cdots, \Omega_k$; cf. [5]). The iteration operator of the multigrid method is denoted by M_k. Under reasonable assumptions, one has

$$\|M_k\| \leqslant \rho < 1, \quad k = 0, 1, \cdots, l \tag{26}$$

from [5]. Then for Algorithm 1 we have

Theorem 1. *Assume $p \geqslant 4$, (9), (12), (18), (25) and (26). Let operator Π_{k-1}^k be bounded by C, i.e.*

$$\|\Pi_{k-1}^k\| \leqslant C, \quad k = 0, 1, \cdots, l \tag{27}$$

and $C\rho^r < 1/16$. Then we obtain

$$\bar{u}_1 = u + O(h_1^4). \tag{28}$$

Proof. Set

$$\alpha_k = \|\bar{u}_k - u_k\| / h_k^4, \quad \beta_k = \|\bar{\varphi}_k - \varphi_k\| / h_k^4, \ k = 0, \quad 1, \cdots, l.$$

From the definition of Algorithm 1 it is easy to see that

$$\alpha_k \leqslant A + C\rho^r(12\alpha_{k-1} + 4\beta_{k-1}), \quad \beta_k \leqslant A + C\rho^r(8\alpha_{k-1} + 8\beta_{k-1}),$$

where $A = (Kh_k^{p-4} + \bar{C}O(1))\rho^r$ ($\bar{C} = C + M$, $\|f_1(u)\| \leqslant M$). If $p \geqslant 4$ and $C\rho^r < 1/16$, then one may obtain

$$\|\bar{u}_k - u_k\| = O(h_k^4), \quad \|\bar{\varphi}_k - \varphi_k\| = O(h_k^4).$$

Hence (28) results from (9).

In the case of $\xi \neq 2$, we can obtain similar results as in the case of $\xi = 2$.

Proposition 1′. Let $h_{k-1} = \xi h_k$ ($k = 1, \cdots, l$); then

$$\frac{2\xi^2+1}{3\xi^2} u_{k-1} + \frac{\xi^2-1}{3\xi^2}\varphi_{k-1} + \frac{\xi^2-1}{12\xi^2} h_{k-1}^2 f(x, u_{k-1}) = u_k + O(u_k^4), \quad x \in \bar{\Omega}_{k-1}. \tag{12′}$$

Proposition 2′. Let $h_{k-1} = \xi h_k$ ($k = 1, \cdots, l$); then

$$(3\xi^2)^{-1}(2(\xi^2-1)u_{k-1} + (\xi^2+2)\varphi_{k-1} + h_{k-1}^2((\xi^2+2)f(x, u_{k-1})$$
$$-3f(x, u_k))/4) = \varphi_k + O(h_k^4), \quad \text{on } \bar{\Omega}_{k-1}. \tag{18′}$$

In Algorithm 1, $\tilde{u}_k$ and $\tilde{\varphi}_k$ are computed by means of the results of Propositions 1′ and 2′. In this case, if the condition $C\rho^r < 1/16$ in Theorem 1 is changed into $C\rho^r < \xi^{-4}$, then we can obtain the same result as Theorem 1.

Note 1. For $k = 1, \cdots, l$, we may change (5) into the linearization

$$\begin{cases} L_k u_k \equiv \Delta_k u_k - f_1(\tilde{u}_k) u_k = f(x, \tilde{u}_k) - f_1(\tilde{u}_k)\tilde{u}_k, & \text{on } \Omega_k, \\ u_k = g - \dfrac{1}{12} h_k^2 f(x, g), & \text{on } \partial\Omega_k, \end{cases} \tag{5′}$$

where $\tilde{u}_k$ $(k=1, \cdots, l)$ are the solutions of the difference equation (5) with second order accuracy. We can construct the same algorithm and obtain the same results as above.

Note 2. In a similar way, we can discuss the following nonlinear problem

$$\begin{cases} Lu \equiv \Delta u - f(x, u, u_x, u_y) = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega. \end{cases} \tag{29}$$

§ 3. The Multigrid Method with Another Correction Approach

We consider again the model problem (1), assuming that

$$u_k = u + h_k^2 v_1 + h_k^4 v_2 + O(h_k^6). \tag{30}$$

Let $\delta_{x_i}^k$ $(i=1, 2, 3)$ be the two-point approximation defined by

$$\delta_{x_i}^k u = (u(x_1, \cdots, x_i + h_k, \cdots, x_3) - u(x_1, \cdots, x_i - h_k, \cdots, x_3))/2h_k.$$

Then we have

$$\delta_{x_i x_i}^k \delta_{x_j x_j}^k u_k = u_{x_i x_i x_j x_j} + O(h_k^2), \quad i \neq j \tag{31}$$

from (30). Let $\bar{u}_k$ be the solution of the following linearization

$$\begin{cases} (\Delta_k - f_1(u_k))\bar{u}_k = f(x, u_k) - f_1(u_k)u_k + h_k^2 f_1(u_k) f(x, u_k)/12 \\ \qquad\qquad - h_k^2 \sum_{i<j} \delta_{x_i x_i}^k \delta_{x_j x_j}^k u_k/6, \quad \text{on } \Omega_k, \\ \bar{u}_k = g - h_k^2 f(x, g)/12, \quad \text{on } \partial\Omega_k. \end{cases} \tag{32}$$

Since in [4] it has been proved that

$$\bar{u}_k + h_k^2 f(x, u_k)/12 = u + O(h_k^4), \tag{33}$$

it is easy to show.

Proposition 3. Let $h_{k-1} = \xi h_k$ $(k=1, \cdots, l)$; then

$$\bar{u}_k + (\xi^2 - 1) h_k^2 f(x, u_k)/12\xi^2 = \bar{u}_{k+1} + O(h_k^4), \quad \text{on } \bar{\Omega}_k. \tag{34}$$

As in [4], one may prove

$$\bar{u}_k = u_k + O(h_k^2), \quad \text{on } \bar{\Omega}_k. \tag{35}$$

From (35) it is easy to see.

Proposition 4. Let $h_{k-1} = \xi h_k$ $(k=1, \cdots, l)$; then

$$\bar{u}_k + \frac{1}{12} h_k^2 f(x, \bar{u}_k) = u + O(h_k^4), \quad \text{on } \bar{\Omega}_k, \tag{36}$$

$$\bar{u}_k + \frac{\xi^2 - 1}{12\xi^2} h_k^2 f(x, \bar{u}_k) = \bar{u}_{k+1} + O(h_k^4), \quad \text{on } \bar{\Omega}_k. \tag{37}$$

The algorithm of the multigrid method with another correction approach reads as follows. $\bar{L}_k \bar{u}_k = \bar{F}_k$ is an abbreviation of (32).

Algorithm 2.

compute $\tilde{u}_0 = \bar{u}_0$
for $k=1$ (1) l *do*
begin

$u_k^0 = \mathrm{INT}(\tilde{u}_{k-1} + h_{k-1}^2 (\xi^2 - 1) f(x, \tilde{u}_{k-1})/12\xi^2, k)$
$\tilde{u}_k = \mathrm{MGI}^r(u_k^0, k, \bar{L}_k, \bar{F}_k)$;

end

$$\bar{u}_1=\tilde{u}_1+h_1^2 f(x, \tilde{u}_1)/12$$

end;

Theorem 2. *Assume* (36), (37), (25), (26), (27) *and*

$$p\geqslant 4, \quad C\rho^r<\xi^{-4}.$$

Then we have

$$\bar{u}_1=u+O(h_1^4). \tag{38}$$

Proof. By the definition of Algorithm 2, we know that

$$\tilde{u}_k-\bar{u}_k=M^r(u_k^0-\bar{u}_k).$$

According to Proposition 4, we have

$$\begin{aligned} u_k^0-\bar{u}_k &= \Pi_{k-1}^k\left(\tilde{u}_{k-1}+\frac{\xi^2-1}{12\xi^2}h_{k-1}^2 f(x, \tilde{u}_{k-1})\right)+w_k-\bar{u}_k \\ &= \Pi_{k-1}^k\left(\bar{u}_{k-1}+\frac{\xi^2-1}{12\xi^2}h_{k-1}^2 f(x, \bar{u}_{k-1})+(\tilde{u}_{k-1}-\bar{u}_{k-1})\right. \\ &\quad \left.+\frac{\xi^2-1}{12\xi^2}h_{k-1}^2(f(x, \tilde{u}_{k-1})-f(x, \bar{u}_{k-1}))\right)+w_k-\bar{u}_k \\ &= \Pi_{k-1}^k(\bar{u}_k+O(h_k^2)+(\tilde{u}_{k-1}-\bar{u}_{k-1}))+w_k-\bar{u}_k. \end{aligned}$$

Set $\alpha_k=\|\tilde{u}_k-\bar{u}_k\|/h_k^2$. Then

$$\alpha_k\leqslant\rho^r(A+C\xi^2\alpha_{k-1}), \quad A=Kh_k^{p-2}+CO(1).$$

Provided $C\rho^r<\xi^{-2}$, we have

$$\tilde{u}_k-\bar{u}_k=O(h_k^2), \quad k=0, 1, \cdots, l.$$

Similarly, provided $C\rho^r<\xi^{-4}$, it is easy to obtain (38).

Note 3. For the general problem

$$\begin{cases} Lu\equiv\Delta u-bu_x-cu_y-du=f, & \text{in } \Omega, \\ u=g, & \text{on } \partial\Omega, \end{cases} \tag{39}$$

where $d\geqslant 0$, the discussion is the same.

The author wishes to express his appreciation to Professor Lin Qun and Professor Wang Neng-chao for their help in this work.

References

[1] Cai Zhi-qiang, Wang Neng-chao, Kang Li-shan, MGE method for solving BVP, *Math. Numer. Sinica*, 8: 1 (1986), 82—89.

[2] Cai Zhi-qiang, Wang Neng-chao, The revised MGE method, *Applied Math. and Computation*, to be submitted.

[3] Cai Zhi-qiang, The splitting extrapolation and MGM for solving BVP of nonlinear equations, *J. Engin. Math.*, to be submitted.

[4] Lin Qun, Lu Tao, Correction and splitting procedures for solving PDEs, *J. Computational Math.*, **2**: 1 (1984), 56—69.

[5] W. Hackbusch, U. Trottenberg (ed.), Multigrid method, in Lecture Notes in Math. **960**, Springer, 1982.