ON THE MINIMUM PROPERTY OF THE PSEUDO $\varkappa$-CONDITION NUMBER FOR A LINEAR OPERATOR*

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Abstract

It is well known that the $\varkappa$-condition number of a linear operator is a measure of ill condition with respect to its generalized inverses and a relative error bound with respect to the generalized inverses of operator T with a small perturbation operator E, namely,

$$\frac{\|(T+E)^+-T^+\|}{\|T^+\|}\leqslant\frac{\varkappa(T)\frac{\|E\|}{\|T\|}}{1-\varkappa(T)\frac{\|E\|}{\|T\|}},$$

where $\varkappa(T)=\|T\|\cdot\|T^+\|$. The problem is whether there exists a positive number $\mu(T)$ independent of E but dependent on T such that the above relative error bound holds and $\mu(T)<\varkappa(T)$.

In this paper, an answer is given to this problem. The main result is

Theorem. *Let X, Y be two Banach spaces, $T, E\in B[X, Y]$ and $\|E\|\cdot\|T^+\|<1$. Suppose*

$$\frac{\|(T+E)^+-T^+\|}{\|T^+\|}\leqslant\mu(T)\frac{\|E\|}{\|T\|}.$$

Then $\varkappa(T)\leqslant\mu(T)$, where $\mu(T)$ is a positive number independent of E but dependent on T and $(I_Y+ET^+)^{-1}(T+E)$ maps $\mathscr{N}(T)$ into $\mathscr{R}(T)$. This theorem shows that $\varkappa(T)$ is minimum in the above sence.

§1. Introduction

In [1], the author showed the minimum property of ω-condition number for a linear operator, and extended the results of [2]. The results of [1] are only related to the relative error bound of an inverse linear operator with a small perturbation operator, or the relative error bound of the a regular solution of linear equations with small perturbation.

In this paper, we will discuss the relative error bound of a generalized inverse of a linear operator from a Banach space to another Banach space and a generalized solution of liear equations whose operator has a small perturbation. In addition, we will show the minimum property of the pseudo $\varkappa$-condition number. The results are very extensive and the results of [1] and [2] are the obvious corollaries.

§2. Generalized Inverses of a Linear Operator in a Banach Space

In general, the letters X, Y denote the Banach space, $B[X, Y]$ is the Banach space consisting of all bounded linear operators from X into Y, $\mathscr{D}(T)$ and $\mathscr{R}(T)$ denote the domain and range of T respectively, and $\mathscr{N}(T)$ denotes the null of T.

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We assume that the closed subspace $\mathscr{N}(T)$ of X has a topological complement $\mathscr{N}(T)^c$ and the closed subspace $\overline{\mathscr{R}(T)}$ of Y has a topological complement $\overline{\mathscr{R}(T)}^c$, namely

$$X=\mathscr{N}(T)\oplus\mathscr{N}(T)^c;\quad Y=\overline{\mathscr{R}(T)}\oplus\overline{\mathscr{R}(T)}^c.$$

In this case, $\mathscr{N}(T)$ and $\overline{\mathscr{R}(T)}$ are closed, however a closed subspace does not necessarily have a topological complement. A subspace $\mathscr{N}(T)$ $(\overline{\mathscr{R}(T)})$ has a topological complement if and only if there exists a projector $P(Q)$ of $X(Y)$ onto $\mathscr{N}(T)$ $(\overline{\mathscr{R}(T)})$, i.e., $PX=\mathscr{N}(T)$ $(QY=\overline{\mathscr{R}(T)})$, see [7]. Nashed pointed out that if the decompositions

$$X=\mathscr{N}(T)\oplus\mathscr{N}(T)^c;\ Y=\overline{\mathscr{R}(T)}\oplus\overline{\mathscr{R}(T)}^c$$

exist, then there exists uniquely the generalized inverse $T^+\equiv T^+_{P,Q}$ ($T^+_{P,Q}$ implies that the operator T^+ depends on the projectors P and Q) such that

$$\begin{cases}\mathscr{D}(T^+)=\mathscr{R}(T)\oplus\overline{\mathscr{R}(T)}^c;\ \mathscr{N}(T^+)=\overline{\mathscr{R}(T)}^c,\\ \mathscr{R}(T^+)=\mathscr{N}(T)^c;\ TT^+T=T;\ T^+TT^+=T^+ \text{ on } \mathscr{D}(T^+),\\ T^+T=I-P;\ TT^+=Q|_{\mathscr{D}(T^+)},\end{cases}\tag{1}$$

where $Q|_{\mathscr{D}(T^+)}$ is the restriction of Q on $\mathscr{D}(T^+)$. T^+ is bounded if and only if $\mathscr{R}(T)$ is closed in Y. In this paper, we consider the case that $\mathscr{R}(T)$ is closed; then we have obviously

$$\begin{cases}X=\mathscr{N}(T)\oplus\mathscr{N}(T)^c;\ Y=\mathscr{R}(T)\oplus\mathscr{R}(T)^c,\\ \mathscr{D}(T^+)=Y;\ \mathscr{N}(T^+)=\mathscr{R}(T)^c,\\ \mathscr{R}(T^+)=\mathscr{N}(T)^c,\end{cases}\tag{2}$$

$$\begin{cases}TT^+T=T;\ T^+TT^+=T^+,\\ T^+T=P_{\mathscr{N}(T)^c};\ TT^+=P_{\mathscr{R}(T)}.\end{cases}\tag{3}$$

From (3) we can obtain easily

$$\begin{cases}T^+P_{\mathscr{R}(T)}=T^+;\ P_{\mathscr{N}(T)^c}T^+=T^+,\\ TP_{\mathscr{N}(T)^c}=T;\ P_{\mathscr{R}(T)}T=T.\end{cases}\tag{4}$$

In the following section, we consider the case that the perturbation $S=T+E$ of T has a generalized inverse and estimate the error bound between S^+ and T^+. We suppose that $y_0\in Y$, $y_0=y_1+y_2$ and $\|y_0\|=1$ imply $\|y_1\|\leqslant1$.

§ 3. The Minimum Property of the Pseudo $\varkappa$-Condition Number

Lemma 1. *Let $T\in B[X, Y]$ and suppose $X=\mathscr{N}(T)\oplus\mathscr{N}(T)^c$ and $Y=\mathscr{R}(T)\oplus\mathscr{R}(T)^c$. Let $T^+_{\mathscr{N}(T)^c,\mathscr{R}(T)^c}$ be the generalized inverses of T with respect to these decompositions. Let $E\in B[X, Y]$ and $S=T+E$. Suppose*

$$\|ET^+\|<1\tag{5}$$

and

$$(I_y+ET^+)^{-1}S \text{ maps } \mathscr{N}(T) \text{ into } \mathscr{R}(T).\tag{6}$$

Then

$$X=\mathscr{N}(S)\oplus\mathscr{R}(T^+);\quad Y=\mathscr{R}(S)\oplus\mathscr{N}(T^+)$$

and

$$S^+ = S^+_{\mathscr{R}(T^+),\mathscr{N}(T^+)}$$

exists. Moreover,

$$S^+ = T^+(I_y + ET^+)^{-1} = (I_x + T^+E)^{-1}T^+. \tag{7}$$

If in addition $\|E\| \cdot \|T^+\| < 1$, *then*

$$\frac{\|(T+E)^+ - T^+\|}{\|T^+\|} \leqslant \frac{\varkappa(T)\dfrac{\|E\|}{\|T\|}}{1-\varkappa(T)\dfrac{\|E\|}{\|T\|}}, \tag{8}$$

where

$$\varkappa(T) \equiv \|T\| \cdot \|T^+\|$$

is the pseudo condition number of T.

Remark. Suppose (5) holds. Then (6) also holds if either $\mathscr{N}(S) \supseteq \mathscr{N}(T)$ (equivalently $\mathscr{N}(E) \supseteq \mathscr{N}(T)$) or $\mathscr{R}(S) \subseteq \mathscr{R}(T)$ (equivalently $\mathscr{R}(E) \subseteq \mathscr{R}(T)$). In fact, if $\mathscr{N}(S) \supseteq \mathscr{N}(T)$, then $(I_y + ET^+)^{-1}S$ maps $\mathscr{N}(T)$ into $\{0\} \subseteq \mathscr{R}(T)$. Similarly, if $\mathscr{R}(E) \subseteq \mathscr{R}(T)$, then on $\mathscr{N}(T)$, $(I_y + ET^+)^{-1}S = (I_y + ET^+)^{-1}E = E(I_x + T^+E)^{-1}$ and its range is contained in $\mathscr{R}(E)$, and hence in $\mathscr{R}(T)$.

Lemma 1 can be found in [7].

Lemma 2. *Let* X, Y *be Banach spaces and* $x_0 \in X$, $y_0 \in Y$, $\|x_0\| = \|y_0\| = 1$ *be any two points. Then there is a bounded linear operator* $H \in B[X, Y]$ *such that* $Hx_0 = y_0$ *and* $\|H\| = 1$.

Proof. By the Hahn-Banach theorem, there is a bounded linear functional $f \in X'$ such that

$$f(x_0) = \|x_0\| \text{ and } \|f\| = 1.$$

We define the operator H: $X \to Z = \text{span } \{y_0\}$

$$Hx = f(x)y_0.$$

Thus

$$Hx_0 = f(x_0)y_0 = \|x_0\| \cdot y_0 = y_0.$$

The operator H is linear since f is linear. In fact,

$$\begin{aligned} H(\alpha x + \beta y) &= f(\alpha x + \beta y)y_0 = (\alpha f(x) + \beta f(y))y_0 \\ &= \alpha f(x)y_0 + \beta f(y)y_0 = \alpha Hx + \beta Hy. \end{aligned}$$

Moreover,

$$\|H\| = \sup_{\|x\|=1} \|Hx\| = \sup_{\|x\|=1} |f(x)| \cdot \|y_0\| = \|f\| = 1$$

and this completes the proof.

Lemma 3. *Let* X, Y *be Banach spaces and* $\tilde{T}$, $\tilde{S} \in B[Y, X]$. *Suppose* $\varepsilon > 0$ *is a given sufficiently small positive number. Then there exists a bounded linear operator* $H \in B[X, Y]$ *such that* $\|H\| = 1$ *and*

$$\|\tilde{S}H\tilde{T}\| \geqslant (\|\tilde{T}\| - \varepsilon')(\|\tilde{S}\| - \varepsilon),$$

where $0 < \varepsilon' \leqslant \varepsilon$.

Proof. For a sufficiently small positive number $\varepsilon > 0$, there are y_0, $z_0 \in Y$ such that $\|x_0\| = \|y_0\| = 1$ and

$$\|\tilde{S}y_0\| \geqslant \|\tilde{S}\| - \varepsilon,$$

$$\|\tilde{T}z_0\| \geqslant \|\tilde{T}\| - \varepsilon$$

by the definition of the operator norm. We can choose a positive number $0<\varepsilon'\leqslant\varepsilon$ such that

$$\|\tilde{T}z_0\| = \|\tilde{T}\| - \varepsilon'.$$

Let

$$x_0 = \frac{\tilde{T}z_0}{\|\tilde{T}\| - \varepsilon'}.$$

Then $\|x_0\| = 1$ and $\tilde{T}z_0 = (\|\tilde{T}\| - \varepsilon')x_0$. For the x_0 and y_0, there is an operator $H \in B[X, Y]$ such that $Hx_0 = y_0$ and $\|H\| = 1$ by Lemma 2. Moreover, we have

$$\begin{aligned}\|\tilde{S}H\tilde{T}\| &\geqslant \|\tilde{S}H\tilde{T}z_0\| = (\|\tilde{T}\| - \varepsilon')\|\tilde{S}Hx_0\| \\ &= (\|\tilde{T}\| - \varepsilon')\cdot\|\tilde{S}y_0\| \geqslant (\|\tilde{T}\| - \varepsilon')(\|\tilde{S}\| - \varepsilon)\end{aligned}$$

and this completes the proof.

Theorem 1. *Let T and E be as in Lemma* 1 *and suppose* $\|E\|\cdot\|T^+\|<1$. *If there exists a positive number $\mu(T)$ dependent on T but independent of E such that*

$$\frac{\|(T+E)^+ - T^+\|}{\|T^+\|} \leqslant \frac{\mu(T)\dfrac{\|E\|}{\|T\|}}{1-\mu(T)\dfrac{\|E\|}{\|T\|}} \tag{9}$$

or

$$\frac{\|(T+E)^+ - T^+\|}{\|T^+\|} \leqslant \mu(T)\frac{\|E\|}{\|T\|}, \tag{10}$$

then

$$\varkappa(T) \leqslant \mu(T). \tag{11}$$

Proof. We only need to prove the conclusion when (9) is satisfied. The remaining part is similar.

By (5) and the Banach lemma, $(I_y + ET^+)^{-1}$ exists. Using the Neumann series it follows that

$$(I_y + ET^+)^{-1} = \sum_{i=0}^{\infty} P^i,$$

where $P = -ET^+$. Moreover, using (7), we have

$$(T+E)^+ = T^+(I_y + ET^+)^{-1} = (I_x + T^+E)^{-1}T^+.$$

Consequently,

$$\begin{aligned}(T+E)^+ - T^+ &= ((I_x + T^+E)^{-1}T^+ - T^+) \\ &= ((I_x + T^+E)^{-1} - I_x)T^+, \\ \|(T+E)^+ - T^+\| &= \|((I_x + T^+E)^{-1} - I_x)T^+\| \\ &= \left\|\sum_{i=1}^{\infty} \tilde{P}^i T^+\right\| \\ &\geqslant \|T^+ET^+\| - \|T^+\|^3\|E\|^2 \sum_{i=0}^{\infty} \|T^+E\|^i,\end{aligned}$$

namely

$$\|T^+ET^+\| \leqslant \|(T+E)^+ - T^+\| + \|T^+\|^3\|E\|^2 \cdot \sum_{i=0}^{\infty} \|T^+E\|^i. \tag{12}$$

For a sufficiently small positive number $\varepsilon>0$, there exists a $y_0\in Y$ such that

$$\|T^+y_0\|\geq\|T^+\|-\varepsilon>0.$$

Hence $T^+y_0\neq 0$ and using the decompositions (2), we obtain

$$y_0=y_1+y_2;\ y_1\in\mathscr{R}(T),\ y_2\in\mathscr{R}(T)^c=\mathscr{N}(T^+).$$

Thus

$$\|T^+y_0\|=\|T^+(y_1+y_2)\|=\|T^+y_1\|\geq\|T^+\|-\varepsilon>0.$$

We change the definition of H in Lemma 2 by

$$Hx=\mathfrak{f}(x)y_1$$

and then $\|H\|=\|y_1\|$. Thus, letting

$$E=\varepsilon H/(\|y_1\|+\|y_2\|). \tag{13}$$

We have obviously $\|E\|=\varepsilon\|y_1\|/(\|y_1\|+\|y_2\|)$,

$$\mathscr{R}(E)=\mathscr{R}(H)=\text{span }\{y_1\}\subseteq\mathscr{R}(T).$$

Thus the operator E in (13) satisfies the condition of Lemma 1.

From Lemma 3, it follows that

$$\|T^+HT^+\|\geq(\|T^+\|-\varepsilon')(\|T^+\|-\varepsilon)$$

and

$$\|T^+\varepsilon/(\|y_1\|+\|y_2\|)\cdot HT^+\|\geq\frac{\varepsilon}{\|y_1\|+\|y_2\|}(\|T^+\|-\varepsilon')(\|T^+\|-\varepsilon)$$

namely

$$\|T^+ET^+\|\geq\frac{\varepsilon}{\|y_1\|+\|y_2\|}(\|T^+\|-\varepsilon')(\|T^+\|-\varepsilon)$$

or

$$\|T^+\|^2-(\varepsilon+\varepsilon')\|T^+\|+(\varepsilon+\varepsilon')\leq\frac{\|y_1\|+\|y_2\|}{\varepsilon}\|T^+ET^+\|$$

$$\leq\frac{\|y_1\|+\|y_2\|}{\varepsilon\|y_1\|}\|T^+ET^+\|=\frac{\|T^+ET^+\|}{\|E\|}.$$

Let $\eta=(\varepsilon+\varepsilon')\|T^+\|-\varepsilon\varepsilon'$. Then we have

$$\|T^+\|^2\leq\frac{\|T^+ET^+\|}{\|E\|}+\eta. \tag{14}$$

Thus

$$\varkappa(T)=\|T\|\cdot\|T^+\|=\frac{\|T^+\|^2\|T\|}{\|T^+\|}\leq\frac{\|T\|}{\|T^+\|}\left\{\frac{\|T^+ET^+\|}{\|E\|}+\eta\right\}.$$

Using (12), it follows that

$$\varkappa(T)\leq\frac{\|T\|}{\|T^+\|}\left(\frac{\|(T+E)^+-T^+\|+\|T^+\|^3\|E\|^2\cdot\sum_{i=0}^{\infty}\|T^+E\|^i}{\|E\|}+\eta\right)$$

$$=\frac{\|(T+E)^+-T^+\|}{\|T^+\|}\frac{\|T\|}{\|E\|}+\|E\|\cdot\|T\|\cdot\|T^+\|^2\cdot\sum_{i=0}^{\infty}\|T^+E\|^i+\frac{\|T\|}{\|T^+\|}\eta$$

$$\leq\frac{\mu(T)}{1-\mu(T)\frac{\|E\|}{\|T\|}}+\|E\|\cdot\|T\|\cdot\|T^+\|^2\cdot\sum_{i=0}^{\infty}\|T^+E\|^i+\frac{\|T\|}{\|T^+\|}\cdot\eta.$$

In the last expression, if $\varepsilon\to 0$ and noting $\|E\|\to 0$, $\eta\to 0$, we have

$$\varkappa(T) \leqslant \mu(T)$$

and this completes the proof.

Corollary 1. Let X and Y be Hilbert spaces and their dimension be finite. Suppose the operator (matrix) T, $E \in B[X, Y]$ and $\operatorname{rank}(T+E) = \operatorname{rank}(T)$. If $\|E\| \cdot \|T^+\| < 1$, then $(T+E)^+$ exists and

$$\frac{\|(T+E)^+ - T^+\|}{\|T^+\|} \leqslant \frac{\varkappa(T) \frac{\|E\|}{\|T\|}}{1 - \varkappa(T) \frac{\|E\|}{\|T\|}}, \tag{15}$$

where T^+ is the Moore–Penrose inverse of T.

Proof. We need only to examine condition (6) in Lemma 1 and to prove $\mathscr{R}(E) \subseteq \mathscr{R}(T)$. If this is not true, then it implies $\mathscr{R}(T+E) \supsetneqq \mathscr{R}(T)$ and this contradicts the supposition (see [7]).

Corollary 2. Let X, Y be Hilbert spaces of finite dimension and T, $E \in B[X, Y]$. Suppose $\operatorname{rank}(T+E) = \operatorname{rank}(T)$ and $\|E\| \|T^+\| < 1$. If there exists a positive number $\mu(T)$ independent of E such that

$$\frac{\|(T+E)^+ - T^+\|}{\|T^+\|} \leqslant \frac{\varkappa(T) \frac{\|E\|}{\|T\|}}{1 - \varkappa(T) \frac{\|E\|}{\|T\|}} \tag{16}$$

holds, then $\varkappa(T) \leqslant \mu(T)$, where $\varkappa(T) = \|T\| \cdot \|T^+\|$.

§ 4. The Regular Inverses

In this section, we consider the case that $T \in B[X]$ and T^{-1} exists. Suppose the small perturbation operator $E \in B[X]$ and $\|T^{-1}\| \|E\| < 1$. We need only to examine condition (6) in Lemma 1. Notice that

$$(I + ET^{-1})^{-1}(T+E) = (I + ET^{-1})^{-1}(I + ET^{-1})T = T,$$

so T maps $\mathscr{N}(T)$ into $\{0\} \subseteq \mathscr{R}(T)$, i.e.,

$$(I + ET^{-1})^{-1}(T+E) \text{ maps } \mathscr{N}(T) \text{ into } \mathscr{R}(T).$$

Thus we have

Corollary 3. Let X and Y be Banach spaces and T, $E \in B[X]$. Suppose $\|E\| \|T^{-1}\| < 1$ and there exists a positive number $\mu(T)$ independent of E such that

$$\frac{\|(T+E)^{-1} - T^{-1}\|}{\|T^{-1}\|} \leqslant \frac{\mu(T) \frac{\|E\|}{\|T\|}}{1 - \mu(T) \frac{\|E\|}{\|T\|}}$$

or

$$\frac{\|(T+E)^{-1} - T^{-1}\|}{\|T^{-1}\|} \leqslant \mu(T) \frac{\|E\|}{\|T\|}.$$

Then $\varkappa(T) \leqslant \mu(T)$, where $\varkappa(T) = \|T\| \cdot \|T^{-1}\|$.

§ 5. The Generalized Solution of Operator Equations

In this section, we consider the relative error bound of generalized solutions of the operator equations

$$Tx=b, \tag{17}$$

where $T\in B[X, Y]$ and $b\in Y$. If $b\bar{\in}\mathscr{R}(T)$, then (17) has no regular solution. We consider the generalized equation of (17)

$$Tx=P_{\mathscr{R}(T)}b. \tag{18}$$

If $x^*=T^+b$, then x^* satisfies (18) and it is said to be the generalized solution of (17). We suppose $E\in B[X, Y]$ is a small perturbation operator and $(T+E)^+$ exist. In addition, suppose $\|E\|\cdot\|T^+\|<1$. From the proof of Theorem 1 we have

$$(T+E)^+b-T^+b=((I_x+T^+E)^{-1}-I_x)T^+b.$$

If we take $x^*=T^+b$ and $\hat{x}^*=(T+E)^+b$, then

$$\|x^*-\hat{x}^*\|\leqslant\left\|\sum_{k=0}^{\infty}(-T^+E)^k-I_x\right\|\cdot\|x^*\|$$

or

$$\frac{\|x^*-\hat{x}^*\|}{\|x^*\|}\leqslant\sum_{k=1}^{\infty}\|T^+E\|^k\leqslant\frac{\|T^+E\|}{1-\|T^+E\|}$$

$$\leqslant\frac{\varkappa(T)\dfrac{\|E\|}{\|T\|}}{1-\varkappa(T)\dfrac{\|E\|}{\|T\|}}, \tag{19a}$$

where $\varkappa(T)=\|T\|\|T^+\|$.

Theorem 2. *Assume X and Y are Banach spaces and* $T, E\in B[X, Y]$. *Let T and E be as in Theorem 1. Let* $x=T^+b$ *and* $\hat{x}=(T+E)^+b$ *and suppose that there exists a positive number* $\nu(T)$ *independent of E such that*

$$\frac{\|x-\hat{x}\|}{\|x\|}\leqslant\frac{\nu(T)\dfrac{\|E\|}{\|T\|}}{1-\nu(T)\dfrac{\|E\|}{\|T\|}} \tag{19}$$

or

$$\frac{\|x-\hat{x}\|}{\|x\|}\leqslant\nu(T)\frac{\|E\|}{\|T\|}. \tag{20}$$

Then $\nu(T)\geqslant1$.

Proof. To simplify the proof, we consider the case where (20) holds. As before, we have

$$(T+E)^+b-T^+b=((I_x+T^+E)^{-1}-I_x)T^+b$$

or

$$\hat{x}-x=\sum_{k=0}^{\infty}P^kx-x\quad(P=-T^+E)=\sum_{k=1}^{\infty}P^kx$$

$$=Px-P^2\sum_{k=0}^{\infty}P^kx.$$

Thus we have

$$\|\hat{x}-x\|\geqslant\|Px\|-\|P\|^2\sum_{k=0}^{\infty}\|P\|^k\|x\|$$

or

$$\|T^+Ex\| \leqslant \|\hat{x}-x\| + \|T^+\|^2 \cdot \|E\|^2 \sum_{k=0}^{\infty} \|T^+E\|^k \cdot \|x\|. \tag{21}$$

Suppose $\varepsilon > 0$ is a given positive number. Then there exists a point $b \in Y$, $\|b\| = 1$, such that

$$\|T^+\| \leqslant \|T^+b\| + \varepsilon = \|T^+P_{\mathscr{R}(T)}b\| + \varepsilon.$$

Assume that x is a generalized solution of (18). We have $Tx = P_{\mathscr{R}(T)}b$. Hence

$$\|T^+\| \leqslant \|T^+Tx\| + \varepsilon.$$

We can choose $E = \varepsilon T$ since it satisfies the requirement of Lemma 1. In fact, $\mathscr{N}(E) = \mathscr{N}(T)$ and $\mathscr{R}(T) = \mathscr{R}(E)$. Substituting them into the above inequality we obtain

$$\|T^+\| \leqslant \frac{\|T^+Ex\|}{\varepsilon} + \varepsilon = \frac{\|T^+Ex\|}{\frac{\|E\|}{\|T\|}} + \varepsilon$$

$$= \frac{\|T\|}{\|E\|} \left(\|T^+Ex\| + \varepsilon \frac{\|E\|}{\|T\|} \right).$$

Using (21) and noticing $\|x\| = \|T^+b\| \leqslant \|T^+\| \cdot \|b\| = \|T^+\|$, we have

$$\|T^+\| \leqslant \frac{\|T\|}{\|E\|} \left\{ \|\hat{x}-x\| + \|T^+\|^2 \|E\|^2 \sum_{k=0}^{\infty} \|P\|^k \|x\| + \varepsilon \frac{\|E\|}{\|T\|} \right\}$$

$$\leqslant \nu(T)\|x\| + \|T\| \|T^+\|^2 \|E\| \sum_{k=0}^{\infty} \|P\|^k \|x\| + \varepsilon$$

$$\leqslant \nu(T)\|T^+\| + \|T\| \|T^+\|^3 \|E\| \sum_{k=0}^{\infty} \|P\|^k + \varepsilon.$$

Thus

$$\varkappa(T) \leqslant \nu(T)\varkappa(T) + \|T\|^2 \|T^+\|^3 \|E\| \sum_{k=0}^{\infty} \|P\|^k + \varepsilon \|T\|.$$

Letting $\varepsilon \to 0$ ($\|E\| \to 0$), we obtain

$$\varkappa(T) \leqslant \nu(T)\varkappa(T)$$

that is, $\nu(T) \geqslant 1$, and this completes the proof.

Corollary 4. Let X, Y be finite dimensional Hilbert spaces and T, $E \in B[X, Y]$. Assume T^+ and $(T+E)^+$ denote the Moore–Penrose inverses of T and $T+E$ respectively. Let $x = T^+b$ and $\hat{x} = (T+E)^+b$ be the least square solution of the equation $Tx = b$ and $(T+E)x = b$ respectively. In addition, suppose $\operatorname{rank}(T) = \operatorname{rank}(T+E)$. Then

$$\frac{\|\hat{x}-x\|}{\|x\|} \leqslant \frac{\varkappa(T)\frac{\|E\|}{\|T\|}}{1-\varkappa(T)\frac{\|E\|}{\|T\|}}.$$

Corollary 5. Let X and Y be finite dimensional Hilbert spaces and T, $E \in B[X, Y]$. Let T^+ and $(T+E)^+$ be the Moore–Penrose inverses of T and $T+E$ respectively and $x = T^+b$, $\hat{x} = (T+E)^+b$ ($b \in Y$). In addition, suppose $\operatorname{rank}(T) = \operatorname{rank}(T+E)$ and there exists a positive number $\nu(T)$ independent of E such that

$$\frac{\|\hat{x}-x\|}{\|x\|} \leqslant \frac{\nu(T)\frac{\|E\|}{T}}{1-\nu(T)\frac{\|E\|}{\|T\|}}. \tag{22}$$

Then $\nu(T)\geqslant 1$.

Theorem 1 shows that the condition number $\varkappa(T)$ in the relative error bound (8) is optimum in a certain sence. But Theorem 2 shows that if the relative error bound (22) holds, then $\nu(T)\geqslant 1$, and this means that the magnifying multiple of the relative error of the generalized solution is necessarily greater than or equal to 1.

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