A TWO-SIDED INTERVAL ITERATIVE METHOD FOR THE FINITE DIMENSIONAL NONLINEAR SYSTEMS*

YOU ZHAO-YONG (游兆永) CHEN XIAO-JUN (陈小君)
(Xi'an Jiaotong University, Xi'an, China),

Abstract

For the nonlinear system

$$x=g(x)+h(x)+c, \quad x\in R^n,$$

where g and h are isotone and antitone mappings respectively, a two-sided iterative method and the existence theorem of a solution for the system have been given in [2]. In this paper, a two-sided interval iterative method is presented, the initial condition of the two-sided iterative method is relaxed, and the convergence of the two methods are proved.

1.

Consider a nonlinear system

$$x=f(x), \quad x\in R^n, \tag{1.1}$$

where f: $R^n\to R^n$ can be expressed as

$$f(x)=g(x)+h(x)+c, \tag{1.2}$$

where g and h are isotone and antitone mappings respectively, that is, from $x\leqslant y$, we have

$$g(x)\leqslant g(y), \quad h(x)\geqslant h(y).$$

By the two-sided iterative method

$$\begin{aligned} y^{(k+1)}&=g(y^{(k)})+h(z^{(k)})+c,\\ z^{(k+1)}&=g(z^{(k)})+h(y^{(k)})+c, \quad k=0, 1, \cdots \end{aligned} \tag{1.3}$$

the existence of a solution to (1.1) is given in [1] and [2].

Assume that

$$y^{(0)}\leqslant y^{(1)}, \quad z^{(1)}\leqslant z^{(0)}. \tag{1.4}$$

Then there exist points y^*, z^*, such that $y^{(k)}\uparrow y^*$ and $z^{(k)}\downarrow z^*$ as $k\to\infty$. Moreover, any fixed point of the operator $f(x)$ in $[y^{(0)}, z^{(0)}]$ is contained in $[y^*, z^*]$. If $f(x)$ is continuous on $[y^{(0)}, z^{(0)}]$, then there exists a solution of (1.1) in $[y^*, z^*]$.

In general, y^* and z^* are not the solution of (1.1).

A method for finding the initial approximation satisfying (1.4) has been given in [3], which is the key to using the two-sided iterative method.

In order to relax the initial condition of the two-sided iterative method the authors give a two-sided interval iterative method. The initial condition of this method is

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$$[y^{(0)}, z^{(0)}] \not\subseteq [y^{(1)}, z^{(1)}]. \tag{1.5}$$

Clearly condition (1.5) is much weaker than (1.4). Moreover, when (1.4) holds, the two methods will coincide.

In this paper, the convergence of $[y^{(k)}, z^{(k)}]$ to the unique solution of (1.1) is given under condition (1.4). The existence and uniqueness of a solution of (1.1) and the convergence of the two-sided interval iterative method are proved under the condition

$$[y_i^{(0)}, z_i^{(0)}] \not\subset [y_i^{(1)}, z_i^{(1)}], \quad i=1, 2, \cdots, n.$$

Finally, we give a simple example for the two-sided interval iterative method. Under the initial condition which the two-sided iterative method fails to meet, we obtain the existence and uniqueness of a solution of the example after one step of iteration and the approximate solution after 15 steps, with accuracy 10^{-3}.

The notation is as follows. Let $y, z, \bar{y}, \bar{z}, x \in R^n$, $y \leqslant z$, $\bar{y} \leqslant \bar{z}$. Then

$$[y, z] = \{x \mid y \leqslant x \leqslant z\},$$
$$W[y, z] = z - y,$$
$$m[y, z] = 1/2\ (z+y),$$
$$|x| = (|x_1|, |x_2|, \cdots, |x_n|),$$
$$I = \{1, 2, \cdots, n\},$$
$$[\bar{y}, \bar{z}] \subseteq [y, z] \Leftrightarrow y_i \leqslant \bar{y}_i,\ \bar{z}_i \leqslant z_i,\ i=1, 2, \cdots, n,$$
$$[\bar{y}, \bar{z}] \subseteq [y, z] \Leftrightarrow y_i \leqslant \bar{y}_i,\ \bar{z}_i \leqslant z_i,\ i=1, 2, \cdots, n$$

and there is $i \in I$ such that $\bar{y}_i - y_i + z_i - \bar{z}_i > 0$,

$$[\bar{y}, \bar{z}] \subset [y, z] \Leftrightarrow y_i \leqslant \bar{y}_i,\ \bar{z}_i \leqslant z_i, \text{ and } \bar{y}_i - y_i + z_i - \bar{z}_i > 0,\ i=1, 2, \cdots, n.$$

2.

For $f(x)$ we consider an interval operator

$$F[y, z] = G[y, z] + H[y, z] + c, \tag{2.1}$$
$$G[y, z] = [g(y), g(z)], \quad H[y, z] = [h(z), h(y)].$$

Property 1. F is an inclusion monotonic interval extension of f [4].

Property 2. If $f(x)$ has a fixed point $x^* \in [y, z]$, then $x^* \in F[y, z]$.

Property 3. If $[y, z] \cap F[y, z] = \varnothing$, then there is no solution of (1.1) in $[y, z]$.

Property 4. Suppose f is continuous on $[y, z]$. Then there is a solution of (1.1) in $[y, z]$ as $F[y, z] \subseteq [y, z]$.

By these important properties, we can introduce the two-sided interval iterative algorithm.

Initial step

Define the initial interval $[y^{(0)}, z^{(0)}]$.

1. If $[y^{(0)}, z^{(0)}] \cap F[y^{(0)}, z^{(0)}] = \varnothing$, then the algorithm is stopped.
2. If $[y^{(0)}, z^{(0)}] \cap F[y^{(0)}, z^{(0)}] \neq \varnothing$, then define $[y^{(1)}, z^{(1)}] = [y^{(0)}, z^{(0)}] \cap F[y^{(0)}, z^{(0)}]$.

Continuation step

Assume that $[y^{(k)}, z^{(k)}]$ for $k\geqslant 1$ is already defined.

1. If $[y^{(k)}, z^{(k)}]\cap F[y^{(k)}, z^{(k)}]=\varnothing$, then the algorithm is stopped.
2. If $[y^{(k)}, z^{(k)}]\cap F[y^{(k)}, z^{(k)}]\neq\varnothing$, then define

$$[y^{(k+1)}, z^{(k+1)}]=[y^{(k)}, z^{(k)}]\cap F[y^{(k)}, z^{(k)}].$$

When $F[y^{(0)}, z^{(0)}]\subseteq[y^{(0)}, z^{(0)}]$ (i.e. (1.4) holds), from the inclusion monotonicity of F, we have

$$[y^{(k+1)}, z^{(k+1)}]=F[y^{(k)}, z^{(k)}]$$

i.e. the two-sided iterative method and the two-sided interval iterative method are coincident.

3.

In this section, a convergent analysis of the two-sided interval iterative method is given. Let g and h be continuous on $[y^{(0)}, z^{(0)}]$.

Theorem 1. *Let*

$$g(z)-g(y)\geqslant p(z-y),\ z\geqslant y,\ 0<p<1, \tag{3.1}$$

$$F[y^{(0)}, z^{(0)}]\subset[y^{(0)}, z^{(0)}]. \tag{3.2}$$

Then (1) *there exists a solution x^* of* (1.1) *in* $[y^{(0)}, z^{(0)}]$;

(2) *the sequence given by the two-sided interval iterative algorithm satisfies*

$$[y^{(k)}, z^{(k)}]\subset[y^{(k-1)}, z^{(k-1)}], \tag{3.3}$$

$$z^{(k)}-y^{(k)}\leqslant q^{(k-1)}(z^{(k-1)}-y^{(k-1)}),\ 0<q^{(k-1)}<1, \tag{3.4}$$

$$x^*\in\bigcap_{k=1}^{\infty}[y^{(k)}, z^{(k)}]; \tag{3.5}$$

(3) *if series* $\sum_{k=0}^{\infty}\frac{1-q^{(k)}}{q^{(k)}}$ *given by* $q^{(0)}$, $q^{(k+1)}=1-p/q^{(k)}+p$ *is divergent, then* x^* *is unique in* $[y^{(0)}, z^{(0)}]$, $[y^{(k)}, z^{(k)}]$ *converges to* x^*, *and from any starting point* $x^{(0)}$ *in* $[y^{(0)}, z^{(0)}]$, *the real iterative sequence*

$$x^{(k+1)}=g(x^{(k)})+h(x^{(k)})+c \tag{3.6}$$

converges to x^*. *And*

$$|x^{(k)}-x^*|\leqslant q^{(k-1)}(z^{(k-1)}-y^{(k-1)}). \tag{3.7}$$

Proof. (1) has been proven in Property 4.

(2) From (3.2), we have

$$[y^{(1)}, z^{(1)}]=F[y^{(0)}, z^{(0)}]\subset[y^{(0)}, z^{(0)}].$$

Let

$$[y^{(k)}, z^{(k)}]=F[y^{(k-1)}, z^{(k-1)}]\subset[y^{(k-1)}, z^{(k-1)}].$$

From the inclusion monotonicity of F, we have

$$[y^{(k+1)}, z^{(k+1)}]=F[y^{(k)}, z^{(k)}]\subseteq F[y^{(k-1)}, z^{(k-1)}]=[y^{(k)}, z^{(k)}].$$

From (3.1), we have

$$y^{(k+1)}-y^{(k)}+z^{(k)}-z^{(k+1)}\geqslant g(y^{(k)})-g(y^{(k-1)})+g(z^{(k-1)})-g(z^{(k)})$$
$$\geqslant p(y^{(k)}-y^{(k-1)}+z^{(k-1)}-z^{(k)})>0.$$

Therefore

$$[y^{(k+1)}, z^{(k+1)}] \subset [y^{(k)}, z^{(k)}].$$

From (3.2) there exists $q^{(0)} \in (0, 1)$, such that

$$z^{(1)} - y^{(1)} \leqslant q^{(0)}(z^{(0)} - y^{(0)}).$$

From (3.1) we have

$$z^{(1)} - y^{(1)} \geqslant g(z^{(0)}) - g(y^{(0)}) \geqslant p(z^{(0)} - y^{(0)}).$$

Hence $q^{(0)} \geqslant p$. Let

$$z^{(k)} - y^{(k)} \leqslant q^{(k-1)}(z^{(k-1)} - y^{(k-1)}), \quad p \leqslant q^{(k-1)} < 1.$$

Then

$$\begin{aligned} z^{(k+1)} - y^{(k+1)} &= g(z^{(k)}) + h(y^{(k)}) - g(y^{(k)}) - h(z^{(k)}) \\ &\leqslant g(z^{(k-1)}) + h(y^{(k-1)}) - g(y^{(k-1)}) - h(z^{(k-1)}) \\ &\quad - p(z^{(k-1)} - z^{(k)}) - p(y^{(k)} - y^{(k-1)}) \\ &= z^{(k)} - y^{(k)} - p(z^{(k-1)} - y^{(k-1)}) + p(z^{(k)} - y^{(k)}) \\ &\leqslant (1 - p/q^{(k-1)} + p)(z^{(k)} - y^{(k)}). \end{aligned}$$

Let $q^{(k)} = 1 - p/q^{(k-1)} + p$. Then $p \leqslant q^{(k)} < 1$. From Property 2, (3.5) holds.

(3) From (2) there exist y^*, z^*, such that $y^{(k)} \uparrow y^*$, $z^{(k)} \downarrow z^*$ and

$$0 \leqslant z^* - y^* \leqslant \prod_{k=0}^{\infty} q^{(k)}(z^{(0)} - y^{(0)}) = q^{(0)} \prod_{k=0}^{\infty} (1 - p/q^{(k)} + p)(z^{(0)} - y^{(0)}).$$

According to $0 < p/q^{(k)} - p \leqslant 1 - p < 1$ and since series $p \sum_{k=0}^{\infty} (1 - q^{(k)})/q^{(k)}$ is divergent, we have $\prod_{k=0}^{\infty} (1 - p/q^{(k)} + p) = 0$, and $x^* = y^* = z^*$.

For the real sequence defined by (3.6), we prove by induction that

$$x^{(k)} \in [y^{(k)}, z^{(k)}], \quad k = 0, 1, 2, \cdots.$$

This is certainly true for $k = 0$. Suppose that it holds for $k = m$, $m \geqslant 0$. Then

$$x^{(m+1)} = f(x^{(m)}) \in F[y^{(m)}, z^{(m)}] = [y^{(m+1)}, z^{(m+1)}].$$

Therefore

$$x^{(k)} \in [y^{(k)}, z^{(k)}],$$

$$|x^* - x^{(k)}| \leqslant z^{(k)} - y^{(k)} \leqslant q^{(k)}(z^{(0)} - y^{(0)}), \quad \lim_{k \to \infty} |x^* - x^{(k)}| = 0.$$

Remark 1. If the conditions of Theorem 1 are changed to

$$wF[y, z] \leqslant qw[y, z], \quad y \leqslant z, \ 0 < q < 1, \tag{3.8}$$

$$F[y^{(0)}, z^{(0)}] \subseteq [y^{(0)}, z^{(0)}], \tag{3.9}$$

the conclusions of Theorem 1 still hold.

Theorem 2. *Let*

1. $|g(y) + h(z) + c - y| \geqslant \bar{q}(z - y)$, $|g(z) + h(y) + c - z| \geqslant \bar{q}(z - y)$, $z \geqslant y$, $0 < \bar{q} < 1$,
2. $[y_i^{(0)}, z_i^{(0)}] \not\subset F_i[y^{(0)}, z^{(0)}]$, $y_i^{(0)} \neq z_i^{(0)}$, $i \in I$.

Then there exists a unique solution of (1.1) *in* $[y^{(0)}, z^{(0)}]$ *if and only if the two-sided interval iterative algorithm can be continued indefinitely. In this case it yields a sequence* $\{[y^{(k)}, z^{(k)}]\}$ *for which*

(1) $[y^{(k+1)}, z^{(k+1)}] \subseteq [y^{(k)}, z^{(k)}]$,

$[y_i^{(k+1)}, z_i^{(k+1)}] \subset [y_i^{(k)}, z_i^{(k)}]$,

$[y_i^{(k)}, z_i^{(k)}] \not\subset F_i[y^{(k)}, z^{(k)}]$, $y_i^{(k)} \neq z_i^{(k)}$, $i \in I$;

(2) $w[y^{(k+1)}, z^{(k+1)}] \leqslant q w[y^{(k)}, z^{(k)}]$, $0<q<1$,

$$\lim_{k\to\infty} z^{(k)} = \lim_{k\to\infty} y^{(k)} = \bar{x};$$

(3) *there exists a unique solution* $x^* = \bar{x}$ *of* (1.1) *in* $[y^{(0)}, z^{(0)}]$ *and*

$$x^* = \bigcap_{k=0}^{\infty} [y^{(k)}, z^{(k)}],$$

$$|x^* - x^{(k)}| \leqslant 1/2\, q^k (z^{(0)} - y^{(0)}),$$

where $x^{(k)} = m[y^{(k)}, z^{(k)}]$.

Proof. If there exists a solution of (1.1) in $[y^{(0)}, z^{(0)}]$, then we have

$$x^* \in F[y^{(0)}, z^{(0)}],$$

$$F[y^{(0)}, z^{(0)}] \cap [y^{(0)}, z^{(0)}] \neq \varnothing.$$

From the definition of the algorithm, we have $x^* \in [y^{(1)}, z^{(1)}]$. We can easily show by induction that

$$[y^{(k)}, z^{(k)}] \cap F[y^{(k)}, z^{(k)}] \neq \varnothing,$$

$$x^* \in [y^{(k+1)}, z^{(k+1)}].$$

Hence, the algorithm is continued indefinitely and cannot be stopped.

Assume that the algorithm can be continued indefinitely.

Let

$$\bar{y}^{(k+1)} = g(y^{(k)}) + h(z^{(k)}) + c, \quad \bar{z}^{(k+1)} = g(z^{(k)}) + h(y^{(k)}) + c.$$

(1) From the algorithm, we have

$$[y^{(1)}, z^{(1)}] \subseteq [y^{(0)}, z^{(0)}].$$

If there exists $i \in I$ such that

$$[y_i^{(1)}, z_i^{(1)}] = [y_i^{(0)}, z_i^{(0)}]$$

from condition 1, as $y_i^{(0)} \neq z_i^{(0)}$,

$$\bar{y}_i^{(1)} < y_i^{(0)}, \quad \bar{z}_i^{(1)} > z_i^{(0)}$$

i.e.

$$[y_i^{(0)}, z_i^{(0)}] \subset F_i[y^{(0)}, z^{(0)}].$$

It will be contradictory to condition 2; hence

$$[y_i^{(1)}, z_i^{(1)}] \subset [y_i^{(0)}, z_i^{(0)}], \quad z_i^{(0)} \neq y_i^{(0)}.$$

Now let for $k=m$

$$[y^{(m)}, z^{(m)}] \subseteq [y^{(m-1)}, z^{(m-1)}], \tag{3.10}$$

$$[y_i^{(m)}, z_i^{(m)}] \subset [y_i^{(m-1)}, z_i^{(m-1)}], \tag{3.11}$$

$$[y_i^{(m-1)}, z_i^{(m-1)}] \not\subset F_i[y^{(m-1)}, z^{(m-1)}], \quad y_i^{(m-1)} \neq z_i^{(m-1)} \tag{3.12}$$

hold. Then

$$[y^{(m+1)}, z^{(m+1)}] \subseteq [y^{(m)}, z^{(m)}], \tag{3.13}$$

$$[\bar{y}^{(m+1)}, \bar{z}^{(m+1)}] \subseteq [\bar{y}^{(m)}, \bar{z}^{(m)}]. \tag{3.14}$$

If there exists $i \in I$, such that

$$[y_i^{(m+1)}, z_i^{(m+1)}] = [y_i^{(m)}, z_i^{(m)}]$$

then from condition 1, as $y_i^{(m)} \neq z_i^{(m)}$,

$$\bar{y}_i^{(m+1)} < y_i^{(m)} = \max\{\bar{y}_i^{(m)}, y_i^{(m-1)}\},$$

$$\bar{z}_i^{(m+1)} > z_i^{(m)} = \min\{\bar{z}_i^{(m)}, z_i^{(m-1)}\}$$

hold. From (3.14), we have

$$y_i^{(m)}=y_i^{(m-1)}, \quad z_i^{(m)}=z_i^{(m-1)}. \tag{3.15}$$

From (3.10), we have $z_i^{(m-1)}\neq y_i^{(m-1)}$, as $y_i^{(m)}\neq z_i^{(m)}$. Therefore (3.15) will be contradictory to (3.11). Hence

$$[y_i^{(m+1)}, z_i^{(m+1)}]\subset[y_i^{(m)}, z_i^{(m)}], \quad y_i^{(m)}\neq z_i^{(m)}$$

holds. From the algorithm, we have

$$[y_i^{(m)}, z_i^{(m)}]\not\subset[\bar{y}_i^{(m+1)}, \bar{z}_i^{(m+1)}], \quad z_i^{(m)}\neq y_i^{(m)}.$$

(2) From (1), as $z_i^{(0)}\neq y_i^{(0)}$,

$$[y_i^{(1)}, z_i^{(1)}]\subset[y_i^{(0)}, z_i^{(0)}]$$

holds. Hence there exists $q_i\in(0, 1)$, such that

$$w[y_i^{(1)}, z_i^{(1)}]\leqslant q_i w[y_i^{(0)}, z_i^{(0)}].$$

Let $\tilde{q}=\max\limits_{i\in I'} q_i$, $I'=\{i|z_i^{(0)}\neq y_i^{(0)}, i\in I\}$. As $z_i^{(0)}=y_i^{(0)}$,

$$z_i^{(1)}=y_i^{(1)}=z_i^{(0)}=y_i^{(0)},$$

$$w[y_i^{(1)}, z_i^{(1)}]\leqslant\tilde{q}w[y_i^{(0)}, z_i^{(0)}]$$

certainly hold. Let $q=\max\{\tilde{q}, 1-\bar{q}\}$. We have

$$w[y^{(1)}, z^{(1)}]\leqslant qw[y^{(0)}, z^{(0)}].$$

Let for $k=m$

$$w[y^{(m)}, z^{(m)}]\leqslant qw[y^{(m-1)}, z^{(m-1)}]$$

hold. We prove that for $k=m+1$ it holds too.

When $F[y^{(m)}, z^{(m)}]\subseteq[y^{(m)}, z^{(m)}]$, if there exists $i\in I$ such that $F_i[y^{(m)}, z^{(m)}]=[y_i^{(m)}, z_i^{(m)}]$, then

$$[y_i^{(m+1)}, z_i^{(m+1)}]=[y_i^{(m)}, z_i^{(m)}]$$

holds, if and only if $y_i^{(m)}=z_i^{(m)}$ holds. In this case,

$$w[y_i^{(m+1)}, z_i^{(m+1)}]\leqslant qw[y_i^{(m)}, z_i^{(m)}]$$

certainly holds. If $F_i[y^{(m)}, z^{(m)}]\subset[y_i^{(m)}, z_i^{(m)}]$, then

$$[y_i^{(m+1)}, z_i^{(m+1)}]=F_i[y^{(m)}, z^{(m)}].$$

From condition 1, we have

$$\begin{aligned}w[y_i^{(m+1)}, z_i^{(m+1)}]&=wF_i[y^{(m)}, z^{(m)}]=g_i(z^{(m)})+h_i(y^{(m)})-g_i(y^{(m)})-h_i(y^{(m)})\\&\leqslant z_i^{(m)}-\bar{q}(z_i^{(m)}-y_i^{(m)})-y_i^{(m)}-\bar{q}(z_i^{(m)}-y_i^{(m)})\\&\leqslant(1-\bar{q})(z_i^{(m)}-y_i^{(m)})\leqslant q(z_i^{(m)}-y_i^{(m)}).\end{aligned}$$

Therefore, as $F[y^{(m)}, z^{(m)}]\subseteq[y^{(m)}, z^{(m)}]$, we have

$$w[y^{(m+1)}, z^{(m+1)}]\leqslant qw[y^{(m)}, z^{(m)}].$$

As $F[y^{(m)}, z^{(m)}]\subsetneqq[y^{(m)}, z^{(m)}]$, if $y_i^{(m)}=z_i^{(m)}$, then $y_i^{(m+1)}=z_i^{(m+1)}$. In this case,

$$w[y_i^{(m+1)}, z_i^{(m+1)}]\leqslant qw[y_i^{(m)}, z_i^{(m)}]$$

certainly holds. If $y_i^{(m)}\neq z_i^{(m)}$, from

$$[y_i^{(m+1)}, z_i^{(m+1)}]=F_i[y_i^{(m)}, z_i^{(m)}]\cap[y_i^{(m)}, z_i^{(m)}]$$

and (1), we have

$$m[y_i^{(m)}, z_i^{(m)}]>mF_i[y^{(m)}, z^{(m)}] \tag{3.16}$$

or

$$m[y_i^{(m)}, z_i^{(m)}] < mF_i[y^{(m)}, z^{(m)}]. \tag{3.17}$$

As (3.16) holds, we have

$$y_i^{(m)} > \bar{y}_i^{(m+1)}, \quad z_i^{(m)} > \bar{z}_i^{(m+1)}.$$

Hence

$$y_i^{(m+1)} = y_i^{(m)}, \quad z_i^{(m+1)} = \bar{z}_i^{(m+1)},$$

$$\begin{aligned} w[y_i^{(m+1)}, z_i^{(m+1)}] &= z_i^{(m+1)} - y_i^{(m+1)} = \bar{z}_i^{(m+1)} - y_i^{(m)} \\ &= -(z_i^{(m)} - \bar{z}_i^{(m+1)}) + z_i^{(m)} - y_i^{(m)} \leqslant (1-\bar{q})(z_i^{(m)} - y_i^{(m)}) \\ &\leqslant q(z_i^{(m)} - y_i^{(m)}). \end{aligned}$$

As (3.17) holds, we have

$$y_i^{(m)} < \bar{y}_i^{(m+1)}, \quad z_i^{(m)} < \bar{z}_i^{(m+1)}.$$

Hence

$$y_i^{(m+1)} = \bar{y}_i^{(m+1)}, \quad z_i^{(m+1)} = z_i^{(m)},$$

$$\begin{aligned} w[y_i^{(m+1)}, z_i^{(m+1)}] &= z_i^{(m+1)} - y_i^{(m+1)} = z_i^{m} - y_i^{(m+1)} = z_i^{(m)} - y_i^{(m)} - (\bar{y}_i^{(m+1)} - y_i^{(m)}) \\ &\leqslant (1-\bar{q})(z_i^{(m)} - y_i^{(m)}) \leqslant q(z_i^{(m)} - y_i^{(m)}). \end{aligned}$$

As mentioned above, we have

$$w[y^{(m+1)}, z^{(m+1)}] \leqslant qw[y^{(m)}, z^{(m)}].$$

Therefore

$$\lim_{k\to\infty} w[y^{(k)}, z^{(k)}] \leqslant \lim_{k\to\infty} q^k w[y^{(0)}, z^{(0)}] = 0.$$

From (1), $y^{(k)}$, $z^{(k)}$ are monotonically increasing and monotonically decreasing sequences respectively. Hence there exists $\bar{x}$, such that

$$\lim_{k\to\infty} z^{(k)} = \lim_{k\to\infty} y^{(k)} = \bar{x}.$$

(3) Let $x^{(k)} = m[y^{(k)}, z^{(k)}]$; then $y^{(k)} \leqslant x^{(k)} \leqslant z^{(k)}$. From (2), we have

$$\lim_{k\to\infty} x^{(k)} = \bar{x}.$$

Let $\bar{x}^{(k+1)} = m[\bar{y}^{(k+1)}, \bar{z}^{(k+1)}]$. We have

$$|\bar{x}^{(k+1)} - x^{(k)}| \leqslant 1/2\ w[\bar{y}^{(k+1)}, \bar{z}^{(k+1)}] + 1/2\ w[y^{(k)}, z^{(k)}].$$

From (2) and the continuity of g and h, we have

$$\begin{aligned} \lim_{k\to\infty} |\bar{x}^{(k+1)} - x^{(k)}| &= \lim_{k\to\infty} |1/2\ (g(z^{(k)}) + h(y^{(k)}) + c + g(y^{(k)}) + h(z^{(k)}) + c) - x^{(k)}| \\ &= |g(\bar{x}) + h(\bar{x}) + c - \bar{x}| = 0. \end{aligned}$$

Hence $\bar{x} = x^*$ is the unique solution of (1.1) in $[y^{(0)}, z^{(0)}]$. From $x^* \in [y^{(k)}, z^{(k)}]$, we have

$$|x^* - x^{(k)}| \leqslant 1/2\ (z^{(k)} - y^{(k)}) \leqslant 1/2\ q^k (z^{(0)} - y^{(0)}).$$

From the proof of Theorem 2, we can obtain

Corollary 1. If the sequence given by the algorithm satisfies

$$\begin{aligned} &a_i^k (z_i^{(k)} - g_i(z^{(k)}) - h_i(y^{(k)}) - c_i) + b_i^k (g_i(y^{(k)}) + h_i(z^{(k)}) + c_i - y_i^{(k)}) \\ &\geqslant r(z_i^{(k)} - y_i^{(k)}), \quad 0 < r < 1,\ i = 1, 2, \cdots, n, \end{aligned}$$

where

$$a_i^k=\begin{cases}1, & \text{as } z_i^{(k)}-g_i(z^{(k)})-h_i(y^{(k)})-c_i\geqslant 0,\\ 0, & \text{as } z_i^{(k)}-g_i(z^{(k)})-h_i(y^{(k)})-c_i<0;\end{cases}$$
$$b_i^k=\begin{cases}1, & \text{as } g_i(y^{(k)})+h_i(z^{(k)})+c_i-y^{(k)}\geqslant 0,\\ 0, & \text{as } g_i(y^{(k)})+h_i(z^{(k)})+c_i-y^{(k)}<0\end{cases}$$

and $a_i^k+b_i^k\geqslant 1$, the conclusions of Theorem 2 hold.

A special case of Corollary 1 is

Corollary 2. Suppose there exists $r\in(0, 1)$, such that

$$y^{(0)}\leqslant g(y^{(0)})+h(z^{(0)})+c, \quad g(y)+h(z)+c-y\geqslant r(z-y)$$

or

$$g(z^{(0)})+h(y^{(0)})+c\leqslant z^{(0)}, \quad z-g(z)-h(y)-c\geqslant r(z-y).$$

Then the conclusions of Theorem 2 hold, and we have

$$y^{(k)}\leqslant g(y^{(k)})+h(z^{(k)})+c=y^{(k+1)}$$

or

$$z^{(k)}\geqslant g(z^{(k)})+h(y^{(k)})+c=z^{(k+1)}.$$

Remark 2. If the conditions of Theorem 2 are changed to

$$wF[y, z]\leqslant q(z-y), \quad y\leqslant z,\ 0<q<1,$$

the conclusions of Theorem 2 still hold.

Example 1. Let

$$f(x)=\begin{pmatrix}0.1x_1^2-0.1x_2^2+2\\ 0.1x_2^2-0.1x_1^2+2\end{pmatrix}, \quad [y^{(0)}, z^{(0)}]=\left[\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\sqrt{21}\\ \sqrt{21}\end{pmatrix}\right],$$

$$F[y, z]=\left[\begin{pmatrix}0.1y_1^2-0.1z_2^2+2\\ 0.1y_2^2-0.1z_1^2+2\end{pmatrix}, \begin{pmatrix}0.1z_1^2-y_2^2+2\\ 0.1z_2^2-y_1^2+2\end{pmatrix}\right],$$

$$F[y^{(0)}, z^{(0)}]=\left[\begin{pmatrix}-0.1\\-0.1\end{pmatrix}, \begin{pmatrix}4.1\\4.1\end{pmatrix}\right]\subsetneqq[y^{(0)}, z^{(0)}].$$

The last relation does not satisfy the initial condition of the two-sided iterative method. We use the two-sided interval iterative method to give

$$[y^{(1)}, z^{(1)}]=\left[\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}4.1\\4.1\end{pmatrix}\right],$$

$$[y^{(2)}, z^{(2)}]=F[y^{(1)}, z^{(1)}]=\left[\begin{pmatrix}0.3190\\0.3190\end{pmatrix}, \begin{pmatrix}3.6810\\3.6810\end{pmatrix}\right]\subset[y^{(1)}, z^{(1)}].$$

According to Remark 1 there exists a unique solution x^* of this example in $[y^{(1)}, z^{(1)}]$.

$$[y^{(3)}, z^{(3)}]=\left[\begin{pmatrix}0.6552\\0.6552\end{pmatrix}, \begin{pmatrix}3.3447\\3.3447\end{pmatrix}\right],$$

$$\vdots$$

$$[y^{(15)}, z^{(15)}]=\left[\begin{pmatrix}1.9962\\1.9962\end{pmatrix}, \begin{pmatrix}2.0051\\2.0051\end{pmatrix}\right],$$

$$|x^*-m[y^{(15)}, z^{(15)}]|\leqslant 10^{-3}, \quad x^*=(2, 2).$$

Remark 3. When initial values satisfy

$$[y^{(0)}, z^{(0)}] \subseteq F[y^{(0)}, z^{(0)}] \tag{3.18}$$

the above two methods will fail. A method is given in [6] by which one can solve (1.1) under condition (3.18). A method for an arbitrary initial condition is given in [7].

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