# THE CONVERGENCE OF THE MULTIGRID ALGORITHM FOR NAVIER-STOKES EQUATIONS*

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## Abstract

This paper deals with a multigrid algorithm for the numerical solution of Navier-Stokes problems. The convergence proof and the estimation of the contraction number of the multigrid algorithm are given.

## § 1. Introduction

The multigrid method is a new method for working out the numerical solutions of elliptic differential equations. Briefly, it consists of smoothing process and coarse-grid correction procedure such that the operational time can be saved and the convergence rate improved. The multigrid method for solving large systems of linear equations, which arise in the numerical solution of boundary value problems by finite elements, has been discussed by many authors, e.g. Astrachancev[1], Nicolaides[10], Bank & Dupont[2], Hackbusch[6-7], Wesseling[11] and Verfüth[12]. In this paper, we discuss the convergence properties of the multigrid algorithm for the nonlinear Navier-Stokes problem:

$$\begin{cases} -\mu\Delta \boldsymbol{u}+\boldsymbol{u}\cdot\nabla \boldsymbol{u}+\operatorname{grad} p=\boldsymbol{f}, & \text{in } \Omega, \\ -\operatorname{div} \boldsymbol{u}=0, & \text{in } \Omega, \\ \boldsymbol{u}=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\boldsymbol{u}=(u_1, u_2, \cdots, u_d)$ and $p$ are the velocity and the pressure of fluid respectively, $\mu$ is its viscosity, $\Omega\subset R^d$ a sufficiently smooth domain, and

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u}=\left(\sum_{j=1}^{d} u_j\,\partial u_i/\partial x_j\right)_{i=1,\cdots,d}.$$

The general structure of our convergence analysis for the multigrid procedure is similar to that of Hackbusch[6-7], and the convergence result of the multigrid algorithm for Navier-Stokes equation is based on the convergence theorem of nonlinear multigrid methods[7]. It is known that the main sufficient conditions of the convergence of the multigrid method are the smoothing properties and the approximation properties. Therefore in this paper we first give the proof of the smoothing properties in some discrete norms, then prove the approximation property from the usual approximation assumptions in terms of Sobolev spaces and finally

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discuss the convergence of the multigrid algorithm and estimate the contraction number under general assumptions. We have to consider the nonlinearity of equation (1.1) and different orders of differentiability of $\boldsymbol{u}$ and $p$ in (1.1). This will be compensated for by considering the nonlinear multigrid methods inside the neighbourhood of solution $[\boldsymbol{u}, p]$ and by introducing mesh-dependent norms. To simplify the analysis we present a smoothing procedure which is related to the Jacobi iteration for scalar problems[7].

## § 2. Preliminaries

Consider Navier-Stokes equation (1.1) in a smooth enough domain and its variational formulation:

Find $[\boldsymbol{u}, p]\in Z=X\times Y$ such that

$$\begin{cases}\mu a_0(\boldsymbol{u}, \boldsymbol{v})+a_1(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v})+b(\boldsymbol{v}, p)=(\boldsymbol{f}, \boldsymbol{v})_0, & \forall \boldsymbol{v}\in X,\\ b(\boldsymbol{u}, q)=0, & \forall q\in Y,\end{cases} \tag{2.1}$$

where

$$a_0=(\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_0,$$

$$a_1(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v})=((\boldsymbol{w}\cdot\nabla)\boldsymbol{u}, \boldsymbol{v})_0,$$

$$b(\boldsymbol{u}, p)=-(\operatorname{div} \boldsymbol{u}, p)_0,$$

$$X=H_0^1(\Omega)^d,\ X^0=L^2(\Omega)^d,$$

$$Y=L_0^2(\Omega)=\left\{p\in L^2(\Omega): \int_\Omega p\,dx=0\right\},$$

and $(\cdot, \cdot)_0$ denotes the $L^2$-inner product.

Let $\Omega$ be a smooth enough domain such that $a_0$, $a_1$ and $b$ satisfy the continuity, coercivity and Brezzi's conditions[6], respectively. In addition, we assume the following regularity assumptions of problem (1.1) and its duality problem: If $\boldsymbol{f}\in L^2(\Omega)^d$, then $[\boldsymbol{u}, p]$, $[\boldsymbol{w}, q]\in H^2(\Omega)^d\times H^1(\Omega)$ with

$$\|[\boldsymbol{u}, p]\|_{2,1}\leqslant c\|\boldsymbol{f}\|_0, \tag{2.2a}$$

$$\|[\boldsymbol{w}, q]\|_{2,1}\leqslant c\|\boldsymbol{f}\|_0, \tag{2.2b}$$

where $c$ denotes the generic constant and $[\boldsymbol{w}, q]$ satisfies the duality problem of (1.1).

Introduce a Navier-Stokes' operator

$$\mathscr{L}'=\begin{bmatrix}-\mu\Delta+(I\cdot\nabla) & \operatorname{grad}\\ -\operatorname{div} & 0\end{bmatrix}, \tag{2.3a}$$

where $I$ denotes an identity operator; then (2.1) is equivalent to

$$\mathscr{L}[\boldsymbol{u}, p]=\mathscr{L}'[\boldsymbol{u}, p]-[\boldsymbol{f}, 0]=0. \tag{2.3b}$$

Obviously, the linearization of $\mathscr{L}'$ equals

$$L[\boldsymbol{u}, p]=\begin{bmatrix}-\mu\Delta+\delta L_{11}(\boldsymbol{u}) & \operatorname{grad}\\ -\operatorname{div} & 0\end{bmatrix}=\begin{bmatrix}\mu L_{11}+\delta L_{11}(\boldsymbol{u}) & L_{12}\\ L_{21} & 0\end{bmatrix}$$

$$=L'+\begin{bmatrix}\delta L_{11}(\boldsymbol{u}) & 0\\ 0 & 0\end{bmatrix}, \tag{2.4}$$

where $(\delta L_{11}(\boldsymbol{u})\boldsymbol{v})_{i=1,\cdots,d}=\sum_{j=1}^{d}(u_j\partial v_i/\partial x_j+v_j\partial u_i/\partial x_j)$, $L_{11}=-\Delta$, $L_{12}=\text{grad}$ and $L_{21}=-\text{div}$.

Let $X_l\subset X$ and $Y_l\subset Y$ be two families of finite element subspaces with

$$h_0>h_1>\cdots>h_{l-1}>h_l>\cdots,\ X_{l-1}\subset X_l,\ Y_{l-1}\subset Y_l,$$

and the usual approximation assumptions and inverse inequality:

$$\begin{aligned}&\inf_{\boldsymbol{v}_l\in X_l}\|\boldsymbol{v}-\boldsymbol{v}_l\|_\alpha\leqslant ch_l^{\beta-\alpha}\|\boldsymbol{v}\|_\beta,\quad \forall\boldsymbol{v}\in H^\beta(\Omega)^d,\quad 0\leqslant\alpha\leqslant1\leqslant\beta\leqslant2,\\&\inf_{p_l\in Y_l}\|p-p_l\|_\alpha\leqslant ch_l^{\beta-\alpha}\|p\|_\beta,\quad \forall p\in H^\beta(\Omega),\quad 0\leqslant\alpha\leqslant\beta\leqslant1,\end{aligned}\tag{2.5a}$$

$$\|\boldsymbol{v}_l\|_{+1}\leqslant ch_l^{-1}\|\boldsymbol{v}_l\|_0,\quad \forall\boldsymbol{v}_l\in X_l.\tag{2.5b}$$

The spaces $X_l$ and $Y_l$ have to fit together so that the discrete Brezzi's condition of $b$ holds.

Put the product space

$$Z_l=X_l\times Y_l$$

equipped with the discrete norm

$$\|[\boldsymbol{u}_l,p_l]\|_s=\{h_l^{-2}\|\boldsymbol{u}_l\|_0^2+\|p_l\|_0^2\}^{1/2}.\tag{2.6}$$

Define a matrix

$$H_l=\begin{bmatrix}h_lI & 0\\ 0 & 1\end{bmatrix},\tag{2.7}$$

where $I$ is a $d\times d$ identity matrix. Then we get

$$\|[\boldsymbol{u}_l,p_l]\|_s=\|H_l^{-1}[\boldsymbol{u}_l,p_l]\|_0.\tag{2.8}$$

Consider the discrete analogue of (2.1) in $Z_l$: Find $[\boldsymbol{u}_l,p_l]\in Z_l$ such that

$$\begin{cases}\mu a_0(\boldsymbol{u}_l,\boldsymbol{v}_l)+a_1(\boldsymbol{u}_l;\boldsymbol{u}_l,\boldsymbol{v}_l)+b(\boldsymbol{v}_l,p_l)=(\boldsymbol{f},\boldsymbol{v}_l)_0, & \forall\boldsymbol{v}_l\in X_l,\\ b(\boldsymbol{u}_l,q_l)=0, & \forall q_l\in Y_l.\end{cases}\tag{2.9}$$

According to Galerkin's finite element method, problem (2.9) has an equivalent matrix-vector system

$$\mathscr{L}_l[\boldsymbol{u}_l,p_l]=0.\tag{2.10}$$

Here we again denote by $[\boldsymbol{u}_l,p_l]$ the vector consisting of its components with the intention of not overloading the presentation with notations. Thanks to the above conditions, (2.1) and (2.9) have, respectively, at least one solution $[\boldsymbol{u}^*,p^*]$ and $[\boldsymbol{u}_l^*,p_l^*]$.

Corresponding to the nonlinear operator $\mathscr{L}_l[\cdot,\cdot]$, there exists the linearization from

$$L_l[\boldsymbol{u}_l,p_l]=\begin{bmatrix}\mu L_{l,11}+\delta L_{l,11}(\boldsymbol{u}_l) & L_{l,12}\\ L_{l,21} & 0\end{bmatrix}=L_l'+\begin{bmatrix}\delta L_{l,11}(\boldsymbol{u}_l) & 0\\ 0 & 0\end{bmatrix},\tag{2.11}$$

where $L_{l,11}$, $\delta L_{l,11}(\boldsymbol{u}_l)$, $L_{l,12}$ and $L_{l,21}$ are the discrete analogues of $L_{11}$, $\delta L_{11}(\boldsymbol{u})$, $L_{12}$ and $L_{21}$ respectively.

Let $\mathscr{F}_l$ be the space consisting of the right-hand grid function $[\boldsymbol{f}_l,g_l]$ on (2.10), equipped with the norm

$$\|[\boldsymbol{f}_l,g_l]\|_{\mathscr{F}}=\{h_l^2\|\boldsymbol{f}_l\|_0^2+\|g_l\|_0^2\}^{1/2}.\tag{2.12}$$

Then from the definition of $H_l$ in (2.7), it follows that

$$\|[\boldsymbol{f}_l, g_l]\|_{\mathscr{F}} = \|H_l[\boldsymbol{f}_l, g_l]\|_0. \quad (2.13)$$

For the vector forms $[\boldsymbol{u}_l, p_l]$ and $[\boldsymbol{f}_l, g_l]$ on (2.10), we again denote by $\|\cdot\|_s$ and $\|\cdot\|_{\mathscr{F}}$ their norms, respectively

$$\|[\boldsymbol{u}_l, p_l]\|_s = \{h_l^{-2}\|\boldsymbol{u}_l\|^2 + \|p_l\|^2\}^{1/2},$$
$$\|[\boldsymbol{f}_l, g_l]\|_{\mathscr{F}} = \{h_l^2\|\boldsymbol{f}_l\|^2 + \|g_l\|^2\}^{1/2},$$

where $\|\cdot\|$ denotes the Euclidean norm independent of $h_l$. Note the equivalence of $\|\cdot\|_0$ and $\|\cdot\|$ in the standard finite element case.

## § 3. A Nonlinear Multigrid Algorithm

We consider a nonlinear multigrid iteration for equation (2.10) inside the neighbourhood of the solution $[\boldsymbol{u}_l^*, p_l^*]$. Let $\varepsilon_l$ be a positive number such that $\mathscr{L}_l[\cdot, \cdot]$ is a homeomorphism from

$$Z_l(\varepsilon_l) = \{[\boldsymbol{u}_l, p_l] : \|[\boldsymbol{u}_l, p_l] - [\boldsymbol{u}_l^*, p_l^*]\|_s \leqslant \varepsilon_l\}$$

to

$$\mathscr{F}_l(\varepsilon_l) = \{[\boldsymbol{f}_l, g_l] = \mathscr{L}_l[\boldsymbol{u}_l, p_l] : [\boldsymbol{u}_l, p_l] \in Z_l(\varepsilon_l)\}.$$

For equation (2.10) we define a smoother

$$\mathscr{S}_l\{[\boldsymbol{u}_l, p_l], [\boldsymbol{f}_l, g_l]\} = [\boldsymbol{u}_l, p_l] - \omega_l^2 H_l^2 L_l^*[\boldsymbol{u}_l, p_l] H_l^2(\mathscr{L}_l[\boldsymbol{u}_l, p_l] - [\boldsymbol{f}_l, g_l]), \quad (3.1)$$

where $L_l^*[\cdot, \cdot]$ is the conjugate operator of $L_l[\cdot, \cdot]$ and $\omega_l$ a smoothing parameter (see below).

Given the initial approximation $[\boldsymbol{u}_l^0, p_l^0] \in Z_l(\varepsilon_l)$ to the solution of (2.10) on $\Omega_l (l \geqslant 2)$, we define the following iteration procedure solving (2.10) inside the above neighbourhoods:

$$\text{Given}: [\tilde{\boldsymbol{u}}_k, \tilde{p}_k] \text{ and } [\tilde{\boldsymbol{f}}_k, \tilde{g}_k] = \mathscr{L}_k[\tilde{\boldsymbol{u}}_k, \tilde{p}_k], \quad k = 0, 1, \cdots, l-1, \quad (3.2a)$$

do steps 1 and 2 for $i = 1, 2, \cdots$

1. Smoothing:

$$[\bar{\boldsymbol{u}}_l^i, \bar{p}_l^i] = \mathscr{S}_l\{[\boldsymbol{u}_l^i\ p_l^i], 0\}. \quad (3.2b)$$

2. Coarse-grid correction:

$$[\boldsymbol{d}_{l-1,1}, d_{l-1,2}] = r\mathscr{L}_l[\bar{\boldsymbol{u}}_l^i, \bar{p}_l^i]; \quad (3.2c)$$

$$s = \begin{cases} \sigma_{l-1}/\|[\boldsymbol{d}_{l-1,1}, d_{l-1,2}]\|_{\mathscr{F}}, & [\boldsymbol{d}_{l-1,1}, d_{l-1,2}] \neq 0, \\ 0, & [\boldsymbol{d}_{l-1,1}, d_{l-1,2}] = 0; \end{cases} \quad (3.2d)$$

$$[\tilde{\boldsymbol{d}}_{l-1,1}, \tilde{d}_{l-1,2}] = [\tilde{\boldsymbol{f}}_{l-1}, \tilde{g}_{l-1}] - s[\boldsymbol{d}_{l-1,1}, d_{l-1,2}]. \quad (3.2e)$$

Compute an approximate solution $[\boldsymbol{v}_{l-1}, q_{l-1}]$ of the defect equation on $\Omega_{l-1}$:

$$\mathscr{L}_{l-1}[\boldsymbol{v}_{l-1}, q_{l-1}] = [\tilde{\boldsymbol{d}}_{l-1,1}, \tilde{d}_{l-1,2}] \quad (3.2f)$$

by performing $\gamma$, $\gamma \geqslant 2$, iterations of (l-1)-grid scheme to (3.2f) with starting value $[\tilde{\boldsymbol{u}}_{l-1}, \tilde{p}_{l-1}]$;

$$[\boldsymbol{u}_l^{i+1}, p_l^{i+1}] = [\bar{\boldsymbol{u}}_l^i, \bar{p}_l^i] + p([\boldsymbol{v}_{l-1}, q_{l-1}] - [\tilde{\boldsymbol{u}}_{l-1}, \tilde{p}_{l-1}])/s. \quad (3.2g)$$

$\sigma_{l-1}$ is some positive number, and $r$ and $p$ stand for the restriction and the prolongation operator:

$$r=\text{blockdiag}\{r_1, r_2\}, \quad p=\text{blockdiag}\{p_1, p_2\}, \tag{3.3}$$

respectively.

At the lowest level $l=0$, we have to solve equation $\mathscr{L}_0[\boldsymbol{v}_0, q_0]=[\tilde{\boldsymbol{d}}_{01}, \tilde{d}_{02}]$ on $\Omega_0$. Assume that these equations are solved by an iteration

$$[\boldsymbol{u}_0^{j+1}, p_0^{j+1}]=\phi_0[\boldsymbol{u}_0^j, p_0^j]$$

converging to $\mathscr{L}_0^{-1}[\tilde{\boldsymbol{d}}_{01}, \tilde{d}_{02}]$ whose contraction number is denoted by $\varphi_0$.

## § 4. Stokes Problem

Consider Stokes equation in the domain $\Omega$:

$$\begin{cases} -\Delta \boldsymbol{u}+\text{grad } p=\boldsymbol{f}, & \text{in } \Omega, \\ -\text{div } \boldsymbol{u}=0, & \text{in } \Omega, \\ \boldsymbol{u}=0, & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

The saddle point problem of (4.1) reads:

Seek $[\boldsymbol{u}, p]\in Z$ such that

$$\begin{cases} a_0(\boldsymbol{u}, \boldsymbol{v})+b(\boldsymbol{v}, p)=(\boldsymbol{f}, \boldsymbol{v})_0, & \forall \boldsymbol{v}\in X, \\ b(\boldsymbol{u}, q)=0, & \forall q\in Y. \end{cases} \tag{4.2}$$

The discrete analogue corresponding to (4.2) in $Z_l$ can be written as follows:

Find $[\boldsymbol{u}_l, p_l]\in Z_l$ such that

$$\begin{cases} a_0(\boldsymbol{u}_l, \boldsymbol{v}_l)+b(\boldsymbol{v}_l, p_l)=(\boldsymbol{f}, \boldsymbol{v}_l)_0, & \forall \boldsymbol{v}_l\in X_l, \\ b(\boldsymbol{u}_l, q_l)=0, & \forall q_l\in Y_l, \end{cases} \tag{4.3}$$

Similarly, problem (4.3) has an equivalent matrix-vector form

$$L_l'[\boldsymbol{u}_l, p_l]=[\boldsymbol{f}_l, 0], \tag{4.4}$$

where $L_l'$ is from (2.11) with $\mu=1$.

Let

$$\hat{L}_l'=H_l L_l' H_l. \tag{4.5}$$

**Lemma 4.1**[7]. *There exists a constant $c$ so that*

$$\|\hat{L}_l'\|\leqslant c, \ \|\hat{L}_l'^{-1}\|\leqslant c,$$

*where $\|\cdot\|$ denotes spectral norm.*

Let

$$1/\omega_l'=\|\hat{L}_l'\|\leqslant c. \tag{4.6}$$

Then we define the smoother for equation (4.4)

$$\mathscr{L}_l'\{[\boldsymbol{u}_l, p_l]-[\boldsymbol{f}_l, g_l]\}=[\boldsymbol{u}_l, p_l]-\omega_l'^2 H_l^2 L_l'^* H_l^2(L_l'[\boldsymbol{u}_l, p_l]-[\boldsymbol{f}_l, g_l]), \tag{4.7a}$$

whose iterate matrix equals

$$S_l'=I-H_l^2 L_l'^* H_l^2 L_l' \omega_l'^2. \tag{4.7b}$$

Here $L_l'^*$ is the conjugate operator of $L_l'$.

**Lemma 4.2**[7]. *Let $U$ be a Hilbert space with scalar product $\langle\cdot,\cdot\rangle_u$ and norm*

*$\|u\|_u=\sqrt{\langle u, u\rangle_u}$, $u\in U$ and $A$ a matrix such that $0\leqslant A=A^*\leqslant I$. Then*

$$\|A(I-A)^\nu\|_{u\leftarrow u}\leqslant\eta_0(\nu),$$

*where*

$$\eta_0(\nu)=\nu^\nu/(1+\nu)^{1+\nu}.$$

**Theorem 4.3.** *The smoother (4.7a—b) satisfies the smoothing property:*

$$\|L_l' S_l'^\nu\|_{\mathscr{F}\leftarrow s}\leqslant\eta'(\nu)=c\sqrt{\eta_0(2\nu)},\ \textit{for all}\ \nu\geqslant 0,\ l\geqslant 0. \tag{4.8}$$

*Proof.* Note that

$$S_l'=I-\omega_l'^2 H_l^2\hat{L}_l'^*\hat{L}_l' H_l^{-1}=I-H_l M_l H_l^{-1}, \tag{4.9}$$

$$\hat{S}_l'=H_l^{-1}S_l' H_l=I-\omega_l'^2\hat{L}_l'^*\hat{L}_l'=I-M_l,$$

where $M_l=\omega_l'^2\hat{L}_l'^*\hat{L}_l'$ with

$$0\leqslant M_l\leqslant I. \tag{4.10}$$

So, using (2.8), (2.13) and Lemma 4.3, it follows that

$$\|L_l' S_l'^\nu\|^2_{\mathscr{F}\leftarrow s}=\|S_l' L_l' S_l'^\nu H_l\|^2=\|H_l L_l' H_l H_l^{-1}S_l'^\nu H_l\|^2=\|\hat{L}_l'\hat{S}_l'^\nu\|^2$$

$$=\|\hat{S}_l'^{*\nu}\hat{L}_l'^* L_l'\hat{S}_l'^\nu\|=\omega_l'^{-2}\|M_l(I-M_l)^{2\nu}\|\leqslant c^2\eta_0(2\nu)$$

which is the desired theorem.

From (4.9) and (4.10), we have the following result:

**Corollary 4.4.** There exists a constant $c$ such that

$$\|S_l'\|_{s\leftarrow s}\leqslant c,\quad \|S_l'^\nu\|_{s\leftarrow s}\leqslant c.$$

## § 5. A Linearization Equation

Consider one more general linear equation

$$\begin{cases} -\mu\cdot\Delta\boldsymbol{u}+\mathbf{c}\cdot\nabla\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{d}+\operatorname{grad}p=\boldsymbol{f}, & \text{in }\Omega,\\ -\operatorname{div}\boldsymbol{u}=0, & \text{in }\Omega,\\ \boldsymbol{u}=0, & \text{on }\partial\Omega, \end{cases} \tag{5.1}$$

where $\mathbf{c}$ and $\boldsymbol{d}$ denote two sufficiently smooth vector functions in $\Omega$. Its equivalent saddle point problem reads:

Find $[\boldsymbol{u}, p]\in Z$ such that

$$\begin{cases} \mu a_0(\boldsymbol{u}, \boldsymbol{v})+a_1(\boldsymbol{u};\ \boldsymbol{d}, \boldsymbol{v})+a_1(\mathbf{c};\ \boldsymbol{u}, \boldsymbol{v})+b(\boldsymbol{v}, p)=(\boldsymbol{f}, \boldsymbol{v})_0, & \forall\boldsymbol{v}\in X,\\ b(\boldsymbol{u}, q)=0, & \forall q\in Y. \end{cases} \tag{5.2}$$

And (5.2) is equivalent to

$$L[\boldsymbol{u}, p]=[\boldsymbol{f}, 0] \tag{5.3a}$$

with

$$L=\begin{bmatrix} -\mu\Delta+\mathbf{c}\cdot\nabla+(\cdot\nabla)\boldsymbol{d} & \operatorname{grad}\\ -\operatorname{div} & 0 \end{bmatrix}$$

$$=\begin{bmatrix} -\mu\Delta & \operatorname{grad}\\ -\operatorname{div} & 0 \end{bmatrix}+\begin{bmatrix} \mathbf{c}\cdot\nabla+(\cdot\nabla)\boldsymbol{d} & 0\\ 0 & 0 \end{bmatrix}=L'+L''. \tag{5.3b}$$

In contrast with (2.4), we get $\mathbf{c}=\boldsymbol{d}=\boldsymbol{u}$ with $\boldsymbol{u}$ from (2.4). Obviously, if $\mu=1$, $L'$ is the principal term of $L$ and Stokes' operator.

Let

$$a(\boldsymbol{u},\ \boldsymbol{v}) = a_0(\boldsymbol{u},\ \boldsymbol{v}) + a_1(\boldsymbol{u};\ \boldsymbol{d},\ \boldsymbol{v}) + a_1(\boldsymbol{c};\ \boldsymbol{u},\ \boldsymbol{v}). \tag{5.4}$$

The discrete analogue of (5.2) in $Z_l$ reads:

Seek $[\boldsymbol{u}_l,\ p_l] \in Z_l$ such that

$$\begin{cases} a(\boldsymbol{u}_l,\ \boldsymbol{v}_l) + b(\boldsymbol{v}_l,\ p_l) = (\boldsymbol{f},\ \boldsymbol{v}_l)_0, & \forall \boldsymbol{v}_l \in X_l, \\ b(\boldsymbol{u}_l,\ q_l) = 0, & \forall q_l \in Y_l, \end{cases} \tag{5.5}$$

or the slightly more general problem

$$a(\boldsymbol{u}_l,\ \boldsymbol{v}_l) + b(\boldsymbol{v}_l,\ p_l) + b(\boldsymbol{u}_l,\ q_l) = F_l(\boldsymbol{v}_l,\ q_l), \quad \forall [\boldsymbol{v}_l,\ q_l] \in Z_l, \tag{5.6}$$

where $F_l$ is a linear functional on $Z_l$. In particular, on the finest grid,

$$F_l(\boldsymbol{v}_l,\ q_l) = (\boldsymbol{f},\ \boldsymbol{v}_l)_0, \quad \forall [\boldsymbol{v}_l,\ q_l] \in Z_l.$$

Similarly, (5.5) has a corresponding matrix vector system:

$$L_l[\boldsymbol{u}_l,\ p_l] = [\boldsymbol{f}_l,\ 0]. \tag{5.7}$$

The discrete analogues of $L'$ and $L''$ are denoted by $L'_l$ and $L''_l$ respectively.

**Theorem 5.1.** *Define the smoothing iteration*

$$\mathscr{L}_l\{[\boldsymbol{u}_l,\ p_l],\ [\boldsymbol{f}_l,\ g_l]\} = [\boldsymbol{u}_l,\ p_l] - \omega_l^2 H_l^2 L_l^* H_l^2 (L_l[\boldsymbol{u}_l,\ p_l] - [\boldsymbol{f}_l,\ g_l])$$

*for problem (5.7). Then $L_l S_l^\nu$ again satisfies the smoothing property (4.8) with $\eta(\nu) = (1+s)c\sqrt{\eta_0(2\nu)}$ and $s$ being any positive number.*

*Proof.* From

$$S'_l = I - \omega_l'^2 H_l^2 L_l'^* H_l^2 L'_l,$$
$$\hat{L}'_l = H_l L'_l H_l, \quad \omega_l'^{-1} = \|\hat{L}'_l\| \leqslant c,$$

and letting $S''_l = 0$, it follows from Theorem 4.3 that $L'_l S_l'^{(\nu)}$ satisfies (4.8) with $\eta'(\nu) = c\sqrt{\eta_0}(2\nu)$. Since

$$\|L''_l[\boldsymbol{u}_l,\ p_l]\|_{\mathscr{F}} = \|L''_{l,11}\boldsymbol{u}_l\|_{\mathscr{F}} = h_l\|L''_{l,11}\boldsymbol{u}_l\|,$$
$$\|L''_{l,11}\boldsymbol{u}_l\|^2 = (L''_{l,11}\boldsymbol{u}_l,\ L''_{l,11}\boldsymbol{u}_l) = a_1(\boldsymbol{u}_l;\ \boldsymbol{d},\ L''_{l,11}\boldsymbol{u}_l) + a_1(\boldsymbol{c};\ \boldsymbol{u}_l,\ L''_{l,11}\boldsymbol{u}_l)$$
$$\leqslant c\|\boldsymbol{u}_l\|_1\|L''_{l,11}\boldsymbol{u}_l\|_0 \leqslant ch_l^{-1}\|\boldsymbol{u}_l\|_0\|L''_{l,11}\boldsymbol{u}_l\|,$$

we have

$$\|L''_l[\boldsymbol{u}_l,\ p_l]\|_{\mathscr{F}} \leqslant c\|\boldsymbol{u}_l\|_0.$$

Consequently, we obtain

$$\|L''_l\|_{\mathscr{F}\leftarrow\mathscr{U}} = \sup_{[\boldsymbol{u}_l,\ p_l]\in Z_l} \|L''_l[\boldsymbol{u}_l,\ p_l]\|_{\mathscr{F}} / \|[\boldsymbol{u}_l,\ p_l]\|_{\mathscr{U}}$$
$$\leqslant c \sup_{\boldsymbol{u}_l \in X_l} \|\boldsymbol{u}\|_0 / (h_l^{-1}\|\boldsymbol{u}_l\|_0) = ch_l \to 0, \quad l \to \infty.$$

Then by Corollary 4.4, $S''_l = 0$, and the criterion of smoothing property [7, Criterion 6.2.7], we can prove the desired result.

## § 6. The Approximation Property

Let $\Omega$ be smooth enough so that the regularity assumptions (2.2a—b) hold for the saddle point problem (5.2) and its duality problem.

Define an affine space

$$Z_l(0) = \{\boldsymbol{v}_l \in X_l : b(\boldsymbol{v}_l,\ \mu_l) = 0,\ \forall \mu_l \in Y_l\}.$$

**Lemma 6.1.** *Let* $[\boldsymbol{u}, p]$ *and* $[\boldsymbol{u}_l, p_l]$ *be the solutions of problem* (5.2) *and* (5.6) *respectively. Then there exist constants* $c_1$ *and* $c_2$ *so that*

$$\|\boldsymbol{u}-\boldsymbol{u}_l\|_1 \leqslant c_1 \inf_{\boldsymbol{v}_l\in Z_l(0)} \|\boldsymbol{u}-\boldsymbol{v}_l\|_1 + c_2 \inf_{\mu_l\in Y_l} \|p-\mu_l\|_0.$$

*Proof.* Assume

$$\boldsymbol{w}_l = \boldsymbol{v}_l - \boldsymbol{u}_l,$$

then

$$a(\boldsymbol{w}_l, \boldsymbol{w}_l) = a(\boldsymbol{v}_l - \boldsymbol{u}, \boldsymbol{w}_l) + a(\boldsymbol{u}-\boldsymbol{u}_l, \boldsymbol{w}_l).$$

By (5.2) and (5.5), we get

$$a(\boldsymbol{u}-\boldsymbol{u}_l, \boldsymbol{w}_l) = b(\boldsymbol{w}_l, p_l - p).$$

Note that for all $\boldsymbol{v}_l \in Z_l(0)$,

$$b(\boldsymbol{v}_l - \boldsymbol{u}_l, \mu_l) = 0, \quad \forall \mu_l \in Y_l.$$

So

$$a(\boldsymbol{u}-\boldsymbol{u}_l, \boldsymbol{v}_l-\boldsymbol{u}_l) = b(\boldsymbol{v}_l-\boldsymbol{u}_l, \mu_l - p), \quad \forall \mu_l\in Y_l, \ \forall \boldsymbol{v}_l \in Z_l(0).$$

From the coercivity and continuity conditions of $a$ and $b$, it follows that

$$\begin{aligned}\alpha\|\boldsymbol{v}_l-\boldsymbol{u}_l\|_1^2 &\leqslant a(\boldsymbol{v}_l-\boldsymbol{u}_l, \boldsymbol{v}_l-\boldsymbol{u}_l) = a(\boldsymbol{v}_l-\boldsymbol{u}, \boldsymbol{v}_l-\boldsymbol{u}_l) + b(\boldsymbol{v}_l-\boldsymbol{u}_l, \mu_l-p) \\ &\leqslant A\|\boldsymbol{v}_l-\boldsymbol{u}\|_1\|\boldsymbol{v}_l-\boldsymbol{u}_l\|_1 + B\|\boldsymbol{v}_l-\boldsymbol{u}_l\|_1\|\mu_l-p\|_0.\end{aligned}$$

Consequently,

$$\|\boldsymbol{v}_l-\boldsymbol{u}_l\|_1 \leqslant A/\alpha\|\boldsymbol{v}_l-\boldsymbol{u}\| + B/\alpha\|\mu_l-p\|_0.$$

Combine this with

$$\|\boldsymbol{u}-\boldsymbol{u}_l\|_1 \leqslant \|\boldsymbol{u}-\boldsymbol{v}_l\|_1 + \|\boldsymbol{v}_l-\boldsymbol{u}_l\|_1$$

and the lemma follows.

**Lemma 6.2.** *There exists a constant* $c$ *such that*

$$\inf_{\boldsymbol{v}_l\in Z_l(0)} \|\boldsymbol{u}-\boldsymbol{v}_l\|_1 \leqslant c \inf_{\boldsymbol{w}_l\in X_l} \|\boldsymbol{u}-\boldsymbol{w}_l\|_1.$$

*Proof.* It suffices that for all $\boldsymbol{w}_l \in X_l$, there exists the corresponding $\boldsymbol{v}_l \in Z_l(0)$ such that

$$\|\boldsymbol{u}-\boldsymbol{v}_l\|_1 \leqslant c\|\boldsymbol{u}-\boldsymbol{w}_l\|_1.$$

Let $\boldsymbol{w}_l \in X_l$ and $\{\boldsymbol{y}_l, \nu_l\} \in Z_l$ satisfy the following equation:

$$\begin{cases} (\boldsymbol{y}_l, \boldsymbol{v}_l)_1 + b(\boldsymbol{v}_l, \nu_l) = 0, & \forall \boldsymbol{v}_l \in X_l, \\ b(\boldsymbol{y}_l, \mu_l) = b(\boldsymbol{u}-\boldsymbol{w}_l, \mu_l), & \forall \mu_l \in Y_l. \end{cases}$$

Using Brezzi's condition of $b$, it follows that

$$\|\nu_l\|_0 \leqslant 1/r\|\boldsymbol{y}_l\|_1, \quad \|\boldsymbol{y}_l\| \leqslant B/r\|\boldsymbol{u}-\boldsymbol{w}_l\|_1.$$

Let $\boldsymbol{v}_l = \boldsymbol{y}_l + \boldsymbol{w}_l$. Then

$$b(\boldsymbol{v}_l, \mu_l) = b(\boldsymbol{u}-\boldsymbol{w}_l, \mu_l) + b(\boldsymbol{w}_l, \mu_l) = b(\boldsymbol{u}, \mu_l).$$

Therefore $\boldsymbol{v}_l \in Z_l(0)$; consequently,

$$\|\boldsymbol{u}-\boldsymbol{v}_l\|_1 \leqslant \|\boldsymbol{u}-\boldsymbol{w}_l\|_1 + \|\boldsymbol{y}_l\|_1 \leqslant (1+B/r)\|\boldsymbol{u}-\boldsymbol{w}_l\|_1.$$

**Lemma 6.3.** *The following error estimate holds*

$$\|p-p_l\|_0 \leqslant c(\|\boldsymbol{u}-\boldsymbol{u}_l\|_1 + \inf_{\mu_l\in Y_l} \|p-\mu_l\|_0)$$

*with c constant.*

It can be proved similarly to the above lemmas.

**Lemma 6.4.** *Let $u_l \in X_l$ and $p_l \in Y_l$ be orthogonal to $X_{l-1}$ and $Y_{l-1}$, respectively, with respect to $(\cdot, \cdot)_0$, and $[v_l, q_l]$ the solution of*

$$a(v_l, z)+b(z, q_l)+b(v_l, s)=(u_l, z)_0+h_l^2(p_l, s)_0, \quad \forall [z, s]\in Z_l. \tag{6.1}$$

*Then*

$$\|[v_l, q_l]\|_s \leqslant ch_l^2\|[u_l, p_l]\|_s. \tag{6.2}$$

*Proof.* Let $[w_l, r_l]\in Z_l$ be the solution of

$$a(w_l, z)+b(z, r_l)+b(w_l, s)=(u_l, z)_0, \quad \forall [z, s]\in Z_l. \tag{6.3}$$

The continuous analogue of this is:

Find $[w, r]\in Z$ such that

$$a(w, z)+b(z, r)+b(w, s)=(u_l, s)_0, \quad \forall [z, s]\in Z_0. \tag{6.4}$$

Note that

$$\|[v_l, q_l]\|_s \leqslant \|[v_l-w_l, q_l-r_l]\|_s+\|[w_l-w, r_l-r]\|_s+\|[w, r]\|_s. \tag{6.5}$$

We estimate each term on the right-hand side of (6.5). Denote by $\Pi_l^0$ the orthogonal projection of $X^0$ (respectively $Y$) onto $X_l$ (respectively $Y_l$) and by $\Pi_l^1$ the orthogonal projection of $X$ onto $X_l$. It is clear that

$$\Pi_{l-1}^0 u_l=\Pi_{l-1}^0 p_l=0.$$

From (6.1), (6.3) and Brezzi's condition of $b$, we have

$$\gamma\|q_l-r_l\|\leqslant \sup_{z\in X_l} b(z, r_l-q_l)/\|z\|_1 \leqslant A\|v_l-w_l\|, \tag{6.6}$$

and

$$\alpha\|v_l-w_l\|_1^2 \leqslant a(v_l-w_l, v_l-w_l)=b(v_l-w_l, r_l-q_l)\leqslant h_l^2\|p_l\|_0\|r_l-q_l\|_0.$$

Let $[\xi, \eta]\in Z$ be the solution of

$$a(z, \xi)+b(z, \eta)+b(\xi, s)=(v_l-w_l, z)_0, \quad \forall [z, s]\in Z.$$

Then by (6.1), (6.3), (2.2a—b) and (2.5a), it follows that

$$\begin{aligned}\|v_l-w_l\|_0^2 &\leqslant a(v_l-w_l, \xi-\Pi_{l-1}^1\xi)+b(\xi-\Pi_{l-1}^l\xi, q_l-r_l)+b(v_l-w_l, \eta-\Pi_{l-1}^0\eta)\\ &\leqslant ch_l\{\|v_l-w_l\|_1+\|r_l-q_l\|_0\}\|v_l-w_l\|_0.\end{aligned} \tag{6.7}$$

From (6.3), (6.4), (2.2a—b), (2.5a) and Lemmas 1—3, we immediately get

$$\|[w-w_l, r-r_l]\|_{1,0}\leqslant c\inf_{[z_l, s_l]\in Z_l}\|[w-z_l, r-s_l]\|_{1,0}\leqslant ch_l\|u_l\|_0.$$

Combining this with a standard duality argument[4], we get

$$\|[w-w_l, r-r_l]\|_s \leqslant ch_l\|u_l\|_0. \tag{6.8}$$

From (2.5a) and (6.4), we have

$$\beta\|r\|_0\leqslant \sup_{z\in X}[(u_l, z-\Pi_{l-1}^0 z)_0-a(w, z)]/\|z\|_1\leqslant ch_l\|u_l\|_0+A\|w\|_1, \tag{6.9}$$

and

$$\alpha\|w\|_1^2\leqslant a(w, w)=(u_l, w-\Pi_{l-1}^0 w)_0\leqslant ch_l\|u_l\|_0\|w\|_1. \tag{6.10}$$

An argument similar to estimating $\|v_l-w_l\|_0$ yields

$$\|w\|_0^2\leqslant ch_l^2\|u_l\|_0\|w\|_0. \tag{6.11}$$

By combining (6.5)—(6.11) the lemma is proved.

**Lemma 6.5.** *Let $[w_l, r_l]$ be any element of $Z_l$, $\hat{w}_{l-1}\in X_{l-1}$ and $\hat{r}_{l-1}\in Y_{l-1}$ denote the orthogonal projection of $w_l$ onto $X_{l-1}$ and of $r_l$ onto $Y_{l-1}$, respectively, with respect to $(\cdot, \cdot)$. Then*

$$\|[w_l-\hat{w}_{l-1},\ r_l-\hat{r}_{l-1}]\|_z^2\leqslant h_l^{-2}\|L_l[w_l,\ r_l]\|_{\mathscr{F}}\|[v_l,\ q_l]\|_s$$

*with* $[v_l,\ q_l]$ *being the solution of*

$$a(v_l,\ z)+b(z,\ q_l)+b(v_l,\ s)=(w_l-\hat{w}_{l-1},\ z)_0+h_l^2(r_l-\hat{r}_{l-1},\ s)_0,\quad \forall [z,\ s]\in Z_l. \tag{6.12}$$

*Proof.* In equality (6.12), let $[z,\ s]$ equal $[w_l,\ r_l]$. Then by the orthogonality from this lemma and the symmetry of $L_{l,11}$, the right-hand side of (6.12) equals

$$h_l^2\|[w_l-\hat{w}_{l-1},\ r_l-\hat{r}_{l-1}]\|_s,$$

and the left-hand one

$$\begin{aligned}a(v_l,\ w_l)+b(w_l,\ q_l)+b(v_l,\ r_l)&=(L_{l,11}v_l,\ w_l)+(w_l,\ L_{l,12}q_l)+(L_{l,21}v_l,\ r_l)\\&=(L_{l,11}w_l+L_{l,12}r_l,\ v_l)+(L_{l,21}w_l,\ q_l)\leqslant\|L_{l,11}w_l+L_{l,12}r_l\|\|v_l\|+\|L_{l,21}w_l\|\|q_l\|\\&\leqslant(h_l^2\|L_{l,11}w_l+L_{l,12}r_l\|^2+\|L_{l,21}w_l\|^2)^{1/2}(h_l^{-2}\|v_l\|^2+\|q_l\|^2)^{1/2}\\&=\|L_l[w_l,\ r_l]\|_{\mathscr{F}}\|[v_l,\ q_l]\|_s.\end{aligned}$$

Therefore, we get

$$\|[w_l-\hat{w}_{l-1},\ r_l-\hat{r}_{l-1}]\|_z^2\leqslant h_l^{-2}\|L_l[w_l,\ r_l]\|_{\mathscr{F}}\|[v_l,\ q_l]\|_s,$$

which is the desired estimate.

**Theorem 6.6.** *There exists a constant c so that*

$$\|L_l^{-1}-pL_{l-1}^{-1}r\|_{s\leftarrow\mathscr{F}}\leqslant c$$

*with* $r=blockdiag\{r_1,\ r_2\}$ *and* $p=blockdiag\{p_1,\ p_2\}$ *from* (3.3).

*Proof.* Let $[w_l,\ r_l]\in Z_l$ with $[\hat{w}_{l-1},\ \hat{r}_{l-1}]$ from Lemma 6.5 and $[u_{l-1}^*,\ p_{l-1}^*]$ be the solution of

$$L_{l-1}[u_{l-1}^*,\ p_{l-1}^*]=rL_l[w_l,\ r_l]-r[f_l,\ 0]. \tag{6.13}$$

An argument similar to estimating

$$\|[w-w_l,\ r-r_l]\|_s$$

in Lemma 6.4 yields

$$\begin{aligned}\|[w_l-p_1u_{l-1}^*-p_1L_{l-1}^{-1}r_1f_l,\ r_l-p_2p_{l-1}^*]\|_s&\leqslant c\|[w_l-\hat{w}_{l-1},\ r_l-\hat{r}_{l-1}]\|_{1,0}\\&\leqslant c\|[w_l-\hat{w}_{l-1},\ r_l-\hat{r}_{l-1}]\|.\end{aligned}\tag{6.14}$$

From Lemmas 6.4—6.5, it follows that

$$\begin{aligned}\|[w_l-\hat{w}_{l-1},\ r_l-\hat{r}_{l-1}]\|_z^2&\leqslant h_l^{-2}\|L_l[w_l,\ r_l]\|_{\mathscr{F}}\|[v_l,\ q_l]\|_s\\&\leqslant c\|L_l[w_l,\ r_l]\|_{\mathscr{F}}\|[w_l-\hat{w}_{l-1},\ r_l-\hat{r}_{l-1}]\|_s.\end{aligned}$$

So by combining this with (6.14), we obtain

$$\|[w_l-p_1u_{l-1}^*-p_1L_{l-1}^{-1}r_1f_l,\ r_l-p_2p_{l-1}^*]\|_s\leqslant c\|L_l[w_l,\ r_l]\|_{\mathscr{F}}. \tag{6.15}$$

From (6.13), we get

$$[u_{l-1}^*,\ p_{l-1}^*]=L_{l-1}^{-1}rL_l[w_l,\ r_l]-L_{l-1}^{-1}r[f_l,\ 0].$$

By this, it follows that

$$[w_l-p_1u_{l-1}^*-p_1L_{l-1}^{-1}rf_l,\ r_l-p_2p_{l-1}^*]=[L_l^{-1}-pL_{l-1}^{-1}r]L_l[w_l,\ r_l].$$

Substituting this into (6.15) yields

$$\|[L_l^{-1}-pL_{l-1}^{-1}r]L_l[w_l,\ r_l]\|_s\leqslant c\|L_l[w_l,\ r_l]\|_{\mathscr{F}}.$$

Since $[w_l,\ r_l]$ is arbitrary, we finally have

$$\|L_l^{-1}-pL_{l-1}^{-1}r\|_{s\leftarrow\mathscr{F}}\leqslant c.$$

## § 7. The Convergence Theorem

We will now consider the convergence theorem about algorithm (3.2) in Section 3. Before this, we will place restriction on some quantities in algorithm (3.2) so that it is well-defined.

Let $[\tilde{u}_k, \tilde{p}_k]$ from (3.2a) satisfy $[\tilde{u}_k, \tilde{p}_k] \in Z_k(\rho_k/6)$ $(k=0, 1, \cdots, l-1)$ with $0<\rho_k<\varepsilon_k$ $(0\leqslant k<l)$, and $\sigma_k$ from (3.2d), $\rho_k/\varepsilon_k$ $(0\leqslant k<l)$ and $\varphi_0$ be sufficiently small. Then if $[u_l^0, p_l^0] \in Z_l(\rho_l)$, we obtain the following convergence theorem:

**Theorem 7.1.** *Consider Navier–Stokes equation* (1.1) *in a domain smooth enough to satisfy the regularity assumptions* (2.2a—d). *Let* $X_l \subset X$ *and* $Y_l \subset Y$ *be two families of finite element subspaces so that the assumptions* (2.5a—b) *and the discrete Brezzi's condition of b hold. Then there exists a number* $\underline{\nu}$ *so that the multigrid algorithm* (3.2) *converges for* $\nu \geqslant \underline{\nu}$ *and its contraction number is bounded by* $c\sqrt{\eta_0(2\nu)}$ *with* $\eta_0(\nu)$ *from Lemma* 4.2.

*Proof.* From (2.10), (2.11) and the smoother (3.1), we have

$$L_l[u_l^*, p_l^*] = \partial \mathscr{L}_l[u_l^*, p_l^*]/\partial[u_l, p_l],$$

$$S_l = \partial \mathscr{L}_l\{[u_l^*, p_l^*], 0\}/\partial[u_l, p_l] = I - \omega_l^2 H_l^2 L_l^*[u_l^*, p_l^*] H_l^2 L_l[u_l^*, p_l^*].$$

Then by Corollary 4.4, Theorems 5.1 and 6.6 and the convergence theorem of nonlinear multigrid methods [7, Thm. 9.5.12], the desired theorem holds.

From the above, it is shown that the multigrid methods can successfully be used to solve Navier–Stokes equation, and for the smoothing iteration (3.1), the contraction number of the algorithm is bounded by $c\sqrt{\eta_0(2\nu)}$.

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