A MIXED STIFFNESS FINITE ELEMENT METHOD FOR NAVIER-STOKES EQUATION*

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Abstract

The paper is devoted to the study and analysis of the mixed stiffness finite element method for the Navier-Stokes equations, based on a formulation of velocity-pressure-stress deviatorics. The method used low order Lagrange elements, and leads to optimal error order of convergence for velocity, pressure, and stress deviatorics by means of the mesh-dependent norms defined in this paper. The main advantage of the MSFEM is that the streamfunction can not only be employed to satisfy the divergence constraint but stress deviatorics can also be eliminated at the element level so that it is unnecessary to solve a larger algebraic system containing stress multipliers, or to develop a special code for computing the MSFE solutions of the Navier-Stokes equations because we can use the computing codes used in solving the Navier-Stokes equations with the velocity-pressure formulation, or even the computing codes used in solving the problems of solid mechanics.

§ 1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^2$ with a sufficiently smooth boundary $\Gamma(=\partial\Omega)$. Then the Navier-Stokes equations governing the flow of the two-dimensional steady incompressible viscous fluid can be written as follows:

$$(u\cdot\nabla)u-\nu\Delta u+\nabla p=f, \quad \text{in } \Omega, \tag{1.1a}$$

$$\nabla\cdot u=0, \quad \text{in } \Omega, \tag{1.1b}$$

$$u=0, \quad \text{on } \Gamma, \tag{1.1c}$$

where $u=(u_1, u_2)$ are the velocities of flow, p is pressure, $f=(f_1, f_2)$ are body forces, and $\nu(>0)$ is the kinematic viscosity coefficient.

If the nonlinear convection terms in (1.1a) are cut out, then we obtain the so-called Stokes equations:

$$-\nu\Delta u+\nabla p=f, \quad \text{in } \Omega, \tag{1.2a}$$

$$\nabla\cdot u=0, \quad \text{in } \Omega, \tag{1.2b}$$

$$u=0, \quad \text{on } \Gamma. \tag{1.2c}$$

It is well known that considerable efforts have been made by both engineers and mathematicians concerning the construction of finite element solutions of the Stokes problem and the Navier-Stokes problem, see, e.g., [3, 6—13, 17—18, and 23]. What is worth mentioning is Zhou's paper, [23], which considers a new variational formulation of the MSFEM in [20] and [22] to avoid the trouble encountered in solving a larger system of equations owing to the additional viscous stress multiplier variables.

The purpose of this paper is to extend the results about the Stokes problem in

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[23] to the case of the Navier–Stokes problem. By treating the nonlinear terms with the upwind–diffusion scheme presented in [7], we prove the existence and uniqueness of the MSFE solutions of the Navier–Stokes problem, and obtain the optimal error estimates for velocity pressure and stress by virtue of special mesh–dependent norms. And L^2–estimates of velocity and pressure are also optimal. Moreover, stress multipliers can be eliminated at element level. An important step in practical computation is that velocity is first calculated in the divergencefree space, then pressure is found by the velocity obtained. To the author's knowledge, with the velocity–pressure–stress deviatorics formulation of the Navier–Stokes equations, the optimal error estimates in this paper are obtained for the first time.

An outline of the paper is as follows. The remaining part of the present section is to describe some definitions and symbols. Section 2 is devoted to the description of the MSFEM for the Navier–Stokes equations. In Section 3, we discuss the construction of the FE subspaces and prove their properties. Section 4 deals with the abstract results of the saddle–point problem. We get, in Sections 5 and 6, the error estimates of the solutions of the MSFEM for the Navier–Stokes problem in the sense of the mesh–dependent norms and L^2–norm.

Throughout this paper, we use the Sobolev space

$$H^m(\Omega)=\left\{v\in L^2(\Omega);\ \partial^\alpha v=\frac{\partial^{|\alpha|}v}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}\in L^2(\Omega),\ |\alpha|=\alpha_1+\alpha_2\leqslant m\right\}$$

equipped with the following norm and semi–norm:

$$\|v\|_{m,\Omega}=\{\sum_{|\alpha|\leqslant m}\|\partial^\alpha v\|^2_{0,\Omega}\}^{1/2},$$

$$|v|_{m,\Omega}=\{\sum_{|\alpha|=m}|\partial^\alpha v|^2_{0,\Omega}\}^{1/2},$$

where m (>0) is an integer. We denote by $H^{1/2}(\Gamma)$ the trace space which consists of functions defined on the boundary Γ. Moreover, some special spaces will be defined when they appear. As to the details of Sobolev spaces, see [1, 8], and [13—14].

For convenience, we do not make distinction between the vector–valued function and the scalar–valued function. The standard summation convention is empolyed. We denote by n and t the unit normal and tangent vector on some boundary respectively. And c stands for the generic constant unless particular explanation is given.

§ 2. The MSFE Formulation

To facilitate the analysis below, we introduce the following relations:

$$\mu=2\nu,$$

$$\varepsilon(v)=\{\varepsilon_{ij}(v)\}_{1\leqslant i,j\leqslant 2},$$

$$\varepsilon_{ij}(v)=(\partial_i v_j+\partial_j v_i)/2,\quad 1\leqslant i,\ j\leqslant 2.$$

Then the Navier–Stokes equations (1.1) can be rewritten as

$$\sigma=\mu\varepsilon(u),\quad \text{in } \Omega, \tag{2.1a}$$

$$\nabla\cdot u=0,\quad \text{in } \Omega, \tag{2.1b}$$

$$(u\cdot\nabla)u-\nabla\cdot\sigma+\nabla p=f,\quad \text{in } \Omega, \tag{2.1c}$$

$$u=0, \quad \text{on } \Gamma, \tag{2.1d}$$

where $\nabla\cdot\sigma=(\partial_j\sigma_{1j}, \partial_j\sigma_{2j})$.

Remark 2.1. From equations (2.1a) and (2.1b), obviously, we have

$$\operatorname{tr}(\sigma)=\sigma_{11}+\sigma_{22}=0. \tag{2.2}$$

In order to motivate the MSFE formulation of the Navier-Stokes problem, let us first recall the MSFE formulation of the Stokes problem presented in [23].

Consider the Stokes problem:

Find the exact solution (σ, u, p) satisfying

$$\sigma=\mu\varepsilon(u), \quad \text{in } \Omega, \tag{2.3a}$$

$$\nabla\cdot u=0, \quad \text{in } \Omega, \tag{2.3b}$$

$$-\nabla\cdot\sigma+\nabla p=f, \quad \text{in } \Omega, \tag{2.3c}$$

$$u=0, \quad \text{on } \Gamma. \tag{2.3d}$$

By using the following generalized Green's formulas

$$(\tau, \varepsilon(v))=\int_\Gamma v\cdot(\tau\cdot n)ds-(\nabla\cdot\tau, v), \quad \forall v\in[H^1(\Omega)]^2, \forall\tau\in\hat{H}(\mathrm{div};\Omega), \tag{2.4}$$

$$(v, \nabla q)=\int_\Gamma q(v\cdot n)ds-(\nabla\cdot v, q), \quad \forall v\in H(\mathrm{div};\Omega), \forall q\in H^1(\Omega), \tag{2.5}$$

where

$$H(\mathrm{div};\Omega)=\{v\in[L^2(\Omega)]^2;\ \nabla\cdot v\in L^2(\Omega)\},$$

$$\hat{H}(\mathrm{div};\Omega)=\{\tau\in[L^2(\Omega)]^4;\ \partial_j\sigma_{ij}\in L^2(\Omega),\ i=1,2\},$$

we get the weak form of problem (2.3):

(P1) Find solutions (σ, u, p) such that

$$\mu^{-1}(\sigma,\tau)-B(\tau,u)=0, \quad \forall\tau\in H_V, \tag{2.6a}$$

$$B(\sigma,v)-(p,\nabla\cdot v)=(f,v), \quad \forall v\in H_U, \tag{2.6b}$$

$$(q,\nabla\cdot u)=0, \quad \forall q\in H_W, \tag{2.6c}$$

where

$$H_V=\{\tau\in\hat{H}_V;\ \operatorname{tr}(\tau)=0, \text{ in } \Omega\},$$

$$\hat{H}_V=\{\tau=\{\tau_{ij}\}_{1\le i,j\le 2},\ \tau_{12}=\tau_{21}\in L^2(\Omega),\ \tau|_{\Omega_j^*}\in[H^1(\Omega_j^*)]^4, \forall\Omega_j^*\in\mathscr{T}_n^*\},$$

$$H_U=\{v\in H(\mathrm{div};\Omega);\ v\cdot n|_\Gamma=0,\ v|_{\Omega_i}\in[H^1(\Omega_i)]^2, \forall\Omega_i\in\mathscr{T}_n\},$$

$$H_W=\{q\in L^2(\Omega);\ q|_{\Omega_i}\in H^1(\Omega_i), \forall\Omega_i\in\mathscr{T}_n\},$$

$$B(\tau,v)=\sum_j\Big[\int_{\Gamma_j^*}v\cdot(\tau\cdot n)ds-(\nabla\cdot\tau,v)_{\Omega_j^*}\Big], \quad \forall(\tau,v)\in H_V\times H_U,$$

$$(p,q)=\int_\Omega pq\,ds, \quad \forall p,q\in L^2(\Omega),$$

while $\mathscr{T}_n=\{\Omega_i\}$ is, in the sense of [5], a regular triangulation corresponding to velocity and $\mathscr{T}_n^*=\{\Omega_j^*\}$ is the so-called dual partition with respect to stress deviatorics, which may consist of the interior quadrilaterals and the boundary triangles generated by connecting the center of gravity of each velocity element Ω_i with its vertices, as shown in Fig. 1. Other forms of the dual element can be found in [23—24].

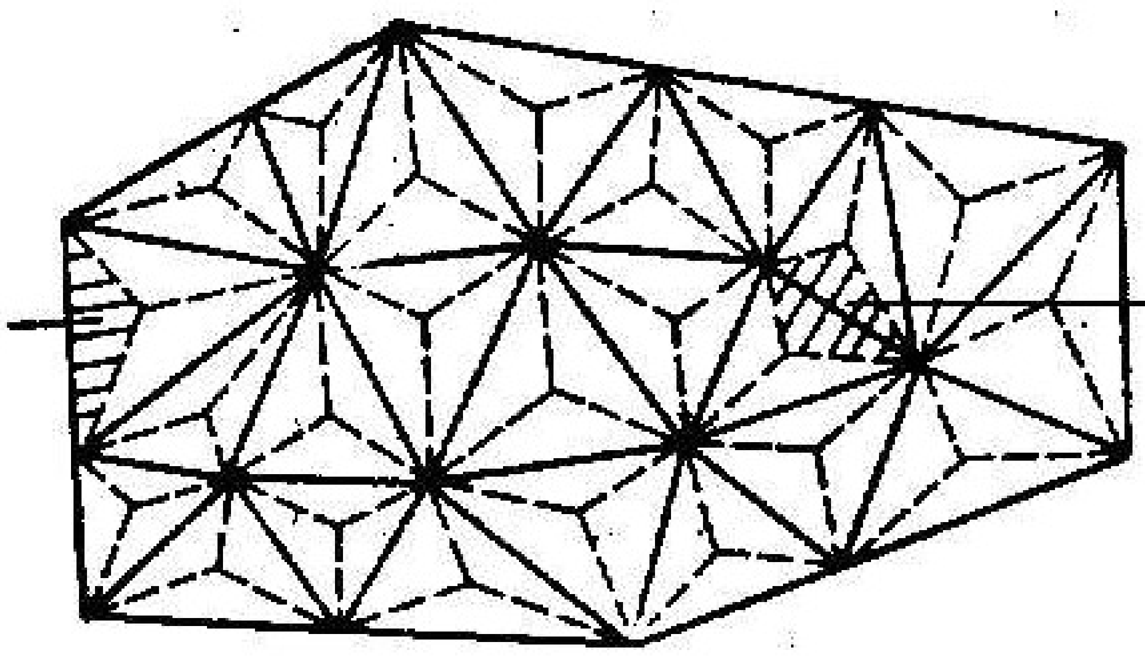

Fig 1
—— velocity partition; —— stress partition

Introducing the divergence-free space

$$\mathring{H}_U=\{v\in H_U;\ \nabla\cdot v=0,\ \text{in}\ \Omega\},$$

we obtain from problem (P1) another variational problem not containing the pressure p:

(P2) Find solution $(\sigma, u)\in H_V\times \mathring{H}_U$, such that

$$\mu^{-1}(\sigma, \tau)-B(\tau, u)=0,\quad \forall\tau\in H_V, \tag{2.7a}$$

$$B(\sigma, v)=(f, v),\quad \forall v\in \mathring{H}_U. \tag{2.7b}$$

Remark 2.2. In problems (P1) and (P2), the velocity solution u does not have to satisfy the condition that $(u\cdot t)$ vanishes on Γ, where t is the unit tangent vecror on Γ, because the relations (2.6a) and (2.7a) imply it.

Define the finite dimensional subspaces $V_h\subset H_V$, $U_h\subset H_U$ and $W_h\subset H_W$, which will be discussed in Section 3. The approximate problems corresponding to problems (P1) and (P2) are respectively:

(P_h1) Find approximate solutions (σ_h, u_h, p_h) satisfying

$$\mu^{-1}(\sigma_h, \tau)-B(\tau, u_h)=0,\quad \forall\tau\in V_h, \tag{2.8a}$$

$$B(\sigma_h, v)-(p_h, \nabla\cdot v)=(f, v),\quad \forall v\in U_h, \tag{2.8b}$$

$$(q, \nabla\cdot u_h)=0,\quad \forall q\in W_h; \tag{2.8c}$$

(P_h2) Find $(\sigma_h, u_h)\in V_h\times \mathring{U}_h$ satisfying

$$\mu^{-1}(\sigma_h, \tau)-B(\tau, u_h)=0,\quad \forall\tau\in V_h, \tag{2.9a}$$

$$B(\sigma_h, v)=(f, v),\quad \forall v\in \mathring{U}_h, \tag{2.9b}$$

where the subspace $\mathring{U}_h$ of U_h is defined by

$$\mathring{U}_h=\{v\in U_h;\ (q, \nabla\cdot v)=0,\ \forall q\in W_h\}.$$

In order to clarify the difference of the property of the MSFEM from general mixed FEMs, we rewrite the formulations (2.8) and (2.9) into:

(P_h1') Find approximate solutions $(u_h, p_h)\in U_h\times W_h$ such that

$$B(\pi_\sigma u_h, v)-(p_h, \nabla\cdot v)=(f, v),\quad \forall v\in U_h, \tag{2.8b'}$$

$$(q, \nabla\cdot u_h)=0,\ \forall q\in W_h; \tag{2.8c'}$$

(P_h2') Find approximate solution $u_h\in \mathring{U}_h$ such that

$$B(\pi_\sigma u_h, v)=(f, v)\quad \forall v\in \mathring{U}_h, \tag{2.9b'}$$

where the mapping π_σ: $u_h\to\sigma_h(=\pi_\sigma u_h)$ is defined element by element such that

$$\mu^{-1}(\sigma_h, \tau)_{\Omega_j^*}=B(\tau, u_h)_{\Omega_j^*},\ \forall\tau\in V_h(\Omega_j^*).$$

It will be seen that the MSFEM makes it possible to generate $\mathring{U}_h$ by using the streamfunctions, and, on the other hand, to find the velocity u_h by solving problem (P_h2') in the same way as the standard FEMs of solid mechanics.

In order to extend equations (2.8) and (2.9) to the case of the Navier-Stokes problem, let us first describe the "upwind" dissipative scheme of [7].

Given a function $v\in U_h$, distinguish, for each element $\Omega_i\in\mathscr{T}_h$, between the part Γ^v_{i-} of the boundary Γ_i of Ω where the flow is in,

$$\Gamma^v_{i-}=\{x\in\Gamma_i;\ v\cdot n(x)\leqslant 0\},$$

and the part Γ^v_{i+} where the flow is out

$$\Gamma^v_{i+}=\{x\in\Gamma_i;\ v\cdot n(x)>0\}.$$

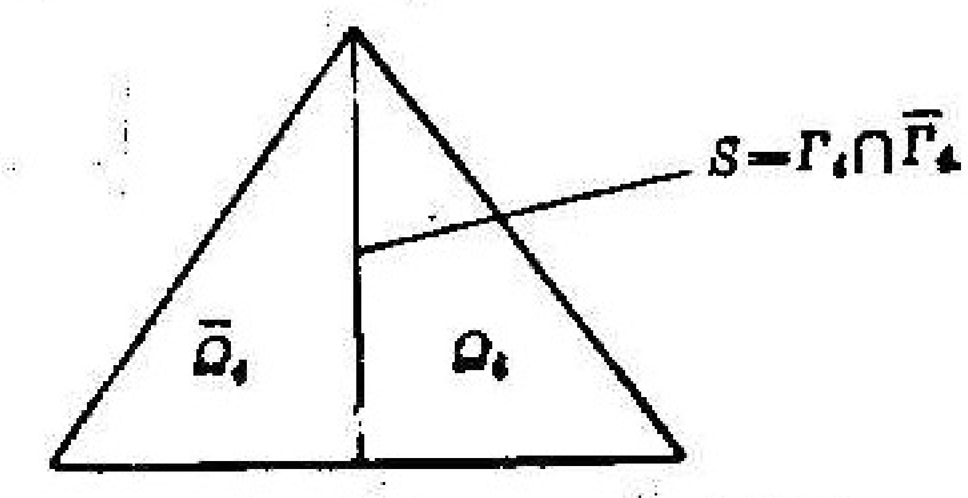

Fig. 2 A pair of adjacent elements

Notice that if $v\in U_h\subset H_U$, $v\cdot n$ is continuous across the interelement boundary. For two adjacent elements Ω_i and $\bar\Omega_i$ with the common side S (see Fig. 2), we have

$$\Gamma^v_{i-}\cap S=\bar\Gamma^v_{i+}\cap S.$$

The trilinear form generated by the convection terms can be defined by using the idea of the "upwind" dissipative scheme:

$$b^*(w,u,v)=\sum_k\Big[\int_{\Gamma_k} u\cdot n\hat w_j v_j\,ds-\int_{\Omega_k}(\partial_i w_i u\cdot v+w_i u_j\partial_i v_j)dx\Big],\quad \forall u,v,w\in H_U,\tag{2.10}$$

$$\hat w|_{\Gamma_k}=\begin{cases}\text{trace of } w|_{\Gamma_k} \text{ on } \Gamma^w_{k+},\\ w^e=\text{trace of } w|_{\bar\Omega_k} \text{ on } \Gamma^w_{i-}\cap S.\end{cases}\tag{2.11}$$

We can now write out the MSFE formulation of the Navier-Stokes equations:

(M_h1) Find $(\sigma_h, u_h, p_h)\in V_h\times U_h\times W_h$ satisfying

$$\mu^{-1}(\sigma_h,\tau)-B(\tau,u_h)=0,\quad \forall\tau\in V_h,\tag{2.12a}$$

$$b^*(u_h,u_h,v)+B(\sigma_h,v)-(p_h,\nabla\cdot v)=(f,v),\quad \forall v\in U_h,\tag{2.12b}$$

$$(q,\nabla\cdot u_h)=0,\quad \forall q\in W_h;\tag{2.12c}$$

(M_h2) Find (σ_h, u_h) $V_h\times\mathring{U}_h$ such that

$$\mu^{-1}(\sigma_h,\tau)-B(\tau,u_h)=0,\quad \forall\tau\in V_h,\tag{2.13a}$$

$$b^*(u_h,u_h,v)+B(\sigma_h,v)=(f,v),\quad \forall v\in\mathring{U}_h.\tag{2.13b}$$

We say that the MSFE scheme (2.12) is strongly consistent, if the exact solutions (σ, u, p) of the continuous problem (P1) strickly satisfy (2.11). Then it is easy to prove that scheme (2.12) is strongly consistent if $f\in[L^2(\Omega)]^2$, and so is scheme (2.13) if we further assume that $\mathring{U}_h\subset\mathring{H}_h$ holds.

Remark 2.3. Because the basic functions of subspace $V_h(\subset H_V)$ have a support limited to one element, the degrees of freedom corresponding to the stress deviatorics can be eliminated at the element level, so that we can avoid the trouble encountered in solving a larger system of equations including Lagrange multiplier. Undoubtedly, the MSFEM is quite attractive to those engaged in practical computation.

Remark 2.4. Since the subspace $\mathring{U}_h$ can be generated by using streamfunctions, approximate velocity u_h should be obtained through solving problem (M_h2) by eliminating σ_h at the element level.

§ 3. Finite Element Subspaces

In this section, we discuss the definitions and some properties of the FE subspaces V_h, U_h, and W_h introduced in Section 2.

First, we define for each interior element $\Omega_j^* \subset \mathscr{T}_h^*$ and a pair of elements $(\Omega_L, \Omega_R) \subset \mathscr{T}_h$ (see Fig. 3):

$$H_{\Omega_j^*} = \left\{\tau \in [c^0(\Omega_j^*)]^4; \tau_{11} = -\tau_{22} = \begin{cases} c_{11}+c_{12}^R l_j, & \text{in } \Omega_j^* \cap \Omega_R, \\ c_{11}+c_{12}^L l_j, & \text{in } \Omega_j^* \cap \Omega_L, \end{cases}\right.$$

$$\left.\tau_{12} = \tau_{21} = \begin{cases} c_{21}+c_{22}^R l_j & \text{in } \Omega_j^* \cap \Omega_R \\ c_{21}+c_{22}^L l_j, & \text{in } \Omega_j^* \cap \Omega_L, \end{cases} \forall c_{ij},\ c_{ij}^R,\ c_{ij}^L \in R, \right\}$$

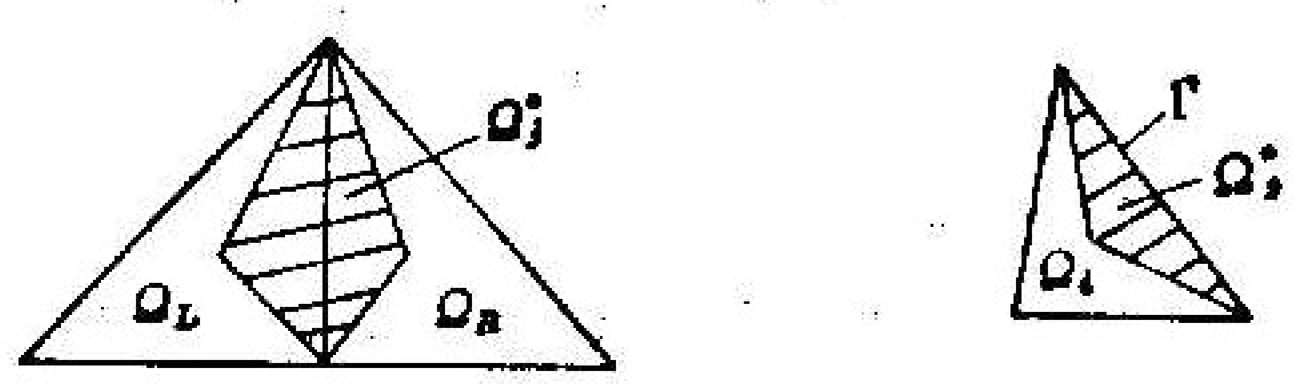

Fig. 3 Structure of Ω_j^*

where $l_j(x)$ is a linear function satisfying

$$l_j(x) = 0, \quad \forall x \in \Gamma_R \cap \Gamma_L.$$

Then, for each boundary element Ω_j^*, we define

$$H_{\Omega_j^*} = \{\tau \in [c^0(\Omega_j^*)]^4;\ \tau_{11} = -\tau_{22} = c_{11}+c_{12}l_j,\ \tau_{12} = \tau_{21} = c_{21}+c_{22}l_j,\ \forall c_{ij} \in R\},$$

where the linear function l_j satisfies

$$l_j(x) = 0, \quad \forall x \in \Gamma_j^* \cap \Gamma.$$

We now define the subspaces

$$V_h = \{\tau \in H_V;\ \tau|_{\Omega_j^*} \in H_{\Omega_j^*},\ \forall \Omega_j^* \in \mathscr{T}_h^*\},$$

$$U_h = \{v \in H_U;\ v|_{\Omega_i} \in [P_1(\Omega_i)]^2,\ \forall \Omega_i \in \mathscr{T}_h\},$$

$$W_h = \{q \in H_W;\ q|_{\Omega_i} \in R,\ \forall \Omega_i \in \mathscr{T}_h\},$$

equipped with the following mesh-dependent norms, respectively,

$$\|\tau\|_V = \{\|\tau\|_{0,\Omega}^2 + \sum_j h_j^{*2} |\tau|_{1,\Omega_j^*}^2\}^{1/2}, \quad h_j^* = \operatorname{diam}(\Omega_j^*),$$

$$\|v\|_{U_1} = \left\{\sum_i \left[\|\varepsilon(v)\|_{0,\Omega_i}^2 + h_i^{-1}\int_{\Gamma_i} |(v_+ - v_-)\cdot t|^2 ds\right]\right\}^{1/2}, \quad h_i = \operatorname{diam}(\Omega_i),$$

$$\|v\|_{U_2} = \{\|v\|_{0,\Omega}^2 + \sum_i |v|_{1,\Omega_i}^2\}^{1/2},$$

$$\|v\|_{U_3} = \{\|v\|_{0,\Omega}^2 + \sum_i h_i^2 |v|_{1,\Omega_i}^2\}^{1/2},$$

$$\|q\|_W = \left\{\sum_i \left[\|\nabla q\|_{0,\Omega_i}^2 - h_i^{-1}\int_{\Gamma_i\setminus\Gamma} |q_+ - q_-|^2 ds\right]\right\}^{1/2},$$

where $(\phi_+ - \phi_-)$ is the jump value of discontinuous function ϕ across the interior element boundary, and

$$\phi_+ - \phi_- = \phi, \text{ on } \Gamma_i \cap \Gamma, \quad \forall \Omega_i \in \mathscr{T}_h.$$

Moreover, we always think that FE Ω_i and Ω_j^* have all features of regular FE in the sense of [5]. Then, we have the following results.

Proposition 3.1. A function $v|_{\Omega_i}$ $(v \in U_h)$ is uniquely determined by six degrees of freedom, i.e., the values of $(v \cdot n)$ at two distinct points of each side of triangle $\Omega_i \in \mathscr{T}_h$.

Proof. For any $\Omega_i \in \mathscr{T}_h$, and any $v \in U_h$, $v|_{\Omega_i}$ is a linear function. Since $\dim[P_1(\Omega_i)]^2 = 6$, the dimension of the space $[P_1(\Omega_i)]^2$ is equal to the given number of degrees of freedom, and we only need to prove that a function $\bar{v}_h \in [P_1(\Omega_i)]^2$ will vanish, if it satisfies the condition that the values of $(\bar{v}_h \cdot n)$ at two distinct points on each side of the element Ω_i are zero, where $\bar{v}_h = v|_{\Omega_i}$.

In fact, from $\bar{v}_h \cdot n \in P_1(\Omega_i)$, we know that there are only six unknown parameters on Γ_i. Therefore, $\bar{v}_h \cdot n|_{(\Gamma_i)_k=0}$, $1 \leqslant k \leqslant 3$, implies $\bar{v}_h \cdot n|_{\Gamma_i=0}$. Using Green's formula, we have

$$\int_{\Omega_i} q_h \nabla \cdot \bar{v}_h \, dx = \int_{\Gamma_i} q(\bar{v}_h \cdot n) ds - \int_{\Omega_i} \bar{v}_h \cdot \nabla q_h = 0, \quad \forall q_h \in W_h.$$

Hence, because $\nabla \cdot \bar{v}_h \in P_1(\Omega_i)$, we have

$$\nabla \cdot \bar{v}_h = 0, \quad \text{in } \Omega_i.$$

Furthermore, there exists a streamfunction $\phi \in P_2(\Omega_i)$, uniquely determined up to an additional constant, such that

$$\bar{v}_h = \text{curl } \phi = (\partial_2 \phi, \ -\partial_1 \phi).$$

Note that $\bar{v}_h \cdot n = \partial_t \phi = 0$, where ∂_t stands for the tangential derivative along Γ_i. Thus we may assume that $\phi = 0$ on Γ_i and

$$\hat{\phi} = c\lambda_1 \lambda_2 \lambda_3,$$

where $\hat{\phi}$ is a function defined in $\hat{\Omega}$ by means of the affine transform for ϕ, λ_i, $1 \leqslant i \leqslant 3$, are area coordiantes, and c is a constant to be determined. Since $\lambda_1 \lambda_2 \lambda_3 \in P_3(\Omega)$, c must be zero. Hence $\hat{\phi} = 0$ and $\hat{v} = 0$. Returing to the practical element Ω_i, we have proved that $\bar{v}_h = 0$.

Proposition 3.2. We have the following relation

$$\{v_h \in U_h;\ \nabla \cdot v_h = 0\} \equiv \{v_h = \text{curl } \phi_h;\ \phi_h \in \Phi_h\},$$

where the streamfunction subspace Φ_h consists of piecewise polynomials of degree $\leqslant 2$.

Proof. Obviously, we have

$$\{v_h = \text{curl } \phi_h;\ \phi_h \in \Phi_h\} \subset \{v_h \equiv U_h;\ \nabla \cdot v_h = 0\}.$$

We now prove the inverse result. For any $v_h \in U_h$, we have

$$\partial_1 v_{h1} + \partial_2 v_{h2} = 0,$$

$$v_{h1} = -\int \partial_2 v_{h2} \, dx_1 + \lambda(x_2) = -\partial_2 \left[\int v_{h2} \, dx_1 - \int \lambda(x_2) dx_2 \right].$$

Considering that

$$v_{h1} = a_1 x_1 + a_2 x_2 + a_3,$$
$$v_{h2} = b_1 x_1 + b_2 x_2 + b_3,$$

and defining

$$\phi_h = \int^{x_1} v_{h2} \, dx_1 - \int^{x_2} \lambda(x_2) dx_2$$

we get

$$\partial_1 \phi_h = v_{h2},$$
$$-\partial_2 \phi_h = v_{h1}.$$

We can set

$$\phi_h = c_1 x_1^2 + c_2 x_1 x_2 + c_3 x_2^2 + c_4 x_1 + c_5 x_2 + c.$$

Then, the corresponding velocities v_h are

$$v_{h1} = -(2c_1x_1 + c_2x_2 + c_4),$$
$$v_{h2} = c_2x_1 + 2c_3x_2 + c_5.$$

Since $v_h|_{\Omega_i}$ are uniquely determined by six values of $(v_h \cdot n)$ at the sides of the element Ω_i, we obtain a streamfunction ϕ_h, exactly to an additional constant, and the following relation

$$v_h = \text{curl } \phi_h.$$

The proposition is proved.

The proofs of the following Lemmas can be found in [20—22].

Lemma 3.1. *There exists a constan c, independent of h, such that*

$$\int_{\Gamma_i} v^2 ds \leqslant c h_i^{-1} \|v\|_{0,\Omega_i} (h_i |v|_{1,\Omega_i} + \|v\|_{0,\Omega_i}), \quad \forall v \in [H^1(\Omega_i)]^2. \tag{3.1}$$

Lemma 3.2. *There exists a constant c independent of h such that*

$$\|v\|_{0,\Omega} \leqslant c \|v\|_{U_i}, \quad \forall v \in H_{U_i}, \ 1 \leqslant i \leqslant 3. \tag{3.2}$$

Lemma 3.3. *There exist constants $\bar{c}_1$ and $\bar{c}_2$ independent of h such that*

$$\|v\|_{U_i} \leqslant \bar{c}_i h^{-1} \|v\|_{U_3}, \quad \forall v \in H_{U_3}, \ i = 1, 2. \tag{3.3}$$

Lemma 3.4. *There exists a constant c, independent of h, such that*

$$\|q\|_{L^2(\Omega)/R} \leqslant c \|q\|_W = c \|q\|_{W/R}, \ \forall q \in H_W. \tag{3.4}$$

Lemma 3.5. *From the definition of Section 2, we have the following results:*

(i) *the bilinear functionals $B(\cdot, \cdot)$ and $(q, \nabla \cdot v)$ are continuous on $H_V \times H_{U_1}$ and $H_W \times H_{U_2}$ respectively, i.e., there exist different constants c independent of h such that*

$$|B(\tau, v)| \leqslant c \|\tau\|_V \|v\|_{U_1}, \quad \forall (\tau, v) \in H_V \times H_{U_1}, \tag{3.5}$$

$$|(q, \nabla \cdot v)| \leqslant c \|q\|_W \|v\|_{U_2}, \quad \forall (q, v) \in H_W \times H_{U_2}; \tag{3.6}$$

(ii) *the trilinear form $b^*(\cdot, \cdot, \cdot)$ continues on $[H_{U_1}]^3$, i.e., there exists a constant c, independent of h, such that*

$$|b^*(u, v, w)| \leqslant c \|u\|_{U_1} \|v\|_{U_1} \|w\|_{U_1}, \ \forall u, v, w \in H_{U_1}. \tag{3.7}$$

Proof. (i) From the definition, for any $(\tau, v) \in H_V \times H_{U_1}$, we have

$$B(\tau, v) = \sum_j \left[\int_{\Gamma_j^*} v(\tau \cdot n) ds - (\nabla \cdot \tau, v)_{\Omega_j^*} \right]$$

$$= \sum_i \left[\sum_j (\tau, \varepsilon(v))_{\Omega_j^* \cap \Omega_i} - \int_{\Gamma_i} (v \cdot t)(\tau \cdot n \cdot t) ds \right],$$

where the conditions that $(v \cdot n)$ continues on $\Gamma_i \backslash \Gamma$ and $(v \cdot n)$ vanishes on Γ are used. Using the Schwarz inequality and Lemma 3.1, we get

$$B(\tau, v) \leqslant \sum_i \left[\sum_j \|\tau\|_{0,\Omega_j^* \cap \Omega_i} \|\varepsilon(v)\|_{0,\Omega_j^* \cap \Omega_i} + \left(h_i^{-1} \int_{\Gamma_i} |(v_+ - v_-) \cdot \tau|^2 ds \right)^{1/2} \left(h_i \int_{\Gamma_i} \tau^2 ds \right)^{1/2} \right]$$

$$\leqslant c \left\{ \sum_i \left[\|\varepsilon(v)\|_{0,\Omega_i}^2 + h_i^{-1} \int_{\Gamma_i} |(v_+ - v_-) \cdot t|^2 ds \right] \right\}^{1/2} \{ \|\tau\|_{0,\Omega}^2 + \sum_j h_j^* |\tau|_{1,\Omega_j^*}^2 \}^{1/2}$$

$$\leqslant c \|\tau\|_V \|v\|_{U_1}, \quad \forall (\tau, v) \in H_V \times H_{U_1},$$

which gives the desired inequality (3.5).

On the other hand, we have

$$|(q,\ \nabla\cdot v)|=\left|\sum_i\left[\int_{\Gamma_i\setminus\Gamma} q(v\cdot n)ds-(v,\ \nabla q)_{\Omega_i}\right]\right|$$
$$\leqslant\Big(\sum_i\int_{\Gamma_i\setminus\Gamma}|q_+-q_-|^2ds\Big)^{1/2}\Big(\int_{\Gamma_i}v^2ds\Big)^{1/2}+(\|v\|^2_{0,\Omega})^{1/2}(\sum_i\|\nabla q\|^2_{0,\Omega_i})^{1/2}$$
$$\leqslant c\Big(\sum_i h_i^{-1}\int_{\Gamma_i\setminus\Gamma}|q_+-q_-|^2ds\Big)^{1/2}(\|v\|^2_{0,\Omega}+\sum_i h_i^2|v|^2_{1,\Omega_i})^{1/2}$$
$$+\|v\|_{0,\Omega}(\sum_i\|\nabla q\|^2_{0,\Omega_i})^{1/2}$$
$$\leqslant c\|q\|_W\|v\|_{U_2},\quad \forall(q,\ v)\in H_W\times H_{U_2}.$$

This proves (3.6).

(ii) For any $u,\ v,\ w\in H_{U_2}$, $u|_{\Omega_k}$, $v|_{\Omega_k}$ and $w|_{\Omega_k}$ belong to $[H^1(\Omega_k)]^2$, $\forall\Omega_k\in\mathscr{T}_h$. Using Sobolev inequalities and the trace theorem (see [1]), we obtain

$$|b^*(v,\ u,\ w)|\leqslant\sum_k\left[\int_{\Gamma_k}|u\cdot n|\,|v\cdot w|ds+\int_{\Omega_k}|u_iv_j\partial_iw_j+u_iw_i\partial_jw_j|dx\right]$$
$$\leqslant c\sum_k[\|\nabla\cdot u\|_{0,\Omega_k}\|v\|_{0,4,\Gamma_k}\|w\|_{0,4,\Gamma_k}+\|u\|_{0,4,\Omega_k}\|v\|_{0,4,\Omega_k}\|\nabla w\|_{0,\Omega_k}$$
$$+\|u\|_{0,4,\Omega_k}\|\nabla v\|_{0,\Omega_k}\|w\|_{0,4,\Omega_k}]$$
$$\leqslant c(\sum_k\|\nabla\cdot u\|^2_{0,\Omega_k})^{1/2}(\sum_k\|v\|^2_{0,4,\Gamma_k})^{1/2}(\sum_k\|w\|^2_{0,4,\Gamma_k})^{1/2}$$
$$+(\sum_k\|u\|^2_{0,4,\Omega_k})^{1/4}[(\sum_k\|v\|^2_{0,4,\Omega_k})^{1/4}(\sum_k\|\nabla w\|^2_{0,\Omega_k})^{1/2}$$
$$+(\sum_k\|\nabla v\|_{0,\Omega_k})^{1/2}(\sum_k\|w\|^2_{0,4,\Omega_k})^{1/4}]$$
$$\leqslant c(\sum_k\|u\|^2_{1,\Omega_k})^{1/2}(\sum_k\|v\|^2_{1,\Omega_k})^{1/2}(\sum_k\|w\|^2_{1,\Omega_k})^{1/2}$$
$$\leqslant c\|u\|_{U_2}\|v\|_{U_2}\|w\|_{U_2},\ \forall u,\ v,\ w\in H_{U_2},$$

which implies inequality (3.7). Therefore, the Lemma is proved.

Lemma 3.6. *There exist different constants c such that*

$$\inf_{v\in U_h}\|u-v\|_{U_3}\leqslant ch^s\|u\|_{s,\Omega},\ s=1,\ 2,\ \forall u\in H_{U_3}\cap[H^s(\Omega)]^2, \tag{3.8}$$
$$\inf_{v\in U_h}\|u-v\|_{U_2}\leqslant ch^{s-1}\|u\|_{s,\Omega},\ s=1,\ 2,\ \forall u\in H_{U_2}\cap[H^s(\Omega)]^2, \tag{3.9}$$
$$\inf_{v\in U_h}\|u-v\|_{U_1}\leqslant ch^{s-1}\|u\|_{s,\Omega},\ s=1,\ 2,\ \forall u\in H_{U_1}\cap[H^s(\Omega)]^2, \tag{3.10}$$
$$\inf_{\tau\in V_h}\|\sigma-\tau\|_V<ch(\sum_j\|\sigma\|^2_{1,\Omega_j})^{1/2},\quad \forall\sigma\in H_V, \tag{3.11}$$
$$\inf_{q\in W}(\|p-q\|_{L^2(\Omega)/R}+h\|p-q\|_W)\leqslant ch\|p\|_{H^1(\Omega)/R},\ \forall p\in H^1(\Omega)/R, \tag{3.12}$$

where c is independent of h.

Proof. By using Lemma 3.3 and the standard results of approximate theory in Sobolev spaces (see, e.g., [5]), we can obtain the interpolant error estimates (3.8)—(3.12) immediately.

§ 4. Some Abstract Results

Let U, V and W be three real vector spaces, and let $H_{U_i}=\{U;\ \|\cdot\|_{U_i}\}$, $1\leqslant i\leqslant3$, $H_V=\{V;\ \|\cdot\|_V\}$ and $H_W=\{W;\ \|\cdot\|_W\}$ be five Hilbert spaces equipped with scalar product norms $\|\cdot\|_{U_i}$, $1\leqslant i\leqslant3$, $\|\cdot\|_V$ and $\|\cdot\|_W$ respectively. We also assume that

$$H_{U_1}\subsetneq H_{U_2}\subsetneq[L^2(\Omega)]^2,\quad i=1,\ 2,$$

$$H_W \subsetneq L^2(\Omega)/R.$$

Define that $d(\cdot, \cdot)$, $e_1(\cdot, \cdot)$ and $e_2(\cdot, \cdot)$ are continuous bilinear forms on $H_V\times H_V$, $H_V\times H_{U_1}$, and $H_W\times H_{U_2}$ respectively, while $b^*(\cdot, \cdot, \cdot)$ a continuous trilinear form on $[H_{U_2}]^3$, i.e., there exist constants β and c, independent of h, such that

$$d(\sigma, \tau)\leqslant\beta^2\|\sigma\|_V\|\tau\|_V, \quad \forall(\sigma, \tau)\in H_V\times H_V,$$

$$e_1(\tau, v)\leqslant c\|\tau\|_V\|v\|_{U_1}, \quad \forall(\tau, v)\in H_V\times H_{U_1},$$

$$e_2(q, v)\leqslant c\|q\|_W\|v\|_{U_2}, \quad \forall(q, v)\in H_W\times H_{U_2},$$

$$b^*(u, v, w)\leqslant c\|u\|_{U_2}\|v\|_{U_2}\|w\|_{U_2}, \quad (u, v, w)\in H_{U_2}\times H_{U_2}\times H_{U_2}.$$

For any given $(\sigma, u, p)\in H_V\times H_U\times H_W$, we consider the following abstract approximate saddle-point problem:

Find $(\sigma_h, u_h, p_h)\in V_h\times U_h\times W_h$, where V_h, U_h, and W_h are finite dimensional subspaces of H_V, H_U and H_W respectively, such that

$$d(\sigma_h, \tau)-e_1(\tau, u_h)=\langle f_1, \tau\rangle_V, \quad \forall\tau\in V_h, \tag{4.1a}$$

$$b^*(u_h, u_h, v)+e_1(\sigma_h, v)-e_2(p_h, v)=\langle f_2, v\rangle_U, \quad \forall v\in U_h, \tag{4.1b}$$

$$e_2(q, u_h)=0, \quad \forall q\in W_h, \tag{4.1c}$$

where $\langle\cdot, \cdot\rangle_X$ is the duality product on $X'\times X$, and h is an approximating parameter.

In order to obtain the existence, uniqueness and convergence error estimates of the solution of problem (4.1), we make the following hypotheses:

(H. 1) there exist constants c_i (>0), $1\leqslant i\leqslant 3$, independent of h, such that

$$\sup_{\tau\in V_h}\frac{e_1(\tau, v)}{\|\tau\|_V}\geqslant c_1\|v\|_{U_1}, \quad \forall v\in Z_{2h}, \tag{4.2a}$$

$$\sup_{v\in U_h}\frac{e_2(q, v)}{\|v\|_{U_2}}\geqslant c_2\|q\|_W, \quad \forall q\in W_h, \tag{4.2b}$$

$$\|v\|_{U_2}\leqslant c_3\|v\|_{U_1}, \quad \forall v\in Z_{2h}, \tag{4.2c}$$

where the subspace Z_{2h} of U_h is defined by

$$Z_{2h}=\{v\in U_h;\ e_2(q, v)=0, \quad \forall q\in W_h\}.$$

Inequalities (4.2a) and (4.2b) are called the Babuska–Brizzi inequalities (for details of the B–B inequality, see [2, 4, and 22]);

(H. 2) the symmetric bilinear form $d(\cdot, \cdot)$ is Z_{1h}–elliptic, i.e., there exists a constant $\alpha(>0)$ independent of h such that

$$d(\sigma, \sigma)\geqslant\alpha^2\|U\|_V^2, \quad \forall\sigma\in Z_{1h}, \tag{4.3a}$$

where

$$Z_{1h}=\{\tau\in V_h;\ e_1(\tau, v)=0, \forall v\in Z_{2h}\}.$$

And for the trilinear form $b^*(\cdot, \cdot, \cdot)$, we have

$$b^*(v, v, v)\geqslant 0, \quad \forall v\in Z_{2h}; \tag{4.3b}$$

(H. 3) the generalized Allmann–Johnson condition (see [21])

$$Z_{2h}\subset Z_2=\{v\in H_{U_2};\ e_2(q, v)=0, \forall q\in H_W\}$$

holds.

Denote by $\psi_{ih}(1\leqslant i\leqslant 3)$ the orthogonal projectors from H_V onto V_h, H_U onto U_h and H_W onto W_h respectively. Then we have the first identifying criterion of the Babuska-Brizzi inequality[21]. The following results can be found in [21].

Lemma 4.1. *Inequality* (4.2a) *holds iff*

(i) *there exists a linear operator* $\pi_{1h}\in\mathscr{L}(H_V, V_h)$ *such that*

$$\|\Pi_{1h}\tau\|_V\leqslant c\|\tau\|_V,\quad \forall\tau\in H_V, \tag{4.4a}$$

$$e_1(\tau, v_h)=e_1(\Pi_{1h}\tau, v_h),\quad \forall\tau\in H_V, \forall v_h\in U_h; \tag{4.4b}$$

(ii)
$$\sup_{\tau\in H_V}\frac{e_1(\tau, v_h)}{\|\tau\|_V}\geqslant c\|v_h\|_{U_1}, \forall v=U_h, \tag{4.5}$$

where c is a positive constant independent of h, τ and V_h.

Corollary 4.1. The operator Π_{1h} defined in Lemma 4.1 has the property that there exists a constant c independent of h and σ such that

$$\|\sigma-\Pi_{1h}\sigma\|_V\leqslant c\inf_{\tau\in V_h}\|\sigma-\tau\|_V,\quad \forall\sigma\in H_V. \tag{4.6}$$

Lemma 4.2. *Inequality* (4.2b) *holds iff*

(i) *there exists a linear operator* $\Pi_{2h}\in\mathscr{L}(H_{U_2}, U_h)$ *such that*

$$\|\Pi_{2h}v\|_{U_2}\leqslant c\|v\|_{U_2},\quad \forall v\in H_{U_2}, \tag{4.7a}$$

$$e_2(q, v)=e_2(q, \Pi_{2h}v),\quad \forall v\in H_{U_2}, \forall q\in W_h; \tag{4.7b}$$

(ii) *there exists a constant c independent of h such that*

$$\sup_{v\in H_{U_2}}\frac{e_2(q, v)}{\|v\|_{U_2}}\leqslant c\|q\|_W,\quad \forall q\in W_h. \tag{4.8}$$

Corollary 4.2. The operator Π_{2h} has the property that there exists a constant c, independent of h, such that

$$\|u-\Pi_{2h}u\|_{U_2}\leqslant c\inf_{v\in U_h}\|u-v\|_{U_2},\quad \forall u\in H_{U_2}. \tag{4.9}$$

The proof of the following lemma can also be found in [21].

Lemma 4.3. *If the* FE *partition families are regular in the sense of* [5], *then for the saddle-point scheme* (4.1), *the relations* (4.5) *and* (4.8) *always hold as h is small enough.*

We can now obtain the basic abstract results.

Theorem 4.1. *If conditions* (H. 1)—(H. 3) *hold, and if α in* (4.3a) *is sufficiently large, or $\|f_2\|^*$, the norm of f_2 in $(H_{U_2})'$, is small enough, then problem* (4.1) *has unique solutions $(\sigma_h, u_h, p_h)\in V_h\times U_h\times W_h$. And we have the following error estimates:*

$$\|\sigma-\sigma_h\|_V+\|u-u_h\|_{U_1}\leqslant c\Big\{\inf_{\tau\in V_h}\|\sigma-\tau\|_V+\inf_{v\in U_h}[\|u-v\|_{U_1}+|b^*(u_h-v)-b_h^*(u_h-v)|^{1/2}]$$
$$+\sup_{v\in U_h}\frac{|b^*(v)-b_h^*(v)|}{\|v\|_{U_1}}\Big\}, \tag{4.10}$$

$$\|p-p_h\|_W\leqslant c\Big\{\inf_{q\in W_h}\|p-q\|_W+\|\sigma-\sigma_h\|_V\sup_{v_h\in U_h}\frac{\|v\|_{U_1}}{\|v\|_{U_2}}$$
$$+\sup_{v\in U_h}\frac{|b^*(v)-b_h^*(v)|}{\|v\|_{U_1}}\frac{\|v\|_{U_1}}{\|v\|_{U_2}}\Big\}, \tag{4.11}$$

where $(\sigma, u, p)\in H_V\times H_U\times H_W$ are the solutions of the continuous problem corresponding to the approximate problem (4.1), *and*

$$b^*(v)=b^*(u, u, v);\ b_h^*(v)=(u_h, u_h, v).$$

Proof. From Theorem 2.6 of Chapter IV, [8], we only discuss the following problem without the variable p_h.

Find $(\sigma_h, u_h)\in V_h\times Z_{2h}$ such that

$$d(\sigma_h, \tau)-e_1(\tau, u_h)=d(\sigma, \tau)-e_1(\tau, u), \quad \forall\tau\in V_h, \tag{4.12a}$$

$$b_h^*(v)+e_1(\sigma_h, v)=b^*(v)+e_1(\sigma, v), \quad \forall v\in Z_{2h}. \tag{4.12b}$$

We will complete the proof of Theorem 4.1 in five steps.

Step 1. For any $\tau\in V_h$, that condition (H. 1) holds implies that there exists a constant c independent of h and τ such that

$$\|\tau\|_V\leqslant c\left\{(1+\alpha^{-1}\beta)\sup_{v\in U_h}\frac{e_1(\tau, v)}{\|v\|_{U_1}}+\alpha^{-1}d^{1/2}(\tau, \tau)\right\}, \tag{4.13}$$

where α and β are constants defined previously in this section.

$\forall\tau\in V_h$, there exists a unique orthogonal decomposition such that

$$\tau=\tau_1+\tau_2,$$

where

$$\tau_1\in V_h^*=\{\psi_{1h}Tv;\ v\in Z_{2h}\}, \quad \tau_2\in (V_h^*)^\perp\subset Z_{1h}$$

and $(V_h^*)^\perp$ is orthogonally complementary of V_h^*, while operator $T\in\mathscr{L}(H_U, H_V)$ is defined by

$$e_1(\tau, v)=(\tau, Tv)_V, \quad \forall(\tau, v)\in H_V\times H_U.$$

Since

$$e_1(\tau, v)=(\tau, \psi_{1h}Tv)_V$$

implies

$$(\tau_1, \psi_{1h}Tv)_V=(\tau, \psi_{1h}Tv)_V,$$

we obtain

$$\|\tau_1\|_V=\sup_{v\in U_h}\frac{|\tau_1, \psi_{1h}Tv)|}{\|\psi_{1h}Tv\|_V}=\sup_{v\in U_h}\frac{|(\tau, \psi_{1h}Tv)|}{\|\psi_{1h}Tv\|_V}\leqslant c\sup_{v\in U_h}\frac{|e_1(\tau, v)|}{\|v\|_{U_1}}.$$

Consider

$$\|\tau\|_V=\|\tau_1\|_V+\|\tau_2\|_V\leqslant\|\tau_1\|_V+\alpha^{-1}d^{1/2}(\tau-\tau_1, \tau-\tau_1)$$
$$\leqslant\|\tau_1\|_V+\alpha^{-1}[d^{1/2}(\tau, \tau)+d^{1/2}(\tau_1, \tau_2)]\leqslant(1+\alpha^{-1}\beta)\|\tau_1\|_V+\alpha^{-1}d^{1/2}(\tau, \tau).$$

Lnserting $\|\tau_1\|_V$ into the relation above, we obtain inequality (4.13).

Step 2. There exist unique solutions $(\sigma_h, u_h)\in V_h\times Z_{2h}$ satisfying equations (4.12)

We first prove the uniqueness of the solutions.

Since both V_h and U_h are finite dimensional subspaces, we only need to prove that when f_1 and f_2 vanish, problem (4.12) has only null solutions.

Taking $\tau=\sigma_h$ and $v=u_h$ in (4.12a) and (4.12b) respectively, we have from condition (H. 2)

$$0\leqslant\alpha^2\|\sigma_h\|_V^2\leqslant d(\sigma_h, \sigma_h)=e_1(\sigma_h, u_h)-b_h^*(u_h)=0$$

i.e., we have $\sigma_h=0$. Using hypothesis (H. 1), we obtain

$$\|u_h\|_{U_1}\leqslant c_i^{-1}\sup_{\tau\in V_h}\frac{e_1(\tau, u_h)}{\|\tau\|_V}=c_i^{-1}\sup_{\tau\in V_h}\frac{d(\sigma_h, \tau)}{\|\tau\|_V}\leqslant c_i^{-1}\beta^2\|\sigma_h\|_V=0.$$

Therefore we have $u_h=0$.

We now prove the existence of the solutions.

Without loss of generality, we can take $f_1=0$ in equation (4.1a), and denote f_2 in equation (4.1b) by f, i.e., we only consider the following problem:

Find $(\sigma_h, u_h)\in V_h\times Z_{2h}$ such that

$$d(\sigma_h, \tau)-e_1(\tau, u_h)=0, \quad \forall\tau\in V_h, \tag{4.14a}$$

$$b_h^*(v)+e_1(\sigma_h, v)=\langle f, v\rangle_U, \quad \forall v\in Z_{2h}. \tag{4.14b}$$

Evidently, using the coerciveness of the functional $d(\cdot, \cdot)$, from equation (4.14a) we obtain a solution $\sigma_h^{u_h}$. In order to find u_h, we consider, $\forall v\in Z_{2h}$,

$$[\mathscr{P}_N(v), w_i]=b^*(v, v, w)+d(\sigma_h^v, \sigma_h^v)-\langle f, w_i\rangle_U, \quad 1\leqslant i\leqslant N,$$

where N is the dimension of the dimensional subspace Z_{2h}, and $\{w_i\}_{i=1}^N$ are the basic functions of Z_{2h}.

In particular,

$$[\mathscr{P}_N(v), v]=b^*(v, v, v)+d(\sigma_h^v, \sigma_h^v)-\langle f, v\rangle_U.$$

If we denote by $\|f\|^*=\sup\limits_{v\in H_{U_1}}\dfrac{\langle f, v\rangle}{\|v\|_{U_1}}$ the norm of f in $(H_{U_1})'$, then condition (H. 2) implies that

$$[\mathscr{P}_N(v), v]\geqslant\alpha^2\|\sigma_h^v\|^2-\|f\|^2\|v\|_{U_1}.$$

Using inequality (4.2a), we have

$$c_1\|v\|_{U_1}\leqslant\sup_{\tau\in V_h}\frac{e_1(\tau, v)}{\|\tau\|_V}=\sup_{\tau\in V_h}=\frac{d(\sigma_h^v, \tau)}{\|\tau\|_V}\leqslant\beta^2\|\sigma_h^v\|_V.$$

Therefore, we obtain

$$[\mathscr{P}_N(v), v]\geqslant(c_1\alpha^2\beta^{-2}\|v\|_{U_1}-\|f\|^*)\|v\|_{U_1}.$$

Hence, we choose $\xi>\dfrac{\|f\|^*}{c_1\alpha^2\beta^{-2}}$ and for $v\in Z_{2h}$ such that $\|v\|_{U_1}=\xi$, we have

$$[\mathscr{P}_N(v), v]>0. \tag{4.15}$$

Moreover, $\mathscr{P}_N$ is continuous in Z_{2h} by virtue of the properties of $b^*(\cdot, \cdot, \cdot)$ and $d(\cdot, \cdot)$. We can, therefore, make use of the classical Brouwer's fixed point theorem (see, e.g., [8]). Hence, there exists an element u_h of Z_{2h} that satisfies problem (4.12).

Step 3. For the saddle-point scheme satisfying the strongly consistent condition and (H. 1), we have the following inequality:

$$d^{1/2}(\sigma-\sigma_h, \sigma-\sigma_h)\leqslant c\Big\{\inf_{\tau\in V_h}\|\sigma-\tau\|_V+\inf_{v\in U_h}\|u-v\|_{U_1} + \inf_{v\in U_h}|b^*(u_h-v)-b_h^*(u_h-v)|^{1/2}+\sup_{v\in U_h}\frac{|b^*(v)-b_h^*(v)|}{\|v\|_{U_1}}\Big\}. \tag{4.16}$$

In fact, using the consistent condition, from (4.12) we have

$$e_1(\sigma-\sigma_h, u-u_h)-b^*(u_h-v)+b_h^*(u_h-v)=e_1(\sigma-\sigma_h, u-v), \quad \forall v\in Z_{2h}, \tag{4.17a}$$

$$d(\sigma-\sigma_h, \sigma-\sigma_h)=d(\sigma-\sigma_h, \sigma-\tau)+e_1(\sigma-\sigma_h, u-u_h)-e_1(\sigma-\tau, u-u_h), \quad \forall\tau\in V_h. \tag{4.17b}$$

Adding (4.17a) to (4.17b), and taking $\tau=\Pi_{1h}\sigma$, we obtain

$$d(\sigma-\sigma_h, \sigma-\sigma_h)=d(\sigma-\sigma_h, \sigma-\Pi_{1h}\sigma)+e_1(\Pi_{1h}\sigma-\sigma_h, u-v) +b^*(u_h-v)-b_h^*(u_h-v), \quad \forall v\in Z_{2h}, \tag{4.18}$$

where Π_{1h} is the interpolation operator defined by Lemma 4.1.

Since the following relation

$$0=e_1(\sigma-\Pi_{1h}\sigma,\ v)=e_1(\sigma-\sigma_h,\ v)+e_1(\sigma_h-\Pi_{1h}\sigma,\ v),\quad \forall v\in U_h$$

holds, we have

$$|e_1(\sigma_h-\Pi_{1h}\sigma,\ v)|=|b^*(v)-b_h^*(v)|,\quad \forall v\in Z_{2h},\tag{4.19}$$

while

$$\begin{aligned}|e_1(\sigma_h-\Pi_{1h}\sigma,\ u-v)|&\leqslant c\|\sigma_h-\Pi_{1h}\sigma\|_V\|u-v\|_{U_1}\\&\leqslant c\|u-v\|_{U_1}\Big\{(1+\alpha^{-1}\beta)\sup_{v\in U_h}\frac{|e_1(\sigma_h-\Pi_{1h}\sigma,\ v)|}{\|v\|_{U_1}}\\&\quad+\alpha^{-1}d^{1/2}(\sigma_h-\Pi_{1h}\sigma,\ \sigma_h-\Pi_{1h}\sigma)\Big\}\\&\leqslant c\|u-v\|_{U_1}\Big\{(1+\alpha^{-1}\beta)\sup_{v\in U_h}\frac{|b^*(v)-b_h^*(v)|}{\|v\|_{U_1}}\\&\quad+\alpha^{-1}d^{1/2}(\sigma_h-\Pi_{1h}\sigma,\ \sigma_h-\Pi_{1h}\sigma)\Big\},\end{aligned}$$

where the result of Step 1 is used. By considering (4.18) and using Young's inequality, we obtain estimate (4.16) by simple operations.

Step 4. If scheme (4.12) is strongly consistent, and if condition (H. 1) holds, then there exists a constant c independent of h such that estimate (4.10) holds.

Considering that

$$e_1(\tau,\ v-u_h)=e_1(\tau,\ v-u)+d(\sigma-\sigma_h,\ \tau),\quad \forall(\tau,\ v)\in V_h\times U_h,$$

and using condition (H. 1) and the result of Step 3, we have

$$\|u_h-v\|_{U_1}\leqslant c\Big\{\inf_{\tau\in V_h}\|\sigma-\tau\|_V+\inf_{v\in U_h}\|u-v\|_{U_1}+\inf_{v\in U_h}|b^*(u_h-v)-b_h^*(u_h-v)|^{1/2}\\+\sup_{v\in U_h}\frac{|b^*(v)-b_h^*(v)|}{\|v\|_{U_1}}\Big\}.\tag{4.20}$$

Using the triangle inequality and the result of Step 1, we get (4.10).

Step 5. Returning to problem (4.1), we now achieve the statement about pressure p.

For the uniqueness of p, we only need to prove that when f_1 and f_2 vanish, σ_h, u_h, and p_h are zeros. Using the equivalence of problems (4.1) and (4.12), the proof of which is a standard technique (cf. e.g., [8]), and using the result of Step 2, we only have to prove $p_h=0$. From (H. 1) we obtain

$$\|p_h\|_W\leqslant c_2^{-1}\sup_{v\in U_h}\frac{e_2(p_h,\ v)}{\|v\|_{U_1}}=c_2^{-1}\sup_{v\in U_h}\Big\{\frac{b_h^*(v)+e_1(\sigma_h,\ v)}{\|v\|_{U_1}}\Big\}=0.$$

Since $H_W\subsetneq L^2(\Omega)/R$, we get $p_h=0$, up to an additional constant.

The equivalence of problems (4.1) and (4.12) implies that there exists an element $p_h\in H_W$ such that $(\sigma_h,\ u_h,\ p_h)$ are the solutions hf problem (4.1).

Finally, we prove estimate (4.11). Considering that

$$e_2(p-p_h,\ v)=e_1(\sigma-\sigma_h,\ v)+b^*(v)-b_h^*(v),\quad \forall v\in U_h,$$

using (H. 1) and the triangle inequality, we have

$$\begin{aligned}\|p-p_h\| &\leqslant \|p-\psi_{3h}p\|_W+\|\psi_{3h}p-p_h\|_W \\ &\leqslant \|p-\psi_{3h}p\|_W+c_2^{-1}\sup_{v\in U_h}\frac{e_2(p-\psi_{3h}p,\ v)-e_2(p-p_h,\ v)}{\|v\|_{U_1}} \\ &\leqslant c\{\inf_{q\in W_h}\|p-q\|_W+\|\sigma-\sigma_h\|_V\},\end{aligned}$$

where ψ_{3h} is the orthogonal projector defined above. This gives (4.11). The proof of Theorem 4.1 is completed.

§ 5. Convergence of the MSFEM

Under the framework of the abstract theory in Section 4, and from Remark 2.4, and Proposition 3.2, we first have to find the MSFE solutions $(\sigma_h,\ u_h)$ in the divergence-free space, i.e., solve problem (M_h2) of Section 2.

To begin with, we have the following relations corresponding to the abstract framework:

$$\begin{gathered} d(\sigma,\ \tau)=\mu(\sigma,\ \tau),\quad \forall\sigma,\ \tau\in H_V, \\ e_1(\tau,\ v)=B(\tau,\ v),\quad \forall(\tau,\ v)\in H_V\times H_{U_1}, \\ e_2(q,\ v)=(q,\ \nabla\cdot v),\quad \forall(q,\ v)\in H_W\times H_{U_2}, \\ Z_{2h}=\mathring{U}_h,\ Z_2=\mathring{H}_U,\end{gathered}$$

and other symbols remain the same.

In order to employ the results of the abstract theory, we have to check all hypotheses made. Obviously, from Lemmas 3.2—3.5, $B(\cdot,\ \cdot)$, $(q,\ \nabla\cdot v)$, and $b^*(\cdot,\ \cdot,\ \cdot)$ are continuous on their defined spaces respectively, and the following inclusions

$$\begin{gathered} H_{U_i}\subsetneq H_{U_i}\subsetneq[L^2(\Omega)]^2,\quad i=1,\ 2, \\ H_W\subsetneq L^2(\Omega)/R \end{gathered}$$

hold. Considering the following inequalities:

$$\sup_{v_h\in U_h}\frac{(q_h,\ \nabla\cdot v_h)}{\|v_h\|_{U_2}}\geqslant c_2\|q_h\|_W,\quad \forall q_h\in W_h, \tag{5.1}$$

$$\sup_{\tau_h\in V_h}\frac{B(\tau_h,\ V_h)}{\|\tau_h\|_V}\geqslant c_1\|v_h\|_{U_1},\quad \forall v_h\in\mathring{U}_h, \tag{5.2}$$

$$\|V_h\|_{U_2}\leqslant c_3\|v_h\|_{U_1},\quad \forall v_h\in\mathring{U}_h, \tag{5.3}$$

$$b^*(v_h,\ v_h,\ v_h)\geqslant 0,\quad \forall v_h\in\mathring{U}_h, \tag{5.4}$$

we have the propositions below.

Proposition 5.1. There exists a constant c independent of h such that inequality (5.1) holds.

Proof. By using Lemma 4.2, we only need to prove that for any $v\in H_U\cap[H^1(\Omega)]^2$, there exists an element $v_0\in U_h$ and a constant c independent of h such that

$$(q_h,\ \nabla\cdot v_0)=(q_h,\ \nabla\cdot v),\quad \forall q_h\in W_h, \tag{5.5a}$$

$$\|v_0\|_{U_2}\leqslant c\|v\|_{U_2}. \tag{5.5b}$$

For any $\Omega_i\in\mathscr{T}_h$, denote by F_i the affine transform from Ω_i to $\hat{\Omega}$, the conference element in $\xi-\eta$ planet see Fig. 4.

Then we have

$$\Omega_i = F_i(\hat{\Omega}),$$
$$x = F_i\hat{x} = B_i\hat{x} + b_i,$$

where the meanings of B_i and b_i are the same as in [5].

Let

$$\hat{v} = J_i B_i^{-1} v|_{\Omega_i} \circ F_i,$$

where $J_i = \det(B_i)$, the Jacobian.

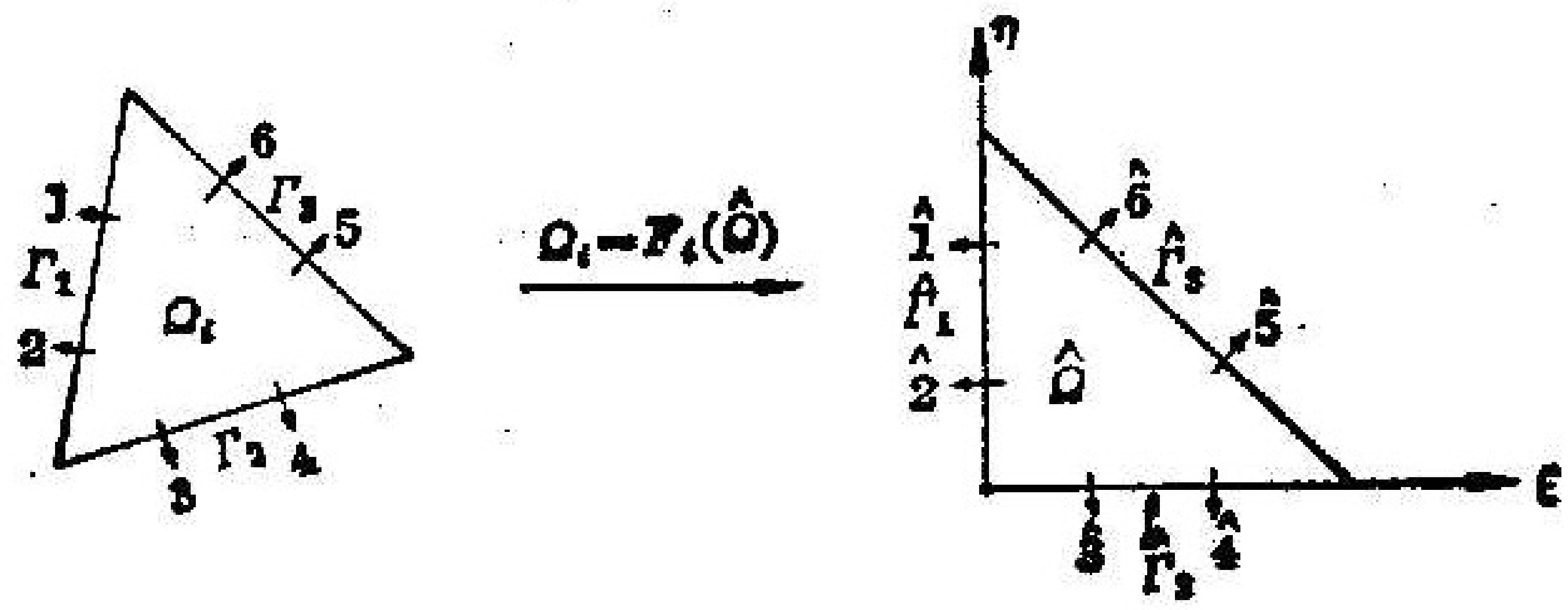

Fig. 4 Practical element Ω_i; conference element $\hat{\Omega}$

From Proposition 3.1, there exists an element $\hat{v}_0 \in [P_1(\hat{\Omega})]^2$ satisfying

$$\int_{\hat{\Gamma}} \hat{q}(\hat{v}_0 \cdot \hat{n})\, d\hat{s} = \int_{\hat{\Gamma}} \hat{q}(\hat{v} \cdot \hat{n})\, d\hat{s}, \quad \forall \hat{q} \in R. \tag{5.6}$$

If we define

$$v_0 = J_i^{-1} B_i \hat{v}_0 \circ F_i^{-1}$$

then from Lemma 2 of [15], i.e., for any $\hat{v} \in [H^1(\Omega)]^2$,

$$\int_{\hat{\Gamma}} \hat{q}(\hat{v} \cdot \hat{n})\, d\hat{s} = \int_{\Gamma_i} q(v \cdot n)\, ds, \quad \forall \hat{q} \in L^2(\hat{\Gamma}), \tag{5.7}$$

we can write (5.6) into

$$\int_{\Gamma_i} q(v_0 \cdot n)\, ds = \int_{\Gamma_i} q(v \cdot n)\, ds, \quad \forall q \in R. \tag{5.8}$$

And by means of Green's formula, (5.8) becomes

$$\int_{\Omega_i} q \nabla \cdot v_0\, dx = \int_{\Omega_i} q \nabla \cdot v\, dx, \quad \forall q \in R. \tag{5.9}$$

Since F_i is an affine transform, we have $v_0 \in U_h$. Summing (5.9) about i, we obtain the relation (5.5a).

Similarly, if $v \in \mathring{H}_U \cap [H^1(\Omega)]^2$, then there exist $v_0 \in \mathring{U}_h$ such that (5.5a) holds.

In order to prove (5.5b), we only require the following relation

$$\|\hat{v}\|_{0,\hat{\Omega}} \leqslant c|\hat{v}|_{1,\hat{\Omega}}. \tag{5.10}$$

In fact, using the result of Theorem 3.1.2 of [5], i.e.,

$$|\hat{v}|_{m,\hat{\Omega}} \leqslant c\|B_i\|^m |\det(B_i)|^{-1/2} |v|_{m,\Omega_i}, \quad \forall v \in H^m(\Omega), \tag{5.11a}$$

$$|v|_{m,\Omega_i} \leqslant c\|B_i^{-1}\|^m |\det(B_i)|^{1/2} |\hat{v}|_{m,\hat{\Omega}}, \quad \forall \hat{v} \in H^m(\hat{\Omega}), \tag{5.11b}$$

and $\|B_i\| \leqslant h_i/\hat{\rho}$, $\|B_i^{-1}\| \leqslant \hat{h}/\rho$, where ρ_i and $\hat{\rho}$ are diameters of internally tangent circles of Ω_i and $\hat{\Omega}$ respectively, and h_i and $\hat{h}$ are diameters of externally tangent circles of Ω_i and $\hat{\Omega}$ respectively, relation (5.10) implies that

$$\|v\|_{0,\Omega_i} \leqslant ch_i|v|_{1,\Omega_i}$$

By empolying the standard inverse inequality, we obtain

$$\|v_0\|^2_{U_h(\Omega_i)} = \|v_0\|^2_{0,\Omega_i} + h_i^2|v_0|^2_{1,\Omega_i} \leqslant c\|v_0\|^2_{0,\Omega_i} \leqslant ch_i^2|v|^2_{1,\Omega_i} \leqslant c\|v\|^2_{U_h(\Omega_i)}. \tag{5.12}$$

Summing the above inequality with respect to Ω_i, we have proved that (5.5b) holds.

We now prove the relation (5.10).

By eliminating the constant q on both sides of (5.6) and using Green's formula, we get

$$\left|\int_{\hat\Gamma} \hat v_0\cdot\hat n\,d\hat s\right| = \left|\int_{\hat\Omega}\nabla\cdot\hat v\,d\hat x\right| \leqslant c\,|\hat v|_{1,\hat\Omega}. \tag{5.13}$$

Setting $\hat\Gamma=\sum_{k=1}^{3}\hat\Gamma_k$ (see Fig. 4), we can write out the following relation

$$\int_{\hat\Gamma}\hat v_0\cdot\hat n\,d\hat s=\sum_{k=1}^{3}\int_{\hat\Gamma_k}\hat v_0\cdot\hat n\,d\hat s.$$

From Proposition 3.1, we have

$$\begin{bmatrix}\hat v_{01}\\ \hat v_{02}\end{bmatrix}=\begin{bmatrix}\hat\alpha_0 & \hat\alpha_1 & \hat\alpha_2\\ \hat\beta_0 & \hat\beta_1 & \hat\beta_2\end{bmatrix}\begin{bmatrix}1\\ \xi\\ \eta\end{bmatrix}$$

and

$$\begin{bmatrix}\hat\alpha_0\\ \hat\alpha_1\\ \vdots\\ \hat\beta_1\\ \hat\beta_2\end{bmatrix}=\begin{bmatrix}\hat n_{11} & \xi_1\hat n_{11} & \eta_1\hat n_{11} & \hat n_{21} & \xi_1\hat n_{21} & \eta_1\hat n_{21}\\ \hat n_{21} & \xi_2\hat n_{21} & \eta_2\hat n_{21} & \hat n_{22} & \xi_2\hat n_{22} & \eta_2\hat n_{22}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \hat n_{51} & \xi_5\hat n_{51} & \eta_5\hat n_{51} & \hat n_{52} & \xi_5\hat n_{52} & \eta_5\hat n_{52}\\ \hat n_{61} & \xi_6\hat n_{61} & \eta_6\hat n_{61} & \hat n_{62} & \xi_6\hat n_{62} & \eta_6\hat n_{62}\end{bmatrix}^{-1}\begin{bmatrix}(\hat v_0\cdot\hat n)_1\\ (\hat v_0\cdot\hat n)_2\\ \vdots\\ (\hat v_0\cdot\hat n)_5\\ (\hat v\cdot\hat n)_6\end{bmatrix},$$

where $\hat n_i=(\hat n_{i1},\hat n_{i2})^T$, $\hat x_i=(\xi_i,\eta_i)$, $1\leqslant i\leqslant 6$.

Considering that

$$\int_{\hat\Omega}\hat v_0\,d\hat x\leqslant(\text{meas}(\hat\Omega))^{1/2}\Big(\int_{\hat\Omega}|\hat v_0|^2\,dx\Big)\leqslant c\left|\sum_{i=1}^{6}\tilde c_i(\hat v_0\cdot\hat n)_i\right|,$$

where $\hat c_i$, $1\leqslant i\leqslant 6$, only depend upon the coordinate values of the six nodes of the element $\hat\Omega$. On the other hand, using the theorem of integral mean value, we have

$$\|\hat v_0\|_{0,\hat\Omega}\leqslant c\left|\sum_{k=1}^{3}\int_{\hat\Gamma_k}\hat v_0\cdot\hat n\,d\hat s\right|=c\left|\int_{\hat\Gamma}\hat v_0\cdot\hat n\,d\hat s\right|.$$

Inserting the result into (5.13), we obtain (5.10). Therefore, the proposition is proved.

Proposition 5.2. There exists a constant c independent of h such that inequality (5.2) holds.

Proof. From the idea in [21], it suffices to prove that for each $\Omega_j^*\subset\mathscr{T}_h^*$, the relation

$$B(\tau,v)_{\Omega_j^*}=0,\quad\forall(\tau,v)\in V_h\times\mathring{U}_h \tag{5.14a}$$

implies

$$\|\varepsilon(v)\|^2_{0,\Delta_{R_j}}+\|\varepsilon(v)\|^2_{0,\Delta_{L_j}}+h_j^{s-1}\int_s|(v_+-v_-)\cdot\tau|^2\,ds=0, \tag{5.14b}$$

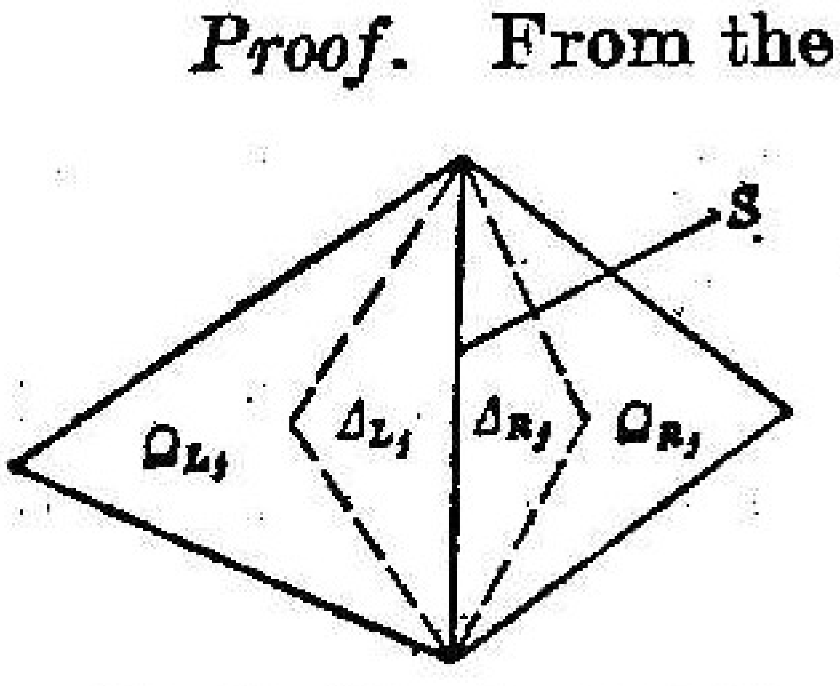

Fig. 5 The structure of subregions Δ_{R_j} and Δ_{L_j}

where

$$\Delta_{R_j}=\Omega_j^*\cap\Omega_{R_j},\ \Delta_{L_j}=\Omega_j^*\cap\Omega_{L_j},\ \Omega_j^*=\Delta_{R_j}\cup\Delta_{L_j},\ s=\Gamma_{R_j}\cap\Gamma_{L_j},$$

while the pair of elements $(\Omega_{R_j},\Omega_{L_j})\subset\mathscr{T}_h$ are two adjacent velocity elements which generate Ω_j^*; see Fig. 5.

In fact, we get, by using Green's formula (2.4) and from (5.14a),

$$\int_s(v_+-v_-)\cdot(\tau\cdot n)ds-(\tau,\varepsilon(v))_{\Delta_{R_j}\cup\Delta_{L_j}}=0,\ \forall(\tau,v)\in V_h\times\mathring{U}_h. \tag{5.15}$$

Without loss of generality, we may assume that

$$\cos 2(n, x) \neq 0, \quad \text{on } S.$$

Recall the structure of the set H_{Dj} in Section 3, and set

$$\tau_{11}=\tau_{22}=0,\ \tau_{12}=\tau_{21}=\begin{cases} c_{21}+c_{22}^R l_j, & \text{in } \Delta_{R_j}, \\ c_{21}+c_{22}^L l_j, & \text{in } \Delta_{L_j}, \end{cases} \tag{5.16}$$

where

$$c_{21}=2[\cos 2(n, x_1)]^{-1}\int_S (v_+-v_-)\cdot t\, ds,$$

$$c_{22}^R=\text{meas}(\Delta_{R_j})[\bar{s}_{21}^R(v)-c_{21}]\Big/\int_{\Delta_{R_j}} l_j\, dx,$$

$$c_{22}^L=\text{meas}(\Delta_{L_j})[\bar{s}_{21}^L(v)-c_{21}]\Big/\int_{\Delta_{L_j}} l_j\, dx,$$

$$\bar{s}(v)=s(v)-(\nabla\cdot v)\delta/2,$$

$$\bar{s}^R(v)=\bar{s}(v)\,|_{\Delta_{R_j}},\ \bar{s}^L(v)=\bar{s}(v)\,|_{\Delta_{L_j}}.$$

The relation (5.15) implies that

$$\|\bar{s}_{21}(v)\|^2_{0,\Delta_{R_j}\cup\Delta_{L_j}}+\left|\int_S (v_+-v_-)\cdot t\, ds\right|^2=0, \tag{5.17}$$

because we have

$$t\cdot(\tau, n)=c_{11}\sin 2(n, x_1)-c_{21}\cos 2(n, x_1), \quad \text{on } S,$$

$$(\tau, s(v))_{\Delta_{R_j}\cup\Delta_{L_j}}=2[(\tau_{11}, s_{11}(v))_{\Delta_{R_j}\cup\Delta_{L_j}}+(\tau_{12}, \bar{s}_{12}(v))_{\Delta_{R_j}\cup\Delta_{L_j}}].$$

In the same way, we have

$$\|\bar{s}_{11}(v)\|_{0,\Delta_{R_j}\cup\Delta_{L_j}}=\|\bar{s}_{22}(v)\|_{0,\Delta_{R_j}\cup\Delta_{L_j}}=0. \tag{5.18}$$

Using (5.17) and (5.18), and since $v\in\mathring{U}_h$, we obtain

$$0\leqslant\|s(v)\|^2_{0,\Delta_{R_j}\cup\Delta_{L_j}}\leqslant\|\bar{s}(v)\|_{0,\Delta_{R_j}\cup\Delta_{L_j}}+\left\|\frac{1}{2}\nabla\cdot v\delta\right\|^2_{0,\Delta_{R_j}\cup\Delta_{L_j}}=0, \tag{5.19}$$

which means that v is a rigid body displacement in each triangle Δ_{R_j} or Δ_{L_j}, i.e., in Δ_{R_j} or Δ_{L_j}, v can be written into

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=\begin{bmatrix} c_1^R-t_R x_2 \\ c_2^R+t_R x_1 \end{bmatrix} \quad \text{in } \Delta_{R_j},$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=\begin{bmatrix} c_1^L-t_L x_2 \\ c_2^L+t_L x_1 \end{bmatrix} \quad \text{in } \Delta_{L_j}.$$

Using the continuity of $(v\cdot n)$ on S, we have

$$[c_1^R-c_1^L-(t_R-t_L)x_2]\cos(n, x_1)+[c_2^R-c_2^L+(t_R-t_L)x_1]\cos(n, x_2)=0, \quad \text{on } S. \tag{5.20}$$

On the other hand, since S is a straight line segment perpendicular to n, we get

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} x_{10}+\bar{t}\cos(n, x_2) \\ x_{20}+\bar{t}\cos(n, \lambda_1) \end{bmatrix}, \quad \bar{t}_1<\bar{t}<\bar{t}_2, \tag{5.21}$$

where (x_{10}, x_{20}) satisfies the relation (5.20) and $\bar{t}$ is a parameter. Then (5.20) and (5.21) imply that

$$t_R-t_L=0. \tag{5.22}$$

Putting (5.22) into (5.20) and the second term of (5.17), we obtain

$$\int_S [(c_1^R - c_1^L)\cos(n, x_2) - (c_2^R - c_2^L)\cos(n, x_1)]ds = 0,$$

$$(c_1^R - c_1^L)\cos(n, x_1) + (c_2^R - c_2^L)\cos(n, x_2) = 0,$$

which imply that

$$c_i^R = c_i^L, \quad i = 1, 2. \tag{5.23}$$

Thus, the relations (5.7), (5.8), (5.22) and (5.23) give

$$\|\varepsilon(v)\|^2_{0,\Delta_k \cup \Delta_{k_j}} + h_j^{*-1}\int_S |(v_+ - v_-)\cdot t|^2 ds = 0.$$

Therefore, the second criterion of the B-B inequality of [21] is checked. The proposition is proved.

Proposition 5.3. There exists a constant c independent of h such that (5.3) holds.

Proof. From the definition of norms $\|\cdot\|_{U_i}$, $i=1, 2$, and using Lemma 3.2, we only have to prove

$$\sum_i |v|^2_{1,\Omega_i} \leqslant c\|v\|^2_{U_i}, \quad \forall v \in \mathring{U}_h. \tag{5.24}$$

Since we have, from the definition,

$$\sum_i |v|^2_{1,\Omega_i} = \sum_i [\|\nabla\cdot v\|^2_{0,\Omega_i} + \|\varepsilon(v)\|^2_{0,\Omega_i}]$$

and

$$\nabla\cdot v = 0, \quad \forall v \in \mathring{U}_h,$$

the inequality is obtained immediately. The proposition is proved.

Proposition 5.4. Inequality (5.4) holds.

Proof. See the proof in [12].

Proposition 5.5. For the finite dimensional subspace U_h defined in the paper, we have

$$\mathring{U}_h \subset \mathring{H}_U.$$

Proof. For any $v_h \subset \mathring{U}_h$, we have by the definition

$$(q_h, \nabla\cdot v_h) = 0, \quad \forall q_h \in W_h.$$

Since q_h is a constant in each element $\Omega_i \in \mathscr{T}_h$, and $\nabla\cdot v_h|_{\Omega_i}$ is also a constant, $\nabla\cdot v|_{\Omega_i}$ equals zero. Furthermore, summing all $\Omega_i \in \mathscr{T}_h$, we have $\nabla\cdot v_h = 0$.

On the other hand, we have

$$v_h \in \mathring{U} \subset U_h \subset H_U,$$

and

$$(q, \nabla\cdot v_h) = 0, \ \forall q \in H_W,$$

which implies that v_h belongs to H_U. The proof is completed.

We now derive the main result.

Theorem 5.1. *For the approximation problem* (M_h2), *if the viscous coefficient* ν *is sufficiently large, or,* $\|f\|^*$, *norm of the body force, is small enough, there exist unique* MSFE *solutions* $(\sigma_h, u_h) \in V_h \times \mathring{U}_h$, *and there exists a constant* c *independent of* h *such that*

$$\|\sigma - \sigma_h\|_V + \|u - u_h\|_{U_1} \leqslant ch\|u\|_{2,\Omega}, \tag{5.25}$$

where we assume that the exact solution of velocity u *belongs to the space* $[H^2(\Omega) \cap H_0^1(\Omega)]$.

Proof. First, we check the hypotheses (H. 1)—(H. 3) required by Theorem 4.1.

(i) Propositions 5.1—5.3 imply that (H. 1) holds.

(ii) By using the standard inverse inequality, i.e.,

$$|\tau|_{1,\Omega_j^*}\leqslant ch_j^{*-1}\|\tau\|_{0,\Omega_j^*},\quad \forall\tau\in H_V,\ \forall\Omega_j^*\in\mathscr{T}_h^*,$$

we have

$$\mu^{-1}(\tau,\tau)=\mu^{-1}\|\tau\|_{0,\Omega}^2\geqslant\mu^{-1}c_4\|\tau\|_V^2,$$

where constant c_4 only depends upon the region Ω.

And considering Proposition 5.4, we have that (H. 2) holds.

(iii) Proposition 5.5 implies that (H. 3) is true.

Using Lemma 3.5, we can now employ Theorem 4.1 directly. Therefore problem (M_h2) has unique solutions (σ_h, u_h) and we obtain the following estimate:

$$\|\sigma-\sigma_h\|_V+\|u-u_h\|_{U_1}$$
$$\leqslant c\Big\{\inf_{\tau\in V_h}\|\sigma-\tau\|_V+\inf_{v\in U_h}[\|u-v\|_{U_1}+|b^*(u_h-v)-b_h^*(u_h-v)|]^{1/2}$$
$$+\sup_{v\in U_h}\frac{|b^*(v)-b_h^*(v)|}{\|v\|_{U_1}}\Big\}. \tag{5.26}$$

It is easy to obtain, from Lemma 3.6,

$$\inf_{\tau\in V_h}\|\sigma-\tau\|_V\leqslant ch\|u\|_{2,\Omega}, \tag{5.27}$$

$$\inf_{v\in U_h}\|u-v\|_{U_1}\leqslant ch\|u\|_{2,\Omega}. \tag{5.28}$$

Evidently, if we have the following relation

$$E=E_1+E_2=\sup_{v\in U_h}\frac{|b^*(v)-b_h^*(v)|}{\|v\|_{U_1}}+\inf_{v\in U_h}|b^*(u_h-v)-b_h^*(u_h-v)|^{1/2}$$
$$\leqslant ch\|u\|_{2,\Omega} \tag{5.29}$$

then the theorem is proved.

We now prove (5.29). Using Lemma 5.1, to be established below, i.e.,

$$\|\sigma_h-\Pi_{1h}\sigma\|_V\leqslant ch\|u\|_{2,\Omega}, \tag{5.30}$$

where Π_{1h} is defined in Lemma 4.1, we obtain (5.29). In fact, from (4.19), (5.30), and the continuity of the bilinear form $B(\cdot,\cdot)$, we have

$$E_1\leqslant ch\|u\|_{2,\Omega},$$
$$E_2\leqslant ch^{1/2}\|u\|_{2,\Omega}^{1/2}(\inf_{v_h\in U_h}\|u_h-v_h\|_{U_1}^{1/2}).$$

Since (H. 1) gives

$$\|u_h-v_h\|\leqslant c_1^{-1}\sup_{\tau\in V_h}\frac{B(\tau,u_h-v_h)}{\|\tau\|_V}\leqslant c(\|\sigma-\Pi_{1h}\sigma\|_V+\|\sigma_h-\Pi_{1h}\sigma\|_V+\|u-v_h\|_{U_1})$$

which implies, from (5.30), and Lemmas 3.6 and 4.1, that

$$E_2\leqslant ch\|u\|_{2,\Omega},$$

the estimate (5.29) is obtained. Combining (5.27)—(5.29), the relation (5.25) is proved.

Lemma 5.1. *Under the conditions of Theorem* 5.1, *inequality* (5.30) *holds, where σ and σ_h are the exact solution and the approximate solution of stress deviatorics respectively.*

Proof. From (H. 1) and Proposition 5.4, we have

$$\|u_h\|_{U_1}\leqslant (c_1\mu)^{-1}\|\sigma_h\|_{0,\Omega},$$
$$\|\sigma_h\|_{0,\Omega}^2=\mu[\mu^{-1}(\sigma_h,\sigma_h)]=\mu[(f,u_h)-b_h^*(u_h)]\leqslant\mu\|f\|^*\|u_h\|_{U_1}.$$

Hence, we get

$$\|u_h\|_{U_1}\leqslant c_1^{-2}\mu^{-1}\|f\|^*. \tag{5.31}$$

Considering that

$$B(\sigma,v)+b^*(v)=(f,v),\quad \forall v\in \mathring{H}_U,$$
$$B(\sigma_h,v_h)+b_h^*(v_h)=(f,v_h),\quad \forall v_h\in \mathring{U}_h,$$

we have

$$B(\sigma-\sigma_h,v_h)+b^*(v_h)-b_h^*(v_h)=0,\quad \forall v_h\in\mathring{U}_h.$$

Taking $v=\Pi_{2h}u-u_h$, where Π_{2h} is an interpolation operator defined in Lemma 4.2, we have

$$d(\Pi_{1h}\sigma-\sigma_h)+\sum_{i=1}^4 R_i=0, \tag{5.32}$$

where

$$R_1=B(\sigma-\Pi_{1h}\sigma,\ \Pi_{2h}u-u_h),$$
$$R_2=B(\Pi_{1h}\sigma-\sigma_h,\ \Pi_{2h}u-u),$$
$$R_3=d(\sigma-\Pi_{1h}\sigma,\ \Pi_{1h}\sigma-\sigma_h),$$
$$R_4=b^*(\Pi_{2h}u-u_h)-b_h^*(\Pi_{2h}u-u_h).$$

Using (H. 2), we have from (5.32)

$$c_4\mu^{-1}\|\Pi_{1h}\sigma-\sigma_h\|_V^2\leqslant c\sum_{i=1}^4|R_i|, \tag{5.33}$$

where c_4 is a constant in the inverse inequality.

Since we have from condition (H. 1)

$$\|\Pi_{2h}u-u_h\|_{U_1}\leqslant c_1^{-1}\sup_{\tau\in V_h}\frac{B(\tau,\Pi_{2h}u-u_h)}{\|\tau\|_V}\leqslant c_1^{-1}(\mu^{-1}\|\sigma-\sigma_h\|_V+c\|\Pi_{1h}\sigma-\sigma_h\|_V$$
$$\leqslant ch\|u\|_{2,\Omega}+(c_1\mu)^{-1}\|\Pi_{1h}\sigma-\sigma_h\|_V,$$

we obtain by means of Young's inequality

$$|R_1|\leqslant c\|\sigma-\Pi_{1h}\sigma\|_V\|\Pi_{2h}u-u_h\|_{U_1}\leqslant ch\|u\|_{2,\Omega}\|\Pi_{2h}u-u_h\|_{U_1}$$
$$\leqslant c\varepsilon_1^{-1}h^2\|u\|_{2,\Omega}^2+\varepsilon_1\mu^{-1}\|\Pi_{1h}\sigma-\sigma_h\|_V^2,$$
$$|R_2|\leqslant c\|\Pi_{1h}\sigma-\sigma_h\|_V\|\Pi_{2h}u-u\|_{U_1}\leqslant ch\|u\|_{2,\Omega}\|\Pi_{1h}\sigma-\sigma_h\|_V$$
$$\leqslant c\mu\varepsilon_2^{-1}h^2\|u\|_{2,\Omega}^2+\varepsilon_2\mu^{-1}\|\Pi_{1h}\sigma-\sigma_h\|_V^2,$$
$$|R_3|\leqslant\mu^{-1}\|\sigma-\Pi_{1h}\sigma\|_V\|\Pi_{1h}\sigma-\sigma_h\|_V\leqslant c\mu^{-1}\varepsilon_3^{-1}h^2\|u\|_{2,\Omega}^2+\varepsilon_3\mu^{-1}\|\Pi_{1h}\sigma-\sigma_h\|_V^2,$$
$$|R_4|\leqslant c\{\|u-\Pi_{2h}u\|_{U_2}\|u\|_{U_2}\|\Pi_{2h}u-u_h\|_{U_2}+\|\Pi_{2h}u\|_{U_2}\|u-\Pi_{2h}u\|_{U_2}\|\Pi_{2h}u-u_h\|_{U_2}$$
$$+\|\Pi_{2h}\|_{U_2}\|\Pi_{2h}u-u_h\|_{U_2}^2+\|u_h\|_{U_2}\|\Pi_{2h}u-u_h\|_{U_2}^2\}$$
$$\leqslant ch^2\|u\|_{1,\Omega}^2+(\varepsilon_5+\varepsilon_6)\mu^{-1}\|\Pi_{1h}\sigma-\sigma_h\|_V^2$$
$$+c_1^{-2}\mu^{-2}(\|u\|_{U_2}+\|u_h\|_{U_1})\|\Pi_{1h}\sigma-\sigma_h\|_V^2,$$

where ε_i, $1\leqslant i\leqslant 6$, are constants which appeared in Young's inequality. Inserting the above results into (5.33), we obtain

$$\mu^{-1}\Big(c_4-\sum_{i=1}^6\varepsilon_i\Big)\|\Pi_{1h}\sigma-\sigma_h\|_V^2\leqslant ch^2\|u\|_{2,\Omega}^2+(c_1\mu)^{-2}(\|u\|_{2,\Omega}+\|u_h\|_{U_1})\|\Pi_{1h}\sigma-\sigma_h\|_V^2. \tag{5.34}$$

Since for the continuous problem, velocity solution u satisfies (see [17])

$$\|u\|_{1,\Omega}\leqslant 2\|f\|^*/\mu,$$

and using the relation (5.31), (5.34) can be written into

$$\mu^{-1}\Big[c_4-\sum_{i=1}^{6}\varepsilon_i-\frac{1}{c_1^2\mu^2}(2+c^{-2})\|f\|^*\Big]\|\Pi_{1h}\sigma-\sigma_h\|_V^2\leqslant ch^2\|u\|_{2,\Omega}. \quad (5.35)$$

Particularly, taking $\sum_{i=1}^{6}\varepsilon_i<c_4/2$, and using the condition that μ is sufficiently large, or $\|f\|^*$ is samll enough, we have

$$0<\frac{1}{2}c_4-(c_1\mu)^{-2}(2+c^{-2})\|f\|^*. \quad (5.36)$$

Therefore, putting (5.36) into (5.35), we get

$$\|\Pi_{1h}\sigma-\sigma_h\|_V\leqslant ch\|u\|_{2,\Omega}.$$

which is (5.30). The lemma is proved.

Remark 5.1. In the procedure of the proof of Lemma 5.1, we require that (5.36) hold, i.e.,

$$\mu^2\geqslant 2(2+c_1^{-2})\|f\|^*/c_4c_1^2.$$

Since we have

$$\mu=2\nu,\quad \mathrm{Re}\propto\nu^{-1},$$

where ν is the kinematic viscosity parameter of fluid, and Re is the Reynolds number, hence we only discuss the case of small Re number, such as problems in hydrodynamics.

§ 6. L^2-Error Estimates

We have previously discussed the MSFE solution of problem (M_h2). In order to find pressure p, we now return to problem (M_h1).

Problems (M_h1) and (M_h2) are equivalent, which can be proved by standard techniques (cf. for example [8], Chep. IV). Therefore, problem (M_h1) has unique solutions $(\sigma_h, u_h, p_h)\in V_h\times U_h\times W_h$.

We now begin to discuss the convergence for the discrete pressure. It is not difficult to see that if the exact solution p of pressure belongs to $H^1(\Omega)/R$, the error estimate can not be obtained by using (4.11). But we have the following result.

Theorem 6.1. *Assume that $p\in H^1(\Omega)/R$. If the conditions of Theorem 5.1 are satisfied, then there exists a constant c independent of h such that*

$$\|p-p_h\|_{L^2(\Omega)/R}\leqslant ch(\|u\|_{2,\Omega}+\|p\|_{H^1(\Omega)/R}), \quad (6.1)$$

where u is the exact solution of velocity.

Proof. Let ψ_h be the orthogonal projector from $L^2(\Omega)/R$ onto W_h. Then, there exists a function $u_0\in H_U\cap[H_0^1(\Omega)]^2$ such that (see, e.g., [6] and [16]):

$$\nabla\cdot u_0=\psi_hp-p_h-c_0,\quad \text{in } \Omega, \quad (6.2a)$$

$$u_0=0,\quad \text{on } \Gamma, \quad (6.2b)$$

where $c=[\mathrm{meas}(\Omega)]^{-1}\int_\Omega(\psi_hp-p_h)dx$.

Moreover, we have

$$\|u_0\|_{1,\Omega}\leqslant c\|\psi_h p-p_h-c_0\|_{0,\Omega}=c\|\psi_h p-p_h\|_{L^2(\Omega)/R}, \tag{6.3}$$

where c is a constant independent of h.

From Lemma 4.2, there exists a function $\Pi_{2h}u_0\in U_h$ satisfying

$$(\nabla\cdot\Pi_{2h}u_0-\nabla\cdot u_0,\ q)=0,\quad \forall q\in W_h, \tag{6.4a}$$

$$\|\Pi_{2h}u_0-u_0\|_{U_*}\leqslant c\inf_{v\in U_h}\|u_0-v\|_{U_*}. \tag{6.4b}$$

Since we have

$$\|\nabla\cdot\Pi_{2h}u_0\|_{0,\Omega}\leqslant\|\nabla\cdot u_0\|_{0,\Omega} \tag{6.5}$$

using Lemma 3.3, Lemma 3.6, and the relation (6.4b), we obtain

$$\|\Pi_{2h}u_0-u_0\|_{U_1}\leqslant ch^{-1}\|\Pi_{2h}u_0-u_0\|_{U_*}\leqslant ch^{-1}\inf_{v\in U_h}\|u_0-v\|_{U_*}\leqslant c\|u_0\|_{1,\Omega}\leqslant c\|\nabla\cdot u_0\|_{0,\Omega}.$$

On the other hand, we have

$$\begin{aligned}\|\psi_h p-p_h-c_0\|_{0,\Omega}&=\frac{(\psi_h p-p_h-c_0,\ \nabla\cdot u_0)}{\|\nabla\cdot u_0\|_{0,\Omega}}\\&\leqslant\|p-\psi_h p-c\|_{0,\Omega}+\frac{(p-p_h,\nabla\cdot\Pi_{2h}u_0)}{\|\nabla\cdot u_0\|_{0,\Omega}},\quad \forall c\in R,\end{aligned} \tag{6.6}$$

which implies that

$$\begin{aligned}\|p-p_h\|_{L^2(\Omega)/R}&\leqslant\|p-\psi_h p\|_{L^2(\Omega)/R}+\|\psi_h p-p_h-c_0\|_{0,\Omega}\\&\leqslant\inf_{q\in W_h/R}\|p-q\|_{L^2(\Omega)/R}+\inf_{c\in R}\|p\cdot\psi_h p-c\|_{0,\Omega}+R(p,p_h,u_0)\\&\leqslant ch\|p\|_{H^1(\Omega)/R}+R(p,p_h,u_0),\end{aligned} \tag{6.7}$$

where

$$R(p,p_h,u_0)=|(p-p_h,\nabla\cdot\Pi_{2h}u_0)|/\|\nabla\cdot u_0\|_{0,\Omega}.$$

We now prove that there exists a constant c independent of h such that

$$R(p,p_h,u_0)\leqslant ch(\|u\|_{2,\Omega}+\|p\|_{H^1(\Omega)/R}). \tag{6.8}$$

Considering

$$|(p-p_h,\nabla\cdot\Pi_{2h}u_0)|\leqslant|b^*(\Pi_{2h}u_0)-b_h^*(\Pi_{2h}u_0)|+|B(\sigma-\sigma_h,\Pi_{2h}u_0)|, \tag{6.9}$$

and using (6.6) and the condition $u_0\in[H_0^1(\Omega)]^2$, we get

$$|B(\sigma-\sigma_h,\Pi_{2h}u_0)|\leqslant ch\|u\|_{2,\Omega}\|\nabla\cdot u_0\|_{0,\Omega}, \tag{6.10}$$

where we have utilized the fact that if $u_0\in[H^1(\Omega)]^2$, $u_0|_{\Omega_i}\in[C^0(\Omega_i)]^2$, then $u_0\in[C^0(\Omega)]^2$ (see [5]).

On the other hand, we have

$$|b^*(\Pi_{2h}u_0)-b_h^*(\Pi_{2h}u_0)|\leqslant\sum_{i=1}^{3}|R_i|, \tag{6.11}$$

where

$$R_1=b^*(u-\Pi_{2h}u,\ u,\ \Pi_{2h}u_0),$$

$$R=b^*(\Pi_{2h}u,\ u-u_h,\ \Pi_{2h}u_0),$$

$$R=b^*(\Pi_{2h}u-u_h,\ u_h,\ \Pi_{2h}u_0).$$

In the same way as in proving (5.33), we can get

$$|R_i|\leqslant ch\|u\|_{2,\Omega}\|\nabla\cdot u\|_{0,\Omega},\ 1\leqslant i\leqslant 3, \tag{6.12}$$

where constant c is only dependent on the exact solution u of velocity. Therefore, the desired inequality (6.8) is obtained by combining (6.9)—(6.12). The proof is completed by considering the relations (6.7) and (6.8).

As the final result, we give the L^2-error estimate of approximate velocity u_h.

Theorem 6.2. *There exists a constant c independent of h such that*

$$\|u-u_h\|_{0,\Omega}\leq ch^2\|u\|_{2,\Omega}. \tag{6.13}$$

Proof. Let us consider an auxiliary linear problem: for any given function $g\in[L^2(\Omega)]^2$, find $(\tilde{\sigma}, \tilde{u})\in H_V\times \mathring{H}_U$, satisfying

$$d(\tau, \tilde{\sigma})-B(\tau, \tilde{u})=0, \quad \forall\tau\in H_V, \tag{6.14a}$$

$$b^*(v, u, \tilde{u})+b^*(u, v, \tilde{u})+B(\tilde{\sigma}, v)=(v, g), \quad \forall v\in \mathring{H}_U, \tag{6.14b}$$

where the definitions of $d(\cdot, \cdot)$, $B(\cdot, \cdot)$ and $b^*(\cdot, \cdot, \cdot)$ are the same as before, and they satisfy all conditions of Theorem 4.1. From the results of the Stokes problem (see [23]), we know that there exist unique solutions for the auxiliary problem (6.14), and

$$\|\tilde{\sigma}\|_{1,\Omega}+\|\tilde{u}\|_{2,\Omega}\leq c\|g\|_{0,\Omega}. \tag{6.15}$$

Denoting by $(\tilde{\sigma}_h, \tilde{u}_h)\in V_h\times \mathring{U}_h$ the MSFE solutions corresponding to (6.14), and taking $v=u_h-u$ in (6.14b), we obtain

$$(u_h-u, g)=b^*(u_h-u, u, \tilde{u})+b^*(u, u_h-u, \tilde{u})+B(\tilde{\sigma}, u_h-u). \tag{6.16}$$

Considering that

$$b^*(u_h, u_h, \tilde{u}_h)-b^*(u, u, \tilde{u}_h)+B(\sigma-\sigma_h, \tilde{\sigma})=0,$$

we can write (6.16) as

$$(u_h-u, g)=\sum_{i=1}^{6}R_i, \tag{6.17}$$

where

$$R_1=b^*(u_h-u, u_h, \tilde{u}-\tilde{u}_h),$$
$$R_2=b(u, u_h-u, \tilde{u}-\tilde{u}_h),$$
$$R_3=-b^*(u_h-u, u_h-u, \tilde{u}),$$
$$R_4=B(\tilde{\sigma}-\tilde{\sigma}_h, u-u_h),$$
$$R_5=B(\sigma-\sigma_h, \tilde{u}_h-\tilde{u}),$$
$$R_6=d(\sigma-\sigma_h, \tilde{\sigma}_h-\tilde{\sigma}).$$

We can prove from the continuities of $d(\cdot, \cdot)$, $B(\cdot, \cdot)$ and $b^*(\cdot, \cdot, \cdot)$, and the result of Theorem 5.1,

$$|R_i|\leq ch^2\|u\|_{2,\Omega}\|g\|_{0,\Omega}, \quad 1\leq i\leq 6, \tag{6.18}$$

where constant c only depends upon u. Therefore, from the definition of dual norm and (6.17)—(6.18), we obtain

$$\|u-u_h\|_{0,\Omega}=\sup_{g\in[L^2(\Omega)]^2}\frac{|(u_h-u, g)|}{\|g\|_{0,\Omega}}\leq ch^2\|u\|_{2,\Omega}.$$

This proves Theorem 6.2.

We shall conclude this paper with the following remark.

Remark 6.1. All results obtained in the paper can be extended to the case of three dimensions. A main difficulty is in two B–B inequalities. Fortunately, [24—25] seem to have provided a strong means for the MSFEM. Moreover, we think it possible to introduce quadrilateral element in the MSFEM for the Navier–Stokes problem.

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