SIMPLICIAL METHODS AND APPROXIMATION OF SEVERAL SOLUTIONS*

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Since Scarf gave in [1] his constructive proof of Brouwer's fixed-point theorem, simplicial methods for solving nonlinear equations have developed rapidly. In 1972, Eaves[2] and Merrill[3] made a substantial improvement on Scarf's algorithm. They approximated zeros of a continuous mapping f by solving a series of piecewise linear approximations of $f(x)=0$. Their work made simplicial methods practical and efficient. Simplicial methods and homotopy continuation methods are closely related. But the former only require that the involved mapping is continuous and need no calculation of derivatives. As a constructive implementation of degree arguments, simplicial methods have a wider range of application than classical iterative methods which are based on the contraction mapping principle. For the survey of simplicial methods and their application, see [4].

There are a lot of discussions on the properties of simplicial methods. It is known that simplicial methods are closely related to the Brouwer degree theory. In this paper we intend to use the properties of piecewise linear approximations of continuous mappings and the Brouwer degree theory to analyze Eaves and Saigal's deformation method (see [2], [6]) and apply our results in discussing the problem of approximating several solutions.

§ 1. Definitions, Notations and Preliminaries

Let T be a triangulation of R^n (for definition, see [4]). For $0\leqslant m\leqslant n$ we denote by T^m the collection of all m-faces (i.e. convex closures of $m+1$ vertices of a simplex in T). Especially $T^n=T$; T^0 is composed of all vertices of elements of T. Define

$$|T^m|=\bigcup_{\sigma\in T^m}\sigma,\quad 0\leqslant m\leqslant n,$$

$$\partial T=\{\tau\in T^{n-1};\ \text{there is only one } \sigma\in T \text{ such that } \tau\subset\sigma\},$$

$$|\partial T|\bigcup_{\tau\in\partial T}\tau.$$

For $\sigma\in T$, define

$$\operatorname{diam}(\sigma)=\max_{x,y\in\sigma}\|x-y\|,$$

$$\theta(\sigma)=\rho/\operatorname{diam}(\sigma),$$

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where $\|\cdot\|$ is a given norm and ρ is the radius of the maximum sphere contained in σ. Define

$$\operatorname{mesh}(T)=\sup_{\sigma\in T}\operatorname{diam}(\sigma),$$

$$\theta(T)=\inf_{\sigma\in T}\theta(\sigma).$$

For $\varepsilon>0$ we define $\varepsilon T=\{\varepsilon\sigma;\ \sigma\in T\}$. It is obvious that $\operatorname{mesh}(\varepsilon T)=\varepsilon\operatorname{mesh}(T)$ and $\theta(\varepsilon T)=\theta(T)$.

We call continuous mapping $f\colon |T|\to R^n$ a labeling. There exists a unique $f_T\colon |T|\to R^n$ such that

(1) $f_T(x)=f(x)$, if $x\in T^0$,

(2) f_T is affine on every $\sigma\in T$.

Hence for every $\sigma\in T$, there are a unique $n\times n$ matrix A_σ and a unique $b_\sigma\in R$ such that $f_T(x)=A_\sigma x+b_\sigma$ for $x\in\sigma$.

Let $\sigma\in T$. We say that σ is f-completely labeled if there is an $\varepsilon_0>0$ such that $(\varepsilon, \varepsilon^2, \cdots, \varepsilon^n)^T\in f_T(\sigma)$ if $0\leqslant\varepsilon\leqslant\varepsilon_0$.

The following lemma is easy to prove by a compact argument.

Lemma 1. *Let $f\colon R^n\to R^n$ be a continuous mapping and $C\subset R^n$ a bounded closed set such that $0\overline{\in} f(C)$. Then there exists an $\eta_0>0$ such that if $\operatorname{mesh}(T)\leqslant\eta_0$, f_T has no zero-point on any $\sigma\in T$ if $\sigma\cap C\neq\emptyset$.*

Let T be a triangulation of R^n and $f\colon R^n\to R^n$ a continuous mapping. Define mapping $I_f\colon T\to\{-1, 1, 0\}$ by

$$I_f(\sigma)=\begin{cases}\operatorname{sign}\det A_\sigma, & \text{if } \sigma \text{ is } f\text{-completely labeled},\\ 0, & \text{otherwise.}\end{cases}$$

Now we consider triangulations of $R^n\times[a, b]$. If T is such a triangulation, it is obvious that $|\partial T|=R^n\times\{a\}\cup R^n\times\{b\}$. We call a mapping $\lambda\colon R^n\times[a, b]\to R^n$ a labeling on $[T]$, and define λ_t by

$$\lambda_t(x)=\lambda(x, t),\quad x\in R^n,\ t\in[a, b].$$

If $\tau\in T^n$ and there exists an $\varepsilon_0>0$, such that $(\varepsilon, \varepsilon^2, \cdots, \varepsilon^n)\in\lambda_T(\tau)$ if $0\leqslant\varepsilon\leqslant\varepsilon_0$, we call τ a λ-completely labeled n-face. For a given $\sigma\in T$, σ contains either two or no λ-completely labeled n-faces (see [4]). We call $\sigma\in T$ an almost λ-completely labeled $n+1$-simplex if it has two λ-completely labeled n-faces.

Let a λ-completely labeled n-face τ_0 be given. Then τ_0 decides a unique chain of λ-completely labeled n-foces $\{\tau_i\}_{i=N_1}^{N_2}$, where $-\infty\leqslant N_1\leqslant 0\leqslant N_2\leqslant+\infty$ and $N_1\neq N_2$. If both N_1 and N_2 are finite, we say that this λ-completely labeled chain is finite. If N_1 (resp. N_2) is finite, we say that τ_{N_1} (resp. τ_{N_2}) is an end face of $\{\tau_i\}_{i=N_1}^{N_2}$.

Lemma 2. *Let T be a triangulation of $R^n\times[a, b]$, let $\lambda\colon |T|\to R^n$ be given and assume that τ_{N_1}, τ_{N_2} are end faces of a finite λ-completely labeled chain $\{\tau_i\}_{i=N_1}^{N_2}$ in T. If both τ_{N_1} and τ_{N_2} are in $R^n\times\{a\}$ (resp. $R^n\times\{b\}$), we have*

$$I_{\lambda_a}(\tau_{N_1})=-I_{\lambda_a}(\tau_{N_2})\ (\textit{resp. } I_{\lambda_b}(\tau_{N_1})=-I_{\lambda_b}(\tau_{N_2})).$$

If $\tau_{N_1}\subset R^n\times\{a\}$ and $\tau_{N_2}\subset R^n\times\{b\}$, we have

$$I_{\lambda_a}(\tau_{N_1})=I_{\lambda_b}(\tau_{N_2}).$$

Lemma 3. *Let $f\colon R^n\to R^n$ be continuous and $\Omega\subset R^n$ a bounded open set such that $0\overline{\in} f(\partial\Omega)$. Then there is an $\eta_0>0$ such that*

$$\deg(f, \Omega, 0) = \sum_{\sigma\subset\Omega,\, \sigma\in T} I_f(\sigma)$$

holds if T, a triangulation of R^n, satisfies $\mathrm{mesh}(T)\leqslant\eta_0$.

For proofs of Lemma 2 and Lemma 3, see [4], [5].

Now we state briefly the deformation algorithm of Eaves and Saigal.

Let $\delta>0$, and let T_k be a triangulation of $R^n\times[1-2^{-k}, 1-2^{-k-1}]$ such that $\mathrm{mesh}(T_k)\leqslant\delta$ $(k=0, 1, \cdots)$, $T=\bigcup_{k=1}^{+\infty} T_k$ is a triangulation of $R\times[0, 1)$ and

$$\lim_{k\to+\infty}\mathrm{mesh}(T_k)=0.$$

We call T a refining triangulation of $R\times[0, 1)$; see [7], [8], for examples.

Define labeling λ: $R^n\times[0, 1)\to R^n$ by

$$\lambda(x, t)=\begin{cases} A(x-x_0), & t=0,\\ f(x), & 0<t<1,\end{cases} \tag{1}$$

where f: $R^n\to R^n$ is a continuous mapping, A is an $n\times n$ nonsingular matrix and $x_0\in R$.

Algorithm 1 (Deformation Algorithm). Under T, a given refining triangulation of $R^n\times[0, 1)$, and the labeling λ defined by (1), there is a unique λ-completely labeled n-face $\tau_0\in\partial T$. Starting from τ_0, the algorithm generates a sequence of λ-completely labeled n-faces and related $n+1$-simplices

$$\tau_0,\ \sigma_0,\ \tau_1,\ \sigma_1,\ \cdots,\ \tau_k,\ \sigma_k,\ \cdots.$$

The algorithm will not end because there is only one λ-completely labeled n-face on ∂T.

It is well known that if $\{\tau_k\}$ remains bounded, and a sequence $\{(x_k, t_k)\}$ is chosen such that $(x_k, t_k)\in\tau_k$, then any limit point of $\{x_k\}$ is a zero-point of f. In the following discussion, we also use the term "limit points of $\{\tau_k\}$" because the limit points we have just mentioned do not depend on the choice of $\{(x_k, t_k)\}$.

For conditions under which $\{\tau_k\}$ is bounded, see [4].

§ 2. Completely Labeled Semichain and Index of Isolated Zero Point

Assume that x^* is an isolated zero-point of f: $R^n\to R^n$, a continuous mapping. Then we can choose a neighborhood of x^*, $N(x^*)$, such that $f^{-1}(0)\cap\overline{N(x^*)}=\{x^*\}$. Hence $\deg(f, N(x^*), 0)$ is well defined and does not depend on the choice of $N(x^*)$. We call it the index of x^*, an isolated zero-point of f, and denote it by index (f, x^*). Given T, a refining triangulation of $R^n\times[0, 1)$, and the labeling defined by (1), we call $\{\tau_i\}_{i=0}^N$ a λ-completely labeled semichain, if τ_i $(i=0, 1, \cdots, N)(N\leqslant+\infty)$, are λ-completely labeled n-faces, $\tau_i\neq\tau_j$ when $i\neq j$, τ_{i-1} and τ_i are two n-faces of the same $n+1$ simplex in T for $i>0$, and $\tau_N\in\partial T$ when N is finite. If $\tau_0\overline{\in}\partial T$, then τ_0 can decide two completely labeled semichains. We say the union of the two λ-completed labeled semichain is a λ-completed labeled chain. If $\tau_0\in T$, we also call the only λ-completely labeled semichain defined by τ_0 a λ-completely labeled chain.

It is easy to see the following relation.

$$\left\{\begin{array}{l}\text{Two } \lambda\text{-completely labeled semichains containing a common} \\ n\text{-face is an equivalence relation.}\end{array}\right. \tag{2}$$

In the following discussion, by "a λ-completely labeled semichain" we always mean an equivalence class.

Theorem 1. *Let f: $R^n \to R^n$ be continuous, and x^* an isolated zero-point of f. Let T be a refining triangulation of $R^n \times [0, 1)$ and λ be defined by* (1). *Then the number of λ-completely labeled semichains converging to x^* is*

$$|\operatorname{index}(f, x)| + 2\nu,$$

where ν is a nonnegative integer depending on T.

Proof. Because x^* is an isolated zero-point of f, there is an $N(x^*)$, a neighborhood of x^*, such that $\overline{N(x^*)} \cap f^{-1}(0) = \{x^*\}$. Define P: $R^n \times [0, 1) \to R^n$ by $P(x, t) = x$, $x \in R^n$, $t \in [0, 1)$. By the refining property of T, there is a positive integer k_0, such that

$$\sum_{\substack{\tau \in T^n \\ \tau \subset N(x^*) \times (1-2^{-k})}} I_f(P(\tau)) = \operatorname{index}(f, x^*)$$

for $k \geq k_0$, and there is no almost λ-completely labeled $n+1$ simplex intersecting $\partial N(x^*) \times [1-2^{-k_0}, 1)$. Assume that in $R^n \times \{1-2^{-k_0}\}$ there are l_1 λ-completely labeled n-faces $\tau_1, \cdots, \tau_{l_1}$ satisfying $I(P(\tau_i)) = +1$ $(i=1, \cdots, l_1)$, and l_2 λ-completely labeled n-faces $\tilde{\tau}_1, \cdots, \tilde{\tau}_{l_2}$ satisfying $I_f(P(\tilde{\tau}_j)) = -1$ $(j=1, \cdots, l_2)$. Then any λ-completely labeled semichain in $T|_{R^n \times [1-2^{-k_0}, 1)}$ which contains some τ_i or $\tilde{\tau}_j$ remains in $N(x^*) \times [1-2^{-k_0}, 1)$, and such semichain has limit point x^* if and only if it has only one end face in $R^n \times \{1-2^{-k_0}\}$. Assume that λ-completely labeled semichains in $T|_{R^n \times [1-2^{-k_0}, 1)}$ decided by $\tau_1, \cdots, \tau_{m_1}, \tilde{\tau}_1, \cdots, \tilde{\tau}_{m_2}$ have limit point x^*. Then by Lemma 2, $l_1 - m_1 = l_2 - m_2$. Hence

$$\operatorname{index}(f, x) = l_1 - l_2 = m_1 - m_2$$

and there is a nonnegative integer ν (depending on T) such that

$$m_1 + m_2 = |\operatorname{index}(f, x^*)| + 2\nu. \quad \blacksquare$$

If f is continuously differentiable and $Df(x^*)$ is nonsingular, then the ν in Theorem 1 is zero. This result can be proved by using algebraic topology, but we will use another method to prove it to obtain a useful intermedium result.

Lemma 4. *Let f: $R^n \to R^n$ be continuously differentiable, T be a triangulation of R^n, and $\sigma \in T$. Then for any $x \in \sigma$,*

$$\|A_\sigma - Df(x)\| \leq 4\omega(Df, \operatorname{diam}(\sigma))/\theta(\sigma), \tag{3}$$

where $\omega(Df, \delta) = \max\limits_{\substack{x, y \in R^n \\ \|x-y\| < \delta}} \|Df(x) - Df(y)\|$.

Proof. First, by formula

$$f(y) - f(x) - Df(x)(y-x) = \int_0^1 (Df(x+t(y-x)) - Df(x))(y-x)dt, \quad x, y \in R^n$$

we have

$$\|f(y) - f(x) - Df(x)(y-x)\| \leq \omega(Df, \|x-y\|)\|x-y\|. \tag{4}$$

From

$$A_\sigma(x_i - x_0) = f(x_i) - f(x_0), \quad i = 0, 1, \cdots, n$$

we know that for any $x\in\sigma$,

$$(A_\sigma-Df(x))(x_i-x_0)=f(x_i)-f(x)-Df(x)(x_i-x)-(f(x_0)-f(x)-Df(x)(x_0-x)),\quad i=0,1,\cdots,n.$$

By (4) we obtain

$$\|(A_\sigma-Df(x))(x_i-x_0)\|\leqslant 2\omega(Df,\operatorname{diam}(\sigma))\cdot\operatorname{diam}(\sigma).$$

It follows that for any $x, u, v\in\sigma$

$$\|(A_\sigma-Df(x))(u-v)\|\leqslant 4\omega(Df,\operatorname{diam}(\sigma))\cdot\operatorname{diam}(\sigma).\tag{5}$$

Denote by ρ the radius of the maximum sphere contained in σ. From (5) we know that

$$\|A_\sigma-Df(x)\|=\sup_{\|w\|=\rho}\|(A-Df(x))w\|/\|w\|\leqslant 4\omega(Df,\operatorname{diam}(\sigma))/\theta(\sigma),\quad x\in\sigma. \blacksquare$$

Lemma 4 gives an estimate of $\|A_\sigma-Df(x)\|$. In Saigal[9] there is a similar result, but he assumed that $Df(x)$ is Lipschitz continuous. For convenience, in Lemma 4 we assume that f is continuously differentiable in R^n, and $\omega(Df,\cdot)$ is also considered in R^n. For practical application, we only need to consider a convex open set which contains the involved set.

Lemma 5. *Let $\Omega\to R^n$ be a bounded open set. Assume that $f:\Omega\to R^n$ is continuously differentiable, $x^*\in f^{-1}(0)$, and $Df(x^*)$ is nonsingular. Let D be an open neighborhood of x^* such that $\bar D\subset\Omega$ and $f^{-1}(0)\cap\bar D=\{x^*\}$. Let T be a triangulation of R^n, $\theta(T)>0$. Then there exists an $\varepsilon_0>0$ such that, if $0<\varepsilon\leqslant\varepsilon_0$, under triangulation $T_\varepsilon=\varepsilon T$ there is a unique f-completely labeled n-simplex σ intersecting D, and $\sigma\subset D$.*

Proof. There is an open convex neighborhood $D'\subset D$ such that $Df(x)$ is nonsingular for $x\in D'$. By Lemma 1 we know that there exists an $\varepsilon_1>0$ such that, if $0<\varepsilon\leqslant\varepsilon_1$, $\bar D\backslash D'$ does not intersect any f-completely labeled simplex in T_ε. And by Lemma 4 we know that there exists an $\varepsilon_2>0$, such that, if $0<\varepsilon\leqslant\varepsilon_2$, then $\det A_\sigma \det Df(x^*)>0$ for $\sigma\in T_\varepsilon$ and $\sigma\subset D'$. Let $\varepsilon_0=\min\{\varepsilon_1,\varepsilon_2\}$. We can obtain the needed conclusion by using Lemma 3. $\blacksquare$

Corollary 1. Let $f: R^n\to R^n$ be continuous and $x^*\in f^{-1}(0)$. Assume that f is continuously differentiable and $Df(x^*)$ is nonsingular. Then the ν in Theorem 1 is 0.

§ 3. Deformation Algorithm for Approximation of Several Solutions

When we attempt to solve a nonlinear equation which has several solutions, it is difficult to approximate different solutions. If f has some smooth and regular properties, theoretically, by using Newton's method we can approximate all solutions to $f(x)=0$ if we choose correct initial points. But if some zero-point has an overwhelming domain of attraction, choosing the correct initial points may be very difficult. Therefore, a generally concerned problem is how to approximate another zero-point of f if we have already obtained a satisfactory approximation of a zero-point of f.

Simplicial methods have been successfully used to approximate all solutions to system of algebraic equations (see [10], [11]). Let $p: C^n\to C^n$ be a polynomial mapping. We identify C^n with R^{2n}. Choose a polynomial mapping $q: C^n\to C^n$ which has $d=\prod_{j=1}^{n} d_j$ (where d_j is the degree of p_j, the jth component of p) simple zeros already known. Define labeling λ by

$$\lambda(x, t)=\begin{cases} p(x), & \text{if } 0<t<1, \\ q(x), & \text{if } t=0. \end{cases}$$

Then all zero-points of p can be approximated by using the deformation algorithm.

But the idea can not be applied to general mappings because in general cases we cannot know the exact number of solutions and global and local Brouwer degrees of the involved mapping. Without such information, we can not make the correct choice of "initial labeling".

In 1972, Jeppson introduced another kind of method (see [4]). In the version of deformation method, Jeppson's method can be described as defining a labeling $\tilde{\lambda}$: $R^n\times[0, 1)\to R^n$ such that there is no $\tilde{\lambda}$-completely labeled n-face at $R^n\times\{0\}$ in order to make different zero-points connected by a $\tilde{\lambda}$-completely labeled chain. Jeppson's method has been used to approximate several solutions to some nonlinear PDEs, but strict theoretical analysis has not been given. In fact, Allgower and Georg[4,12] only gave the conditions under which the involved problem has a second solution in a certain region, but did not give a strict analysis of the algorithm. In the following discussion, we will give a strict theoretical analysis.

Condition 1. Let $\Omega\subset R^n$ be a bounded open set and f: $\bar{\Omega}\to R^n$ continuous. Assume that $x^*\in\Omega$ is an isolated zero-point of f, and there is a $d\in R^n$ such that for every $x\in\partial\Omega$ there exists a $v_x\in R^n$ satisfying $v_x^T d>0$, $v_x^T f(x)>0$.

Define labeling $\tilde{\lambda}$: $R^n\times[0, 1)\to R^n$ by

$$\tilde{\lambda}(x, t)=\begin{cases} f(x), & \text{if } 1-2^{-k_0}\leqslant t<1, \\ d, & \text{if } 0\leqslant t<1-2^{-k_0}, \quad k_0>0. \end{cases} \tag{6}$$

Algorithm 2. Let Condition 1 be satisfied and a refining triangulation of $R^n\times[0, 1)$ be given. Define $\tilde{\lambda}$ by (6). Assume that mesh (T) is small enough such that there is no $\tilde{\lambda}$-completely labeled n-face intersecting $\partial\Omega$. Choose a $\tilde{\lambda}$-completely labeled n-face $\tau_0\subset R^n\times\{1-2^{-k_1}\}$, where $k_1\geqslant k_0$, such that the $\tilde{\lambda}$-completely labeled semichain $\{\tau_k\}_{k=0}^{+\infty}$ in $T|_{R^n\times[1-2^{-k_1},1)}$ converges to x^* (if x^* has already been approximated by using Algorithm 1, such τ_0 is available). Then there exists a unique $\sigma\in T|_{R^n\times[1-2^{-k_1},1)}$ such that $\tau_0\subset\sigma$. Starting from $\tilde{\tau}_0=\tau_0$, generate a sequence of $\tilde{\lambda}$-completely labeled n-faces in $T\setminus\{\sigma\}$:

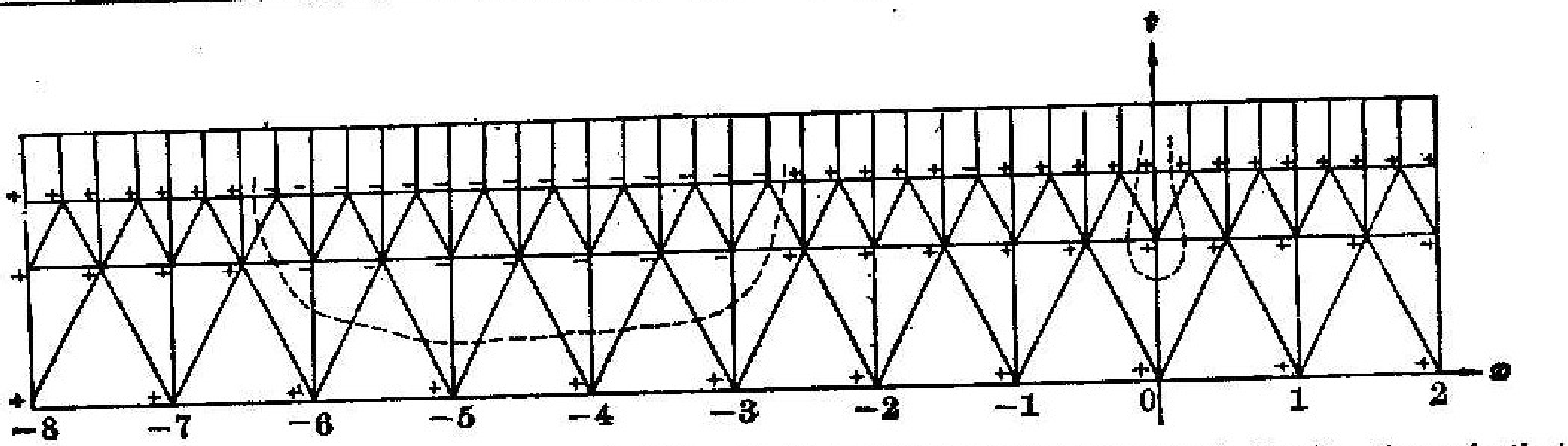
$$\tilde{\tau}_0, \tilde{\tau}_1, \cdots \tilde{\tau}_k, \cdots.$$

The algorithm will not end. If mesh (T) is small enough, $\{\tilde{\tau}_k\}_{k=0}^{+\infty}$ will be contained by $\Omega\times[0, 1)$. Hence any of its limit points is a zero-point of f.

The following example shows that sometimes Algorithm 2 fails to approximate a second zero-point of f.

Example. Let $f(x)=x^2(x+e)(x+2\pi)$, $x\in R$. Let $\Omega=(-8, 2)$, $d=1$, $x^*=0$. Then f, Ω, d satisfy Condition 1. But by using Algorithm 2 we can not approximate a zero-point other than $x^*=0$ if we start near it (see the figure).

Now the problem is when $\{\tilde{\tau}_k\}$ has a limit point which is different from x^*. The following theorem tells us that if f is a C^1 mapping and $Df(x^*)$ is nonsingular,



In the figure, signs of $\tilde{\lambda}(x, t)$ at vertices of siplices in T, a refining triangulation of $R\times[0, 1)$ such that mesh $(T)\leqslant 1$, are marked by "+" or "−" or "0". $\tilde{\lambda}$-completely labeled chains are marked by dotted lines.

Algorithm 2 will succeed.

Theorem 2. *Let $\Omega\subset R^n$ be a bounded open set, and let f: $\overline{\Omega}\to R^n$ be continuous on $\overline{\Omega}$ and continuously differentiable in Ω. Assume that $x^*\in f^{-1}(0)\cap\Omega$, and that $Df(x^*)$ is nonsingular. Moreover, assume that f and Ω satisfy Condition 1. Then any limit point of $\{\tilde{\tau}_k\}_{k=0}^{+\infty}$ generated by Algorithm 2 is a zero-point of f and is different from x^*.*

Proof. Suppose that $\{\tilde{\tau}_k\}$ converges to x^*. Denote max $\{i;\ \tilde{\tau}_i\subset R^n\times\{1-2^{-k_1}\}\}$ by i_0. Then $i_0\neq 0$, and $\tilde{\tau}_0$, τ_{i_0} decide two different $\tilde{\lambda}$-completely labeled semichains in $T|_{R^n\times[1-2^{-k_1},1)}$ and both of the chains converge to x^*. This contradicts Corollary 1. ∎

Under the conditions of Theorem 2, if x^* is approximated by using Algorithm 1, denoting max$\{j;\ \tau_j\subset R^n\times\{1-2^{-k_0}\}\}$ by j_0, we can choose $\tilde{\tau}_0=\tau_{j_0}$, where k_0 is a positive integer. k_0 can be chosen arbitrarily. But considering the efficiency of computation, it seems to be good to choose $k_0=1$ because such choice may make $\{\tilde{\tau}_k\}_{k=0}^{+\infty}$ move from x^* to another zero-point of f faster at lower levels of t.

If we only assume that f is continuous and index $(f, x^*)\neq 0$, we can prove the following ∎

Theorem 3. *Let f, Ω and x^* satisfy Condition 1. Then for sufficiently large positive integer k_1, there are at least* $|$index$(f, x^*)|$ *$\tilde{\lambda}$-completely labeled n-faces in $R^n\times\{1-2^{-k_1}\}$ and near $(x^*, 1-2^{-k_1})$, such that starting from any of them, the $\tilde{\lambda}$-competely labeled n-face sequence generated by Algorithm 2 has the property that any of its limit points is a zero-point of f and is different from x^*.*

Proof. Let N, an open neighborhood of x^*, satisfy that $\overline{N}\subset\Omega$ and $f^{-1}(0)\cap\overline{N}=\{x^*\}$. Then for sufficiently large integer k_1 there is no almost $\tilde{\lambda}$-completely labeled $n+1$ simplex intersecting $\partial N\times[1-2^{-k_1}, 1)$ and for $k\geqslant k_1$

$$\sum_{\tau\subset N\times\{1-2^{-k}\}} I_f(p(\tau))=\text{index}(f, x^*).$$

Suppose that there are m $\tilde{\lambda}$-completely labeled n-faces $\tau_1, \cdots, \tau_m$ in $N\times\{1-2^{-k_1}\}$, such that any $\tilde{\lambda}$-completely labeled n-face sequence in $T|_{R^n\times[1-2^{-k_1},1)}$ decided by τ_i converges to x^*. It is easy to see that

$$\text{index}(f, x^*)=\sum_{i=1}^{m} I_f(P(\tau_i)).$$

Starting from a τ_i $(1\leqslant i\leqslant m)$, perform Algorithm 2. It is possible that after finite steps we reach $\tau_j(1\leqslant j\leqslant m,\ j\neq i)$. Assume that all such τ_i are $\tau_1, \cdots, \tau_l$. According to Lemma 2,

$$\sum_{i=1}^{l} I_f(P(\tau_i))=0.$$

Then starting from any v_i, $l+1\leqslant i\leqslant m$, any limit point of the obtained $\tilde{\lambda}$-completely labeled n-face sequence should be a zero-point of f and be different from x^*. Moreover, we have

$$\operatorname{index}(f, x^*)=\sum_{i=l+1}^{m} I_f(P(v_i)).$$

Hence $m-l\geqslant|\operatorname{index}(f, x^*)|$. ▮

The above discussion offers a theoretical ground for using Algorithm 2 to approximate several solutions. Because of the wide application of the degree theory to proving the existence of several solutions to nonlinear PDEs and other nonlinear problems, the algorithm discussed here can be used to numerically approximate several solutions to these problems. In [4] and [12] we can find such application.

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