GAUSS-NEWTON-REGULARIZING METHOD FOR SOLVING COEFFICIENT INVERSE PROBLEM OF PARTIAL DIFFERENTIAL EQUATION AND ITS CONVERGENCE*

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Introduction

In this paper we define a new nonlinear operator of the coefficient inverse problem of the wave equation and heat equation such that the inverse problem will be reduced to a new nonlinear operator equation. This new nonlinear operator is most useful for extending to 2-D and 3-D inverse problem of the wave equation and heat equation. We used the Gauss-Newton method to solve the regularizing equation of the above nonlinear operator equation of the inverse problem of 2-D elastic wave equation. The iterative process is very stable and yields excellent numerical results ([3], [4]). Here, we study in detail the properties of this nonlinear operator and prove the convergence of this iterative solution to the regularizing solution of the inverse problem in Tikhonov's sense. In particular, we prove that the conditions of Theorem 5.1 are entirely satisfied.

§ 1. A New Statement of the Coefficient Inverse Problem of 1-D Wave Equation and Heat Equation

1.1. The inverse problem of 1-D wave equation and heat equation

It is well known that, after Laplace transformation ([3]), the coefficient inverse problem of 1-D wave equation and heat equation can be reduced to finding a coefficient function $k(x)$ in

$$\Sigma^* = \{k(x) \in C^1[0, 1],\ 0<\gamma_1 \leqslant k(x) \leqslant \gamma_2,\ \|dk(x)/dx\| \leqslant \beta_1\},$$

such that a solution, $u(x, s)$, of the ordinary differential equation

$$d(k(x)du/dx)/dx + s^{\theta}u = 0, \quad s>0,\ 0<x<1, \tag{1.1}$$

$$u(0, s) = F(s),\ du/dx(1, s) = 0, \quad s>0, \tag{1.2}$$

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satisfies the additional condition:

$$du/dx(0, s)=G(s), \quad s>0, \tag{1.3}$$

where $\theta=2$ for the wave equation, $\theta=1$ for the heat equation. The $F(s)$ and $G(s)$ are known data, and satisfy certain compatible conditions such that the above inverse problem has a solution.

In what follows, we will consider the inverse problem of the weak form of (1.1)—(1.3).

1.2. A new nonlinear operator of the coefficient inverse problem

Definition.

$$\Sigma=\{k(x)\in C[0, 1],\ 0<k_0\leqslant k(x)\leqslant k_1\}. \tag{1.4}$$

For fixed $k(x)\in\Sigma$ and $s>0$, let $u_1(x, s; k)$ be a solution of

$$\int_0^1 \{k(x)\cdot du_1/dx\cdot dv/dx+s^\theta\cdot u_1\cdot v\}dx=0, \quad s>0, \tag{1.5}$$

$$u_1(0, s)=F(s),\ s>0, \text{ for all } v\in H^1[0, 1],\ v(0)=0, \tag{1.6}$$

and let $u_2(x, s; k)$ be a solution of

$$\int_0^1 \{k(x)\cdot du_2/dx\cdot dv/dx+s^\theta\cdot u_2\cdot v\}dx=-k(0)G(s)v(0), \tag{1.7}$$

for

$$\text{all} \quad v\in H^1[0, 1], \quad s>0. \tag{1.8}$$

Definition.

$$T(k)=\int_0^1 \{k(x)(d(u_1-u_2)/dx)^2+s^\theta(u_1-u_2)^2\}dx. \tag{1.9}$$

$T(k)$ is a new nonlinear operator of the coefficient inverse problem of 1-D wave equation and heat equation, so that the coefficient inverse problem of weak form of the wave equation and heat equation will be reduced to the following nonlinear operator equation

$$T(k)=0. \tag{1.10}$$

§ 2. Gauss-Newton-Regularizing Method for Solving the Coefficient Inverse Problem of the Wave Equation and Heat Equation

2.1. Hilbert space $L_w(0, \infty)$

In Section 1.2, we took Σ as a domain of the operator $T(k)$, here we have to define a range of $T(k)$.

Definition. *$L_w(0, \infty)$ is a Banach space consisting of all functions $f(s)$ which are defined in $(0, \infty)$ and satisfies*

$$\int_0^\infty f^2(s)w(s)ds<+\infty, \tag{2.1}$$

with a norm

$$\|f\|_w^2=\int_0^\infty f(s)^2\cdot w(s)ds, \tag{2.2}$$

where

$$w(s)=s^{2\theta}/(1+s^2)^{2\theta}. \tag{2.3}$$

Definition. *The inner product of $f(s)$ and $g(s)$ which are in $L_w(0, \infty)$ is*

$$(f, g)_w=\int_0^\infty f(s)\cdot g(s)\cdot w(s)ds. \tag{2.4}$$

2.2. Tikhonov's variational problem[1] of the coefficient inverse problem

Let A be a linear bounded self-adjoint positive defined operator such that $\Omega(k)=(Ak, k)$ is a stable functional in Tikhonov's sense[1], for example,

$$\Omega(k)=(Ak, k)=\int_0^1 (k^2+(dk/dx)^2)dx. \tag{2.5}$$

By the regularizing method[1], the coefficient inverse problem of 1-D wave equation and heat equation will be reduced to finding a function $k_\alpha(x)$ which minimizes the following nonlinear functional, i.e.,

$$J(k_\alpha)=\min_{k\in\Sigma} J(k), \tag{2.6}$$

where

$$\begin{aligned}J(k)&=\|T(k)\|_w^2+\alpha(Ak, k)\\&=\int_0^\infty\left\{\int_0^1 (k(x)\cdot(d(u_1-u_2)/dx)^2+s^\theta(u_1-u_2)^2)dx\right\}^2 w(s)ds+\alpha(Ak, k).\end{aligned} \tag{2.7}$$

2.3. Euler equation

The Euler equation of (2.6) is

$$T'^*(k)\cdot T(k)+\alpha Ak=0. \tag{2.8}$$

2.4. Gauss-Newton method

We use the Gauss-Newton method to solve the nonlinear equation (2.8), so as to obtain the Gauss-Newton-regularizing iteration for solution the coefficient inverse problem. The iterative process is as follows:

(1) If $k_n(x)$ is known by $(n-1)$th iterative step, we can compute $T(k_n)$ and its first order Frechet derivative operator $T'(k_n)$.

(2) Solve the following linear equation

$$\begin{aligned}&(T'^*(k_n)\cdot T'(k_n)+T''^*(k_0)\cdot T(k_0)+\alpha A)\cdot\delta k_n\\&\quad=-(T'^*(k_n)\cdot T(k_n)+\alpha A\cdot k_n),\end{aligned} \tag{2.9}$$

where $T''(k_0)$ was computed in the first step.

(3) $$k_{n+1}=k_n+\delta k_n. \tag{2.10}$$

The iteration (1)—(3) is called the Gauss-Newton-regularizing iteration. In order to compute $T'(k)$ and $T''(k)$ and study their properties, we have to study in detail the generalized Green function of equations (1.5)—(1.6) and (1.7).

§ 3. The Generalized Green Function

3.1. The generalized Green function $G_1(x, \xi)$ and $G_2(x, \xi)$

Definition. *If a function $G_1(x, \xi)$ satisfies the following conditions:*

(1) $G_1(x, \xi)\in H^1[0, 1]$ *with respect to either x or ξ,*

(2) $G_1(x, \xi)=G_1(\xi, x)$,

(3) $G_1(x, \xi)$ *is a solution of the following equation*

$$\int_0^1 \{k(\xi)\cdot\partial G_1/\partial\xi\cdot dv/d\xi + s^\theta G_1\cdot v\}d\xi = v(x), \quad 0<x<1,\ s>0, \tag{3.1}$$

$$G_1(x, 0)=0, \quad \text{for all } v\in H^1[0, 1],\ v(0)=0, \tag{3.2}$$

then $G_1(x, \xi)$ *is called the generalized Green function of equation* (1.5) *with the homogeneous boundary value condition. For simplicity, in what follows we refer to it as the generalized Green function of* (1.5).

Definition. *If a function* $G_2(x, \xi)$ *satisfies the following conditions:*

(1) $G_2(x, \xi)\in H^1[0, 1]$ *with respect to either* x *or* ξ,

(2) $G_2(x, \xi)=G_2(\xi, x)$,

(3) $G_2(x, \xi)$ *is a solution of the following equation*

$$\int_0^1 \{k(\xi)\cdot\partial G_2/\partial\xi\cdot dv/d\xi + s^\theta G_2\cdot v\}d\xi = v(x),\ 0<x<1,\ s>0,\ \text{for all } v\in H^1[0, 1], \tag{3.3}$$

then $G_2(x, \xi)$ *is called the generalized Green function of equation* (1.7).

Remark. For fixed $k\in\Sigma$ and $s>0$, $G_i(x, \xi)$ can be found. In fact, $G_i(x, \xi)$ continuously depend on $k(x)$ and s; so in what follows we often write $G_i(x, \xi; s, k)$.

Let us consider the following equations

$$\int_0^1 \{k(x)\cdot du_1/dx\cdot dv/dx + s^\theta\cdot u_1\cdot v\}dx = \int_0^1 f\cdot v\,dx, \quad s>0, \tag{3.4}$$

$$u_1(0, s)=0,\ s>0, \text{ for all } v\in H^1[0, 1],\ v(0)=0, \tag{3.5}$$

and

$$\int_0^1 \{k(x)\cdot du_2/dx\cdot dv/dx + s^\theta\cdot u_2\cdot v\}dx = \int_0^1 f\cdot v\,dx, \text{ for } v\in H^1[0, 1],\ s>0. \tag{3.6}$$

We use the generalized Green function $G_i(x, \xi)$ $(i=1, 2)$, to represent a solution of (3.4)—(3.5) and of (3.6), respectively.

Lemma 3.1. *Suppose* $G_i(x, \xi)$ *is the generalized Green function of* (1.5) *or* (1.7), $i=1, 2$, *for the fixed* $k(x)\in\Sigma$ *and* $s>0$; *then the solution of* (3.4)—(3.5) *or* (3.6) *can be represented by*

$$u_i(x, s; k) = \int_0^1 G_i(x, \xi)\cdot f(\xi)d\xi. \tag{3.7}$$

Proof. It is obvious, so we omit it.

Lemma 3.2. *Under the conditions of Lemma* 3.1, *if the right hand terms of* (3.4) *and* (3.6) *are changed to* $\int_0^1 \partial f/\partial x\cdot\partial v/\partial x\cdot dx$, *then their solutions can be represented by*

$$u_i(x, s; k) = -\int_0^1 \partial G_i/\partial\xi\cdot\partial f/\partial\xi\cdot d\xi \tag{3.8}$$

$$= -s^\theta \int_0^1 G_i(x, \xi; s, k)\cdot\int_0^\xi \partial f/\partial\eta\cdot 1/k(\eta)d\eta\cdot d\xi + \int_0^x \partial f/\partial\eta\cdot 1/k(\eta)d\eta. \tag{3.8*}$$

3.2. The properties of the generalized function

Lemma 3.3. *If $k(x)\in C^1[0, 1]$ and $G_1(x, \xi)$ is a classical Green function of*

$$d(k(x)du/dx)/dx+s^\theta u=0, \quad s>0,\ 0<x<1, \tag{3.9}$$

$$u(0, s)=0,\ du/dx(1, s)=0, \quad s>0, \tag{3.10}$$

then $G_1(x, \xi)$ is also the generalized Green function of (1.5). If the first condition of (3.10) is changed to $u'(0)=0$, then its classical Green function $G_2(x, \xi)$ is also the generalized Green function of (1.7).

Lemma 3.4. *If $k(x)=k^*$, k^* being constant, then the generalized Green function of (1.5) is*

$$G_1^*(x, \xi)=\begin{cases}[1/(\sqrt{(k^*)})\cdot s^{(\theta/2)}]\mathrm{sh}((s^{(\theta/2)})/\sqrt{(k^*)}\cdot x)\mathrm{ch}((s^{(\theta/2)})/\sqrt{k^*} \\ \quad \cdot(1-\xi))/\mathrm{ch}(s^{(\theta/2)}/\sqrt{(k^*)}), \quad 1\leqslant x\leqslant \xi, \\ [1/(\sqrt{(k^*)})\cdot s^{(\theta/2)}]\,\mathrm{sh}((s^{(\theta/2)})/\sqrt{k^*}\cdot\xi)\,\mathrm{ch}((s^{(\theta/2)})/\sqrt{k^*} \\ \quad \cdot(1-x))/\mathrm{ch}(s^{(\theta/2)}/\sqrt{(k^*)}), \quad \xi\leqslant x\leqslant 1\end{cases} \tag{3.11}$$

and

$$\int_0^1 G_1^*(x, x)dx=1/2(1/(\sqrt{(k^*)}\cdot s^{(\theta/2)}))\,\mathrm{sh}((s^{(\theta/2)})/\sqrt{k^*})/\mathrm{ch}(s^{(\theta/2)}/\sqrt{k^*})$$
$$\leqslant 1/2\cdot(1/(\sqrt{(k^*)}\cdot s^{\theta/2})). \tag{3.12}$$

Proof. Omitted.

Lemma 3.5. *If $k(x)=k^*$, k^* a constant, then the generalized Green function of (1.7) is*

$$G_2^*(x, \xi)=\begin{cases}(1/(\sqrt{(k^*)})\cdot s^{(\theta/2)})\,\mathrm{ch}((s^{(\theta/2)})/\sqrt{k^*}\cdot x)\,\mathrm{ch}((s^{(\theta/2)})/\sqrt{k^*} \\ \quad \cdot(1-\xi))/\mathrm{sh}(s^{(\theta/2)}/\sqrt{(k^*)}), \quad 1\leqslant x\leqslant\xi, \\ (1/(\sqrt{(k^*)})\cdot s^{(\theta/2)})\mathrm{ch}((s^{(\theta/2)})/\sqrt{k^*}\cdot\xi)\,\mathrm{ch}((s^{(\theta/2)})/\sqrt{k^*} \\ \quad \cdot(1-x))/\mathrm{sh}(s^{(\theta/2)}/\sqrt{(k^*)}), \quad \xi\leqslant x\leqslant 1\end{cases} \tag{3.13}$$

and

$$\int_0^1 G_2^*(x, x)dx=1/2(1/(\sqrt{(k^*)}\cdot s^{(\theta/2)})\,\mathrm{ch}((s^{(\theta/2)})/\sqrt{k^*})/\mathrm{sh}(s^{(\theta/2)}/\sqrt{k^*})$$
$$\leqslant 1/2\cdot 1/((\sqrt{k^*})\cdot s^{\theta}). \tag{3.14}$$

Proof. Omitted.

Lemma 3.6. *Suppose $k(x)\in\Sigma$, and $G_i(x, \xi; s, k)$ is the generalized Green function of (1.5) or (1.7), $i=1, 2$, respectively; then*

$$\int_0^\infty w(s)\left\{\int_0^1\int_0^1 (k(\xi)[\partial G_i/\partial\xi]^2+s^\theta\cdot G_i^2)d\xi\, dx\right\}ds<+\infty. \tag{3.15}$$

Lemma 3.7. *Suppose $k_i(x)\in\Sigma$ and $G_i(x, \xi; s, k)$ is the generalized Green function of (1.5) when $k(x)=k_i(x)$, $i=1, 2$, respectively. Then there exists a constant L independent of k_i and s, such that*

$$\int_0^\infty w(s)\int_0^1\int_0^1\left[k_1(\xi)\left(\frac{\partial(G_2-G_1)}{\partial\xi}\right)^2+s^\theta(G_2-G_1)^2\right]d\xi\cdot dx\cdot ds\leqslant L\|k_1-k_2\|_{c^0}^2. \tag{3.16}$$

Lemma 3.8. *Suppose $k_i(x)\in\Sigma$ and $G_i(x, \xi; s, k)$ is the generalized Green function of (1.7) with $k(x)=k_i$, $i=1, 2$, respectively. Then there exists a constant L independent of k_i and s such that (3.16) holds.*

Corollary 3.6. Under the condition of Lemma 3.6, the following inequality holds

$$\int_0^\infty w(s)\left(\int_0^1\int_0^1\left(k(\xi)\left(\frac{\partial G_i}{\partial \xi}\right)^2+s^\theta\cdot G_i^2\right)d\xi\, dx\right)^2 ds<+\infty. \tag{3.17}$$

Corollary 3.7. Under the conditions of Lemmas 3.7 and 3.8 $G_{ij}(x, \xi; s, k_j)$ is the generalized Green function of (1.6) and (1.8), $i=1, 2$, when $k(x)=k_j(x)$. Then the following inequality holds.

$$\int_0^\infty w(s)\left(\int_0^1\int_0^1\left[k_1(\xi)\left(\frac{\partial(G_{12}-G_{11})}{\partial \xi}\right)^2+s^\theta(G_{12}-G_{11})^2\right]d\xi\, dx\right)^2 ds\leqslant L_1\|k_1-k_2\|_{c^0}^2,$$

where L_1 is a constant independent of k_j and $s>0$.

§ 4. The Properties of T defined by (1.9)

Lemma 4.1. *Suppose $F(s)\in L_2(0,\infty)$, $k(x)\in\Sigma$, and $u_1(x, s; k)$ is a solution of* (1.5)—(1.6). *Then*

$$\int_0^1\left[k(x)\left(\frac{du_1}{dx}\right)^2+s^\theta u_1^2\right]dx\leqslant s^\theta(F(s))^2. \tag{4.1}$$

Lemma 4.2. *Suppose $k(x)\in\Sigma$, $G(s)\in L_\infty(0,\infty)$, and $u_2(x, s; k)$ is a solution of* (1.8). *Then*

$$\int_0^1\left[k(x)\left(\frac{du_2}{dx}\right)^2+s^\theta u_2^2\right]dx\leqslant\begin{cases} L_1\|G(s)\|_\infty^2\,\dfrac{k_0+s^\theta}{s^\theta}, & s<1,\\ L^2\|G(s)\|_\infty^2, & s>1,\end{cases} \tag{4.2}$$

where L_1 and L_2 are constants independent of $k(x)$ and $s>0$.

Lemma 4.3. *$T(k)$ in (1.9) maps $k(x)\in\Sigma$ into $L_w(0,\infty)$, i.e.,*

$$\int_0^\infty w(s)T^2(k)dx<+\infty. \tag{4.3}$$

Proof. Since $F(s)\in L_2(0,\infty)\in L_4(0,\infty)$ and $\|G\|_\infty^4$ is also finite, and moreover, $w(s)=\frac{s^{2\theta}}{(1+s^2)^{2\theta}}$, so by (4.1) and (4.2), (4.3) holds.

Definition. *For $k(x)\in\Sigma$, $s>0$,*

$$H_{ks}^1=\left\{u\,\middle|\,\int_0^1\left[k(x)\left(\frac{du}{dx}\right)^2+s^\theta u^2\right]dx<+\infty\right\}. \tag{4.4}$$

Denote the norm of u in H_{ks}^1 by $\|u\|_{ks}$,

$$\|u\|_{ks}^2=\int_0^1\left[k(x)\left(\frac{du}{dx}\right)^2+s^\theta u^2\right]dx.$$

Definition. *For $k(x)\in\Sigma$, $s>0$,*

$$L_w(0,\infty;H_{ks}^1)=\left\{u\int_0^\infty w(s)\left(\int_0^1\left[k(x)\left(\frac{du}{dx}\right)^2+s^\theta u^2\right]dx\right)^2 ds<\infty\right\}. \tag{4.5}$$

Denote the norm of u in $L_w(0,\infty,H_{ks}^1)$ by $\|u\|_{wks}$,

$$\|u\|_{wks}^2=\int_0^\infty w(s)\left(\int_0^1\left[k(x)\left(\frac{du}{dx}\right)^2+s^\theta u^2\right]dx\right)ds.$$

Remark. (a) By Lemmas 4.1 and 4.2, we know that

$$u_i(x, s; k)\in L_w(0,\infty;H_{ks}^1).$$

(b) If $u\in H^1[0, 1]$ and $u(0)=0$, then $u\in H_{ks}^1$.

Lemma 4.4. *If $F(s)\in L_2(0,\infty)$ and $G(s)\in L_\infty(0,\infty)$, $k_i\in\Sigma$, and $u_{ij}(x,s;k)$ is the solution of equation (1.5)—(1.6) or (1.7), $i=1,2$, with $k(x)=k_j(x)$, $j=1,2$, respectively, then*

$$\|u_{i1}-u_{i2}\|^2_{wks}=\int_0^\infty w(s)\left\{\int_0^1\left[k_1(x)\left(\frac{d(u_{i1}-u_{i2})}{dx}\right)^2+s^\theta(u_{i1}-u_{i2})^2\right]dx\right\}^2 ds$$
$$\leqslant L\|k_1-k_2\|^2_{C^0}\leqslant L_1\|k_1-k_2\|^2_{H^1}, \tag{4.6}$$

where L_1 is a Lipschitz constant independent of k_i and s.

Corollary 4.4. Under the conditons of Lemma 4.4,

$$\int_0^\infty w(s)\left(\int_0^1\left[k(x)\left(\frac{d(u_{i1}-u_{i2})}{dx}\right)^2+s^\theta(u_{i1}-u_{i2})^2\right]dx\right)^2 ds$$
$$\leqslant L\|k_1-k_2\|^4_{C^0}\leqslant L_1\|k_1-k_2\|^4_{H^1},$$

L, L_1 are constants independent of k_j and s.

Lemma 4.5. *Suppose $F(s)\in L_2(0,\infty)$ and $G(s)\in L_\infty(0,\infty)$. Then $T(k)$ defined by (1.9) is a Lipschitz continuous map from Σ into L_w, i.e., there exist constants L and L_1 independent of k and s such that*

$$\|T(k_1)-T(k_2)\|_w\leqslant L\|k_1-k_2\|_{C^0}\leqslant L_1\|k_1-k_2\|_{H^1}. \tag{4.7}$$

Next, we will study their first order and second order Frechet derivatives.

Lemma 4.6. *Suppose $F(s)\in L_2(0,\infty)$, $k(x)\in\Sigma$, and $G(s)\in L_\infty(0,\infty)$. Then the nonlinear operator $u_i(x,s;k)$ defined by (1.5)—(1.6) or (1.7), $i=1,2$, has the first order Frechet derivative $u'_{i,k}$, and*

$$u'_{i,k}\cdot h=-\int_0^1 h\,\frac{\partial G_i(x,\xi)}{\partial\xi}\,\frac{du_i}{d\xi}\,d\xi \tag{4.8}$$
$$=s^\theta\int_0^1 G_i\cdot(x,\xi;s,k)\int_0^\xi\frac{du_i}{d\eta}\,\frac{h(\eta)}{k(\eta)}\,d\eta\cdot d\xi-\int_0^x\frac{du_i}{d\eta}\cdot\frac{h(\eta)}{k(\eta)}\,d\eta. \tag{4.9}$$

Moreover, $u'_{i,k}\cdot h\in L_w(0,\infty;H^1_{ks})$.

Lemma 4.7. *Suppose $F(s)\in L_2(0,\infty)$, $G(s)\in L_\infty(0,\infty)$, and $k(x)\in\Sigma$. The nonlinear operator $u_i(x,s;k)$ is defined by (1.5)—(1.6) or (1.7), $i=1,2$, respectively. Then their first order Frechet derivative $u'_{ik}h$ is a Lipschitz continuous map from Σ into $L_w(0,\infty;H^1_{ks})$ and*

$$\|u'_{ik_1}h-u'_{ik_2}h\|^2_{wk_1s}\leqslant L\|h\|^2_{C^0}\|k_1-k_2\|^2_{C^0}, \tag{4.10}$$

where L is a Lipschitz constant independent of k_i, h, s.

Lemma 4.8. *Suppose $F(s)\in L_2(0,\infty)$, and $G(s)\in L_\infty(0,\infty)$. Then the nonlinear operator $T(k)$ defined by (1.9) has the first order Frechet derivative $T'(k)h\in L_w(0,\infty)$,*

$$T'(k)h=\int_0^1 h(x)\left[\left(\frac{du_1}{dx}\right)^2-\left(\frac{du_2}{dx}\right)^2\right]dx. \tag{4.11}$$

Corollary 4.7. Under the condit ions of Lemma 4.7,

$$\int_0^\infty w(s)\left(\int_0^1\left[k_1(x)\left(\frac{d(u'_{i,k_1}h-u'_{2,k_2}h)}{dx}\right)^2\right.\right.$$
$$\left.\left.+s^\theta(u'_{i,k_1}h-u'_{i,k_2}h)^2\,dx\right)^2 ds\leqslant L\|h\|^4_{C^0}\|k_1-k_2\|^4_{C^0}.$$

Lemma 4.9. *Under the conditions of Lemma 4.8, the first order Frechet* *differentiable $T'(k)h$ of the nonlinear operator $T(k)$ defined by (1.9) is a Lipschitz continuous map from Σ into $L_\infty(0, \infty)$, i.e., for any k_1, k_2, $h \in \Sigma$ and $h(0)=0$, there exists a constant L independent of k_i and h, such that*

$$\|T'(k_1)h - T'(k_2)h\| \leqslant L\|k_1 - k_2\|_{c^\theta}\|h\|_{c^\theta}. \tag{4.12}$$

Proof. From the representation of $T'(k)h$ and the proof of Lemma 4.5, we can obtain (4.12) immediately.

Lemma 4.10. *Under the conditions of Lemma 4.8, the nonlinear operator $T(k)$ defined by (1.9) is second order Frechet differentiable, and $T''(k)hh^*$ is a Lipschitz continuous map from Σ into $L_\infty(0, \infty)$.*

Moreover, we have

$$\begin{aligned}
T''(k)hh^* = & 2\int_0^1 \frac{h(x)}{k(x)}\cdot h^*(x)\cdot\left(\left(\frac{du_2}{dx}\right)^2 - \left(\frac{du_1}{dx}\right)^2\right)dx \\
& + 2\cdot s^\theta\cdot\int_0^1\left\{\int_0^x \frac{h(\eta)}{k(\eta)}\cdot\frac{du_1}{d\eta}\,d\eta\cdot\int_0^x\frac{h^*(\eta)}{k(\eta)}\cdot\frac{du_1}{d\eta}\,d\eta\right. \\
& \left. - \int_0^x \frac{h(\eta)}{k(\eta)}\cdot\frac{du_2}{d\eta}\,d\eta\cdot\int_0^x\frac{h^*(\eta)}{k(\eta)}\cdot\frac{du_2}{d\eta}\,d\eta\right\}dx \\
& - 2\cdot s^{2\theta}\int_0^1\int_0^1\left\{\int_0^x\frac{h(\eta)}{k(\eta)}\cdot\frac{du_1}{d\eta}d\eta\cdot G_1(x, \xi)\cdot\int_0^\xi\frac{h^*(\eta)}{k(\eta)}\cdot\frac{du_1}{d\eta}\,d\eta\right. \\
& \left. - \int_0^x\frac{h(\eta)}{k(\eta)}\cdot\frac{du_2}{d\eta}d\eta\cdot G_2(x, \xi)\cdot\int_0^\xi\frac{h^*(\eta)}{k(\eta)}\cdot\frac{du_2}{d\eta}\,d\eta\right\}dx\,d\xi
\end{aligned} \tag{4.13}$$

and for any k_1, k_2, h, $h^* \in \Sigma$ and $h(0)=h^*(0)=0$, there exists a constant L independent of k_i, h and h^*, such that

$$\|T''(k_1)hh^* - T''(k_2)hh^*\|_\infty \leqslant L\|k_1-k_2\|_{c^\theta}\|h\|_{c^\theta}\|h^*\|_{c^\theta} \leqslant L_1\|k_1-k_2\|_{H^1}\|h\|_{H^1}\|h^*\|_{H^1}. \tag{4.14}$$

§ 5. The Convergence of Gauss-Newton-Regularizing Method

5.1. The convergence of quasi-Newton Method

Let

$$\Phi(k) = T'^*(k)\cdot T(k) + \alpha Ak. \tag{5.1}$$

Then (2.8) reduces to

$$\Phi(k) = 0. \tag{5.2}$$

If we assume that equation (1.10) has a unique local solution $k^*(x)$, by [1], then a solution of (5.2), $k_\alpha(x)$, satisfies

$$\|k_\alpha(x) - k^*(x)\|_{c^\theta} \to 0, \text{ as } \alpha \to 0. \tag{5.3}$$

$k_\alpha(x)$ is called the regularizing solution of (1.10). Next, we will prove the convergence of the modified Gauss-Newton iteration (2.9)—(2.10) for solving (5.2).

Let us consider the quasi-Newton iteration for solving (5.2)

$$k_{n+1} = k_n - R_n^{-1}\Phi(k_n), \tag{5.4}$$

where R_n is a sequence of invertible linear operator. Under certain conditions of R_n and the initial point $k_0(x)$, the following theorem holds.

Lemma 5.1. *Suppose $\Phi(k)$ is Frechet differentiable in the Banach space B. Let R_n be a sequence of the invertible linear operator from B into B^*, such that for a $k_0(x)\in B$ and $\rho>0$, the sphere $S(k_0, \rho)\in B$, and for given nonnegative constants λ, ε, β, and η,*

(a) $\|R_n^{-1}\|\leq\lambda$, (5.5)

(b) $\|R_n-\Phi'(k_0)\|\leq\varepsilon$, (5.6)

(c) *if k, h are in sphere $S(k_0, \rho)$, then*

$$\|\Phi(k)-\Phi(h)-\Phi'(k_0)(k-h)\|_{B^*}\leq\beta_2\|k-h\|, \tag{5.7}$$

(d) $\|\Phi(k_0)\|_{B^*}\leq\eta$, (5.8)

$$e=\lambda(\beta_2+\varepsilon)<1,\quad r=\frac{\lambda\eta}{(1-e)}\leq\rho. \tag{5.9}$$

Under these conditions, the iteration (5.4) is well defined in $S(k_0, \rho)$ and converges to a solution $k_a(x)$ of (5.2). Furthermore, $\|k_a-k_0\|\leq r$ and $k_a(x)$ is the only solution of (5.2) contained in this sphere. The rate of the convergence is given by

$$\|k_n-k_a\|_B\leq e^n r. \tag{5.10}$$

The proof of this lemma can be found in [5].

For (5.2); if we take

$$R_n=T'^*(k_n)\cdot T'(k_n)+\alpha A,$$

then the quasi-Newton iteration is the standard Gauss-Newton iteration. If

$$R_n=T'^*(k_n)\cdot T''(k_n)+T'''^*(k_0)\cdot T(k_0)+\alpha A, \tag{5.11}$$

then it is called the modified Gauss-Newton iteration, where k_0 is the initial point,

For convenience. Let

$$N(k)=T'^*\cdot T'(k)+\alpha A. \tag{5.12}$$

It is clear that in order to prove the convergence of the iteration (2.9)—(2.10), it is sufficient to prove that under certain conditions, the conditions of Lemma 5.1 can all be satisfied.

5.2. The proof of the convergence of the Gauss-Newton-Regularizing method

The commutative diagrams of Fig. 1 and Fig. 2 are useful for understanding the range and domain of the operators $T'^*\cdot T'$ and $T'''^*\cdot T$. In Fig. 1, for fixed k, $T'(k)$ is the linear operator which maps $h\in C^0[0, 1]$ into $L_w(0, \infty)$. $T'^*T'(k)$ is the linear operator which maps $h\in C^0[0, 1]$ into $C^{0*}[0, 1]$. In Fig. 2, $T(k)$ is the linear operator for fixed k. It maps $C^0[0, 1]$ into $L_w(0, \infty)\times H^1[0, 1]$. $T'''^*\cdot T$ is the linear operator for fixed k which maps $h\in C^0[0, 1]$ into $C^*[0, 1]$.

Lemma 5.2. *For $g(s)\in L_w(0, \infty)$, we have*

$$T'^*(k)\cdot g(s)(x)=\int_0^\infty\left[\left(\frac{du_1}{dx}\right)^2-\left(\frac{du_2}{dx}\right)^2\right]\cdot g(s)\cdot w(s)\cdot ds. \tag{5.13}$$

Upon substituting (5.13) *into* (5.1), *we have*

$$\Phi(k)=\int_0^\infty\left\{\left[\left(\frac{du_1}{dx}\right)^2-\left(\frac{du_2}{dx}\right)^2\right]\cdot\int_0^1\left[k(\xi)\left(\frac{d(u_1-u_2)}{d\xi}\right)^2+s^e(u_1-u_2)\right]^2 d\xi\cdot w(s)\right\}ds+\alpha Ak. \tag{5.14}$$

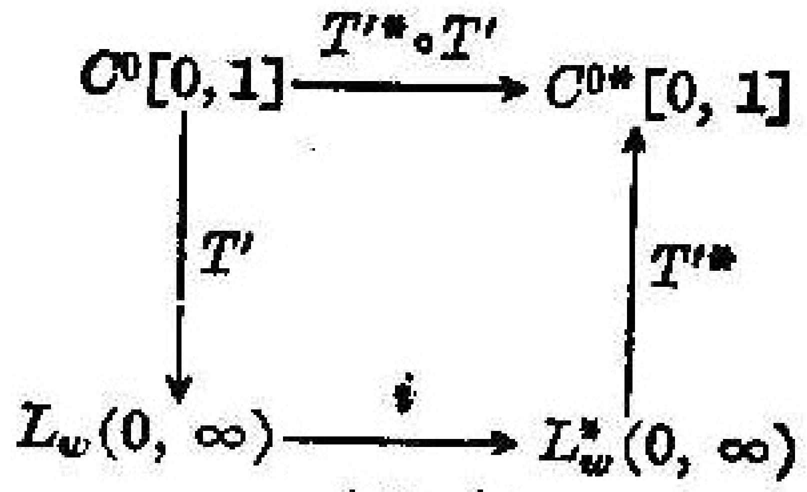

Fig. 1

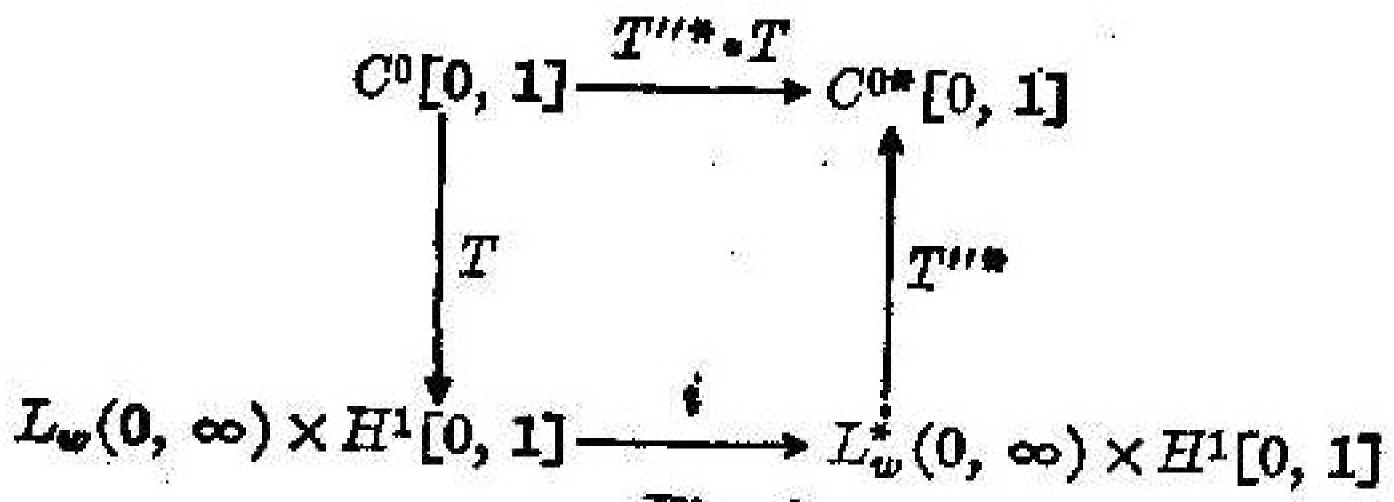

Fig. 2

Lemma 5.3. *Suppose* $k(x)\in\Sigma$ *and* $F(s)\in L_2(0,\infty)$, *and* $G(s)\in L_\infty(0,\infty)$; *then*

$$T'^*(k)\cdot T'(k)\cdot h^*=\int_0^\infty\left\{\left(\frac{du_1(x,s;k)}{dx}\right)^2-\left(\frac{du_2(x,s;k)}{dx}\right)^2\right\}$$

$$\cdot\int_0^1 h^*\left[\left(\frac{du_1(\xi,s;k)}{d\xi}\right)^2-\left(\frac{du_2(\xi,s;k)}{d\xi}\right)^2\right]d\xi\cdot w(s)ds, \qquad (5.15)$$

$$T''^*(k)\cdot T(k)\cdot h^*=2\,\frac{h^*(x)}{k(x)}\cdot\int_0^\infty\left[\left(\frac{du_2}{dx}\right)^2-\left(\frac{du_1}{dx}\right)^2\right]\cdot T(k)\cdot w(s)ds$$

$$+2\,\frac{1}{k(x)}\int_0^\infty\left\{\frac{du_1}{dx}\int_0^1\int_0^\xi\frac{h^*(\eta)}{k(\eta)}\,\frac{du_1}{d\eta}\,d\eta\,d\xi\right.$$

$$-\frac{du_2}{dx}\int_0^1\int_0^\xi\frac{h^*(\eta)}{k(\eta)}\,\frac{du_2}{d\eta}\,d\eta\,d\xi-\frac{du_1}{dx}\int_0^x\int_0^\xi\frac{h^*(\eta)}{k(\eta)}\,\frac{du_1}{d\eta}\,d\eta\,d\xi$$

$$\left.+\frac{du_2}{dx}\int_0^x\int_0^\xi\frac{h^*(\eta)}{k(\eta)}\,\frac{du_2}{d\eta}\,d\eta\,d\xi\right\}\cdot T(k)\cdot s^\theta\cdot w(s)ds$$

$$-2\,\frac{1}{k(x)}\int_0^\infty\left\{\frac{du_1}{dx}\int_0^1\int_0^1 G_1(y,\xi;s,k)\int_0^\xi\frac{h^*(\eta)}{k(\eta)}\,\frac{du_1}{d\eta}\,d\eta\,d\xi\,dy\right.$$

$$-\frac{du_1}{dx}\cdot\int_1^1\int_0^x G_1(\eta,\xi;s,k)d\eta\cdot\int_0^\xi\frac{h^*(\eta)}{k(\eta)}\,\frac{du_1}{d\eta}\,d\eta\,d\xi$$

$$-\frac{du_2}{dx}\int_0^1\int_0^1 G_2(y,\xi;s,k)\int_0^\xi\frac{h^*(\eta)}{k(\eta)}\,\frac{du_2}{d\eta}\,d\eta\,d\xi\,dy$$

$$\left.+\frac{du_2}{dx}\int_0^1\int_0^x G_2(\eta,\xi;s,k)d\eta\int_0^\xi\frac{h^*(\eta)}{k(\eta)}\,\frac{du_1}{d\eta}\,d\eta\,d\xi\right\}$$

$$\cdot T(k)\cdot s^{2\theta}\cdot w(s)ds, \qquad (5.16)$$

$$N'(k)\cdot h^*=(T''^*(k)\cdot T'(k)+T'^*\cdot T''(k))\cdot h^*$$

$$=4\int_0^\infty\frac{h^*(x)}{k(x)}\left[\left(\frac{du_2}{dx}\right)^2-\left(\frac{du_1}{dx}\right)^2\right]\cdot\left[\left(\frac{du_2}{dz}\right)^2-\left(\frac{du_1}{dz}\right)^2\right]w(s)ds$$

$$+4\,\frac{1}{k(x)}\int_0^\infty\left\{\frac{du_1}{dx}\int_x^1\int_0^\xi\frac{h^*(\eta)}{k(\eta)}\,\frac{du_1}{d\eta}\cdot d\eta\cdot d\xi\right.$$

$$\left.-\frac{du_2}{dx}\cdot\int_x^1\int_0^\xi\frac{h^*(\eta)}{k(\eta)}\,\frac{du_2}{d\eta}\,d\eta\,d\xi\right\}\cdot\left\{\left(\frac{du_1}{dz}\right)^2-\left(\frac{du_2}{dz}\right)^2\right\}\cdot s^\theta\cdot w(s)ds$$

$$-4\,\frac{1}{k(x)}\int_0^\infty\left\{\frac{du_1}{dx}\int_0^1\int_x^1 G_1(y,\xi;s,k)\cdot\int_0^\xi\frac{h^*(\eta)}{k(\eta)}\,\frac{du_1}{d\eta}\,d\eta\,d\xi\,dy\right.$$

$$\left.-\frac{du_2}{dx}\int_0^1\int_x^1 G_2(y,\xi;s,k)\int_0^\xi\frac{h^*(\eta)}{k(\eta)}\,\frac{du_2}{d\eta}\,d\eta\,d\xi\,dy\right\}$$

$$\cdot\left\{\left(\frac{du_1}{dz}\right)^2-\left(\frac{du_2}{dz}\right)^2\right\}\cdot s^{2\theta}\cdot w(s)ds. \qquad (5.17)$$

Lemma 5.4. *If* $k(x)\in\Sigma$, $F(s)\in L_2(0,\infty)$, *and* $G(s)\in L_\infty(0,\infty)$, *then* $T''^*\cdot$

$T'(k)$ is a Lipschitz continuous map from $\Sigma\in c^0[0,1]$ into $C^{0*}[0,1]$, i.e., there exists a constant $L>0$ independent of k_i such that

$$\|T''^*\cdot T'(k_1)-T''^*\cdot T'(k_2)\|\leqslant L\|k_1-k_2\|^2_{c^0}. \tag{5.18}$$

Lemma 5.5. *Under the conditions of Lemma 5.3, the operators $T'^*\cdot T'(k)$ and $T'''^*\cdot T(k)$ are also Lipschitz continuous maps from Σ into C^{0*} i.e.*

$$\|T'^*\cdot T'(k_1)-T'^*\cdot T'(k_2)\|\leqslant L_1\|k_1-k_2\|_{c^0}\leqslant L_2\|k_1-k_2\|_{H^1}, \tag{5.19}$$

$$\|T'''^*\cdot T(k_1)-T'''^*\cdot T(k_2)\|\leqslant L_3\|k_1-k_2\|_{c^0}\leqslant L_4\|k_1-k_2\|_{H^1}, \tag{5.20}$$

where L_j are Lipschitz contants independent of k_i and s, $j=1,2,3,4$, $i=1,2$.

Lemma 5.6. *Under the conditions of Lemma 5.3, the operator $T'^*\cdot T$ is Frechet differentiable and we have*

$$(T'^*\cdot T)'h=T'^*\cdot T'\cdot h+T''^*\cdot T\cdot h \tag{5.21}$$

for any $h(x)\in c^0[0,1]$.

Lemma 5.7. *Let L_1 and L_2 be linear operators from Banach space X into Y. Assume that*

(i) *L_1 is invertible and $\|L_1^{-1}\|\leqslant\lambda$,*

(ii) *$\|L_1-L_2\|\leqslant\delta$,*

(iii) *$\lambda\delta<1$*

Then L_2 is also invertible and

$$\|L_2^{-1}\|\leqslant\frac{\lambda}{1-\lambda\delta}.$$

Proof. See [6].

Lemma 5.8. *Let $\Phi(k)$ be a nonlinear operator from Banach space X into Y, and let it be first order Frechet differentiable. For $k_0\in X$ and $\rho>0$, if $k(x)\in S(k_0,\rho)$ and*

$$\|\Phi'(k)-\Phi'(k_0)\|\leqslant\beta,$$

then for all k, $h\in S(k_0,\beta)$, we have

$$\|\Phi(k)-\Phi(h)-\Phi'(k_0)(k-h)\|_Y\leqslant\beta\|k-h\|_X.$$

Now, we are able to prove our main convergence theorems.

Theorem 5.1. *Assume that there exists a $k_\alpha(x)\in\Sigma\in H^1$ for α small enough such that $\Phi(k_\alpha)=0$, and let $\Phi'(k_\alpha)$ be an invertible linear operator which maps $\Sigma\in H^1[0,1]$ into $H^{-1}[0,1]$. Moreover, let R_n be defined by (5.11). Then all of the conditions of Lemma 5.1, (a)—(e), are satisfied, and the iterative sequence $\{k_n\}$ of Gauss-Newton-regularizing method, (2.9)—(2.10), converges to the regularizing solution k_α of (1.10). Furthermore, it has the estimate of (5.10).*

In Theorem 5.1, we assume

(1) there exists a $k_\alpha(x)\in\Sigma\in H^1$, for α small enough, such that

$$\Phi(k_\alpha)=0.$$

(2) $\Phi'(k_\alpha)$ is invertible.

We will prove that under certain conditions, the above hypotheses (1) and (2) hold.

Theorem 5.2. *Suppose $F(s)\in L_2(0,\infty)$ and $G(s)\in L_\infty(0,\infty)$ and there exists a unique solution $k^*(x)$ of equation (1.10) such that*

$$k^*(x) \in H^1[0, 1], \quad and \quad \gamma_1^* \leqslant k^*(x) \leqslant \gamma_2^*.$$

Then, when α is small enough, there is a $k_\alpha(x) \in H^1[0, 1]$ such that

$$\frac{\gamma_1^*}{2} < k_\alpha(x) < 2\gamma_2^*.$$

Moreover,

$$\Phi(k_\alpha) = 0. \tag{5.22}$$

Theorem 5.3. *Under the conditions of Theorem 5.2, $\Phi'(k_\alpha)$ is invertible.*

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