THE COUPLING OF FINITE ELEMENT METHOD AND BOUNDARY ELEMENT METHOD FOR TWO-DIMENSIONAL HELMHOLTZ EQUATION IN AN EXTERIOR DOMAIN*1)

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Abstract

This paper presents a new coupling of the Finite Element Method (FEM) and the Boundary Element Method (BEM) to solve the two-dimensional exterior Helmholtz problems by using the asymptotic radiation conditions in [1], in which the coupling relations are the same as C. Johnson and J. C. Nedelec's[2]. The error estimates are derived and results of numerical calculation in comparison with analytic solution verify the theoretical estimates.

§ 0. Introduction

In practical engineering we often encounter boundary value problems of unbounded domain of PDE, such as the flow around a symmetric body, the acoustic scattering and diffusion or electromagnetic scattering by an arbitrarily shaped body, etc. The numerical computation of the above problems is very important in many areas of cappliation, e.g. the design of wave guides, the study of engine noise, the assessment of damage by an electromagnetic pulse and the biological effects of microwave radiation, etc. Mathematically, the problems have the form of an exterior boundary value problem of PDE, which gives rise to particular difficulties emerging from the facts that the domain is unbounded and the solution is oscillatory for large values of frequency. Specially, the acoustic of electromagnetic scattering by an arbitrary body can be formulated by Dirichlet's (or Neumann) boundary value problem of Helmholtz equation satisfying Sommerfield's radiation conditions at infinity (see [3]). There are varieties of numerical methods for the exterior Helmholtz problem in recent years, such as the BEM; we refer to [4—8]. In [2, 9, 10], the coupling of FEM and BEM is presented and successfully applied to many physical problems. P. Bettess gave an infinite element method in [11]. For more infinite element methods for treating problems of unbounded domain see [12] and [13]. C. I. Goldstein[14, 15] presented a method by which the three dimensional exterior Helmholtz equation is replaced by an approximate problem in a sphere with sufficiently large radius, and the boundary condition on the surface of the sphere is approximated by the Sommerfield condition at infinity (called the artificial boundary condition). This approximate problem is then solved using FEM with

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nonuniform mesh sizes. Error estimates are obtained, provided that certain relationships hold between the frequency mesh size and outer radius. If we, however, want to obtain a highly accurate approximate solution, we must take the diameter of the sphere sufficiently large. In this case high cost of computation is inevitable although nonuniform mesh sizes are used. In order to get a highly accurate numerical solution, the approximate errors of the artificial boundary conditions must be decreased. To this end, with the help of the asymptotic radiation conditions of high degree in [1] a new coupling of FEM is presented in this paper for the two-dimensional exterior Helmholtz problem with Dirichlet's boundary condition. The method does not artificially add any unknown variable as in [16, 17]. Furthermore, the existence, uniqueness and convergence of the solution of the discrete problem for arbitrary wave number k are proved and the error estimates are obtained. For references treating the exterior Helmholtz equation using the FEM, see [18] and [19].

We end this section by outlining the remainder of the paper. Sect. 1 is mainly contributed to the asymptotic radiation condition. In Sect. 2 we give the details of the coupling process of the FEM and BEM. Thus we establish the errors of the asymptotic radiation condition in Sect. 3. In Sect. 4 we prove the existence, uniqueness and convergence of the solution of the corresponding finite element approximate problems, and obtain the error estimates. A numerical example is presented for a given wave number k in Sect. 5; results of numerical computation verify the theoretical estimates. At last we make a simple discussion on results of numerical computation, and on an extension of the method of this paper to the corresponding three-dimensional problems.

§ 1. A Family of Asymptotic Radiation Conditions for the Helmholtz Equation

In this section, we will mainly introduce the forms of the asymptotic radiation conditions of arbitrary order given by Feng Kang in [1].

Let $k>0$ be the wave number, $\forall x=(x_1, x_2)\in R^2$, $r=|x|=(x_1^2+x_2^2)^{1/2}$. The domain $\Omega_R=\{(x_1, x_2)\,|\,r>R\}$ is the exterior to the circle $\Gamma_R=\{(x_1, x_2)\,|\,r=R\}$ of radius $R>0$. Based on the Fourier expansion of the Helmholtz equation in Ω_R, the asymptotic expansion and the properties of Hankel functions, and Laplace-Beltrami operator, we can obtain the asymptotic radiation conditions for the two-dimensional Helmholtz equation on Γ_R as follows:

$$\begin{cases} (\mathrm{F}_0) \quad -\dfrac{\partial u}{\partial r}=F_0u=iku, \\ (\mathrm{F}_1) \quad -\dfrac{\partial u}{\partial r}=F_1u=\left(ik+\dfrac{1}{2R}\right)u, \\ (\mathrm{F}_2) \quad -\dfrac{\partial u}{\partial r}=F_2u=\left(ik+\dfrac{1}{2R}+\dfrac{1}{8kR^2}\right)u+\dfrac{i}{2kR^2}\Lambda_1u, \\ (\mathrm{F}_3) \quad -\dfrac{\partial u}{\partial r}=F_3u=F_2u-\dfrac{1}{2k^2R^3}\left(\dfrac{u}{4}+\Lambda_1u\right), \\ \cdots\cdots \end{cases} \tag{1.1}$$

and for the three-dimensional Helmholtz equation on Γ_R as follows:

$$\begin{cases} (\mathrm{F}_0^*) & -\dfrac{\partial u}{\partial r}=iku, \\ (\mathrm{F}_1^*) & -\dfrac{\partial u}{\partial r}=\left(ik+\dfrac{1}{2R}\right)u, \\ (\mathrm{F}_2^*) & -\dfrac{\partial u}{\partial r}=\left(ik+\dfrac{1}{R}\right)u+\dfrac{i}{2kR^2}\Lambda_2 u, \\ (\mathrm{F}_3^*) & -\dfrac{\partial u}{\partial r}=\left(ik+\dfrac{1}{R}\right)u+\left(\dfrac{i}{2kR^2}-\dfrac{1}{2k^2R^3}\right)\Lambda_2 u, \\ \cdots\cdots \end{cases} \tag{1.2}$$

where $\Lambda_1=\dfrac{\partial^2}{\partial\theta^2}$ and $\Lambda_2=\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial}{\partial\theta}\right)+\dfrac{1}{\sin^2\theta}\dfrac{\partial^2}{\partial\varphi^2}$ are Laplace-Beltrami operators, and $(r, \theta, \varphi)\in R^3$ are the spherical coordinates. Consult [1] for further details.

Finally, we point out that error estimates of the asymptotic radiation conditions (1.1) were not obtained in [1]. In the Sect. 3, we will give an error estimate of (F_1) with order 1, thus obtaining error estimates of (F_p) with arbitrary order.

§ 2. The Coupling Process of FEM and BEM

Consider the two-dimensional Helmholtz equation:

$$\begin{cases} (\Delta+k^2)u=f, & x\in\Omega^C, \\ u|_\Gamma=0, \\ \dfrac{\partial u}{\partial r}+iku=o\left(\dfrac{1}{\sqrt{r}}\right), & \text{as} \quad r\to\infty, \end{cases} \tag{2.1}$$

where $k>0$ is a wave number; Ω^C is the complement in R^2 (the two-dimensional Euclidean space) of a bounded domain Ω with smooth boundary $\partial\Omega=\Gamma$, and $0\in\Omega$; f is a smooth function and has a bounded support on Ω^C. Point x in R^2 will be denoted by $x=(x_1, x_2)$ in Cartesian coordinates. We shall also employ polar coordinates (r, θ), defined by $x_1=r\cos\theta$ and $x_2=r\sin\theta$.

Let

$$u_s=(A+k^2+is)^{-1}f\in L^2(\Omega^C) \quad \text{for} \quad s\neq 0, \tag{2.2}$$

where $A=\Delta$ associated with the Dirichlet boundary condition on $\partial\Omega$ and acting in $L^2(\Omega^C)$ has a real spectrum. The first lemma follows from the results proved in [20] and [21]. These results were obtained by using the Rillich compactness theorem and elliptic regularity theorem.

Lemma 2.1. *Suppose that B is a bounded subset of $\overline{\Omega^C}$, $f\in C^\infty(\overline{\Omega}^C)$,* supp $f\subset B$ *and u_s is defined as (2.2) with $s>0$. Then*

(a) *there exists a unique solution $u\in C^\infty(\Omega^C)$ of problem (2.1),*

(b) $\|u^s-u\|_{H^l(B)}\to 0$ *as $s\downarrow 0$, and*

(c) $\|u\|_{H^l(B)}\leqslant C\|f\|_{H^{l-1}(B)}$ *for each integer $l\geqslant 0$,*

where the constant C is independent of f. However, C increases as $\dim(B)$

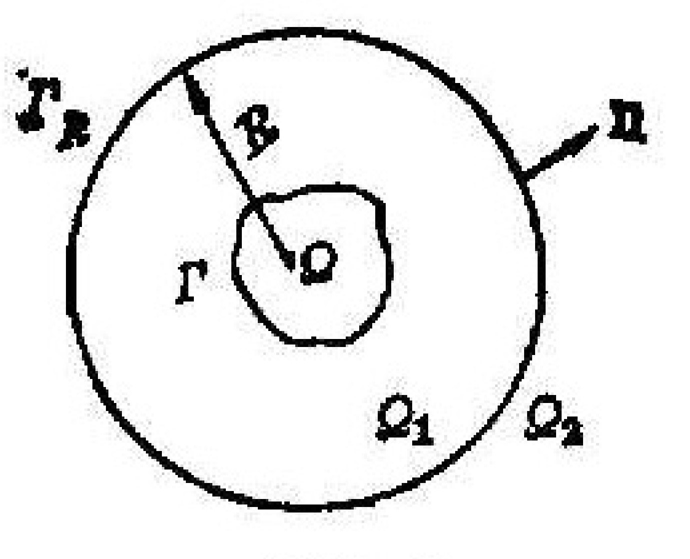

Fig. 1

increases.

Now we are in a position to deduce the details of the coupling of FEM and BEM. Take an appropriately large positive real number R. Let $\Omega_R=\{x\in R^2, \|x\|<R\}$ and denote $\partial\Omega_R$ by Γ_R (usually called "the artificial boundary"). Let $\Omega_1=\Omega_R\cap\Omega^C$ and $\Omega_2=R^2\backslash\overline{\Omega}_R$ such that supp $(f)\subset\Omega_1$. Denote by n the exterior normal vector to the boundary Γ_R of domain Ω_1 (see Fig. 1). Then by the above division of the domain Ω^C, (2.1) is identical with the following problem:

$$\begin{cases} \Delta u_1+k^2u_1=f, & x\in\Omega_1,\\ u_1=0, & x\in\Gamma,\\ \Delta u_2+k^2u_2=0, & x\in\Omega_2,\\ u_2=u_1, & x\in\Gamma_R,\\ \dfrac{\partial u_1}{\partial n}=\dfrac{\partial u_2}{\partial n}=\lambda, & x\in\Gamma_R,\\ \dfrac{\partial u_2}{\partial r}+iku_2=o\left(\dfrac{1}{\sqrt{r}}\right), & r=|x|\to\infty, \end{cases} \tag{2.3}$$

where $u_i=u|_{\Omega_i}$, $i=1, 2$. Thus, the direct coupling of FEM and BEM is that FEM and BEM are respectively employed in the bounded domain Ω_1 and the unbounded domain Ω_2 with the coupling conditions of (2.3) to solve problem (2.3) (see [9, 16]).

As usual, we define $H^m(\Omega_1)$ as a standard Sobolev space and the trace space $H^s(\Gamma)$ on the boundary; for more details see [22].

We define the Hilbert space $H^1_E(\Omega_1)=\{v\in H^1(\Omega_1),\ v|_\Gamma=0\}\subset H^1(\Omega_1)$.

It follows from [22] that norm $\|\cdot\|_{1,\Omega_1}$ is identical with seminorm $|\cdot|_{1,\Omega_1}$ in $H^1_E(\Omega_1)$.

1° Since $\Delta u+k^2u=f$ in Ω_1, the variational formulation of FEM in Ω_1 is

$$\begin{cases} a(u, v)+\langle v, \lambda\rangle_{\Gamma_R}=(f, v)_{\Omega_1}, & \forall v\in H^1_E(\Omega_1),\\ a(u, v)=-\displaystyle\int_{\Omega_1}\nabla u\cdot\overline{\nabla v}\,dx+k^2\int_{\Omega_1}u\bar{v}\,dx, & \\ \langle v, \lambda\rangle_{\Gamma_R}=\displaystyle\int_{\Gamma_R}\lambda\bar{v}\,ds_x, \quad \lambda=\dfrac{\partial u}{\partial n}\Big|_{\Gamma_R}, & \\ (f, v)_{\Omega_1}=\displaystyle\int_{\Omega_1}f\bar{v}\,dx. & \end{cases} \tag{2.4}$$

2° With the help of the fundamental solution $\frac{1}{4i}H_0^{(2)}(k\rho)$ of the two-dimensional Helmholtz equation and Green's formulation and the Sommerfield condition at infinity, in Ω_2 we obtain a boundary-integral equation (see [4, 16])

$$u(x)=\frac{1}{4i}\int_{\Gamma_R}\frac{\partial H_0^{(2)}(k\rho)}{\partial n}u\,ds_{x'}-\frac{1}{4i}\int_{\Gamma_R}\lambda H_0^{(2)}(k\rho)ds_{x'}, \quad \forall x\in\Omega_2. \tag{2.5}$$

Here $\rho=|x-x'|$, $\lambda=\frac{\partial u}{\partial n}\Big|_{\Gamma_R}$ and $H_0^{(2)}$ is a Hankel function. Let x in (2.5) go to the boundary Γ_R, and we have (see [16])

$$u(x)=\frac{1}{2i}\int_{\Gamma_R}\frac{\partial H_0^{(2)}(k\rho)}{\partial n}u\,ds_{x'}-\frac{1}{2i}\int_{\Gamma_R}\lambda H_0^{(2)}(k\rho)ds_{x'},\quad \forall x\in\Gamma_R. \tag{2.6}$$

Here $\rho=|x-x'|$ and $\lambda=\left.\frac{\partial u}{\partial n}\right|_{\Gamma_R}$. Using (2.6) we get the variational formulation of BEM on Γ_R as follows:

$$\begin{cases}\langle u,\mu\rangle_{\Gamma_R}+\langle Gu,\mu\rangle_{\Gamma_R}-b(\lambda,\mu)=0, & \forall\mu\in H^{1/2}(\Gamma_R),\\ Gu=-\frac{1}{2i}\int_{\Gamma_R}\frac{\partial H_0^{(2)}(k\rho)}{\partial n}u\,ds_{x'}, & \\ b(\lambda,\mu)=\frac{1}{2i}\int_{\Gamma_R}\left(\int_{\Gamma_R}\overline{\lambda H_0^{(2)}(k\rho)}ds_{x'}\right)\mu\,ds_{x'}, & \\ \langle u,\mu\rangle_{\Gamma_R}=\int_{\Gamma_R}\mu\bar{u}\,ds_x. & \end{cases} \tag{2.7}$$

Combining (2.4) with (2.7), then the direct coupling of FEM and BEM is formulated as:

$$\begin{cases}\text{Find } (u,\lambda)\in H_E^1(\Omega_1)\times H^{1/2}(\Gamma_R) \text{ such that}\\ a(u,v)+\langle u,\lambda\rangle_{\Gamma_R}=(f,v)_{\Omega_1}, \quad \forall v\in H_E^1(\Omega_1),\\ \langle u,\mu\rangle_{\Gamma_R}+\langle Gu,\mu\rangle_{\Gamma_R}-b(\lambda,\mu)=0, \quad \forall\mu\in H^{1/2}(\Gamma_R).\end{cases} \tag{2.8}$$

Therefore in (2.5), we may utilize the values $u|_{\Gamma_R}$ and $\lambda=\left.\frac{\partial u}{\partial n}\right|_{\Gamma_R}$ of (u,λ) on Γ_R solved from (2.8) and obtain the value of u at an arbitrary point $x\in\Omega_2$. But kernels in $b(\lambda,\mu)$ and $\langle Gu,\mu\rangle$ are singular. When (2.8) is numerically solved, the cost of computation will be large, and the existence, the convergence and the error estimates of the approximate solution are difficult to prove. Furthermore, the required solutions in (2.8) are one more than in (2.1), artificially increasing the computational efforts. In order to overcome these shortcomings we use the first order asymptotic radiation condition $-\left(ik+\frac{1}{2R}\right)u$ of the Helmholtz equation in [1] to approximate $\lambda=\frac{\partial u}{\partial n}$ as in [15]. So we eliminate the required solution λ of (2.8). An unknown variable is taken out, and the problem is simplified and can directly be solved from (2.8) by FEM in Ω_1. As a result we may put the values of $\frac{\partial u}{\partial n}=\lambda\approx -\left(ik+\frac{1}{2R}\right)u$ and u on Γ_R into (2.5) and obtain the approximate value of u at an arbitrary point $x\in\Omega_2$.

By employing the asymptotic radiation condition of order one of [1], the approximate problem of (2.1) in Ω_1 is

$$\begin{cases}\Delta w+k^2w=f, & x\in\Omega_1,\\ w=0, & x\in\Gamma,\\ \frac{\partial w}{\partial r}+\left(ik+\frac{1}{2R}\right)w=0, & x\in\Gamma_R,\end{cases} \tag{2.9}$$

and in Ω_2 is

$$w(x)=\frac{1}{4i}\int_{\Gamma_R}\frac{\partial H_0^{(2)}(k\rho)}{\partial n}w\,ds_{x'}-\frac{1}{4i}\int_{\Gamma_R}\frac{\partial w}{\partial n}H_0^{(2)}(k\rho)ds_{x'},\quad x\in\Omega_2. \tag{2.10}$$

The corresponding adjoint-problem of (2.9) is

$$\begin{cases} \Delta w^*+k^2w^*=f, & x\in\Omega_1, \\ w^*=0, & x\in\Gamma, \\ \frac{\partial w^*}{\partial r}+\left(-ik+\frac{1}{2R}\right)w^*=0, & x\in\Gamma_R. \end{cases} \tag{2.11}$$

We now prove that problems (2.9) and (2.11) are well-posed.

Theorem 2.1. *Suppose that $f\in H^{m-2}(\Omega_1)$. Then (2.9) and (2.11) have respectively a unique solution w and $w^*\in H^m(\Omega_1)$, such that*

(a) $\|w\|_{m,\Omega_1}\leqslant C\|f\|_{m-2,\Omega_1}$,

(b) $\|w^*\|_{m,\Omega_1}\leqslant C\|f\|_{m-2,\Omega_1}$.

Furthermore, if $f\in C^\infty(\Omega_1)$, then $w, w^*\in C^\infty(\Omega_1)$, where the constant C is independent of u, but dependent on R.

Proof. We only prove the theorem for (2.9). The proof of (2.11) is similar. Firstly we show the uniqueness of (2.9). Let w be a solution of (2.9) with f replaced by 0. Then

$$0=\int_{\Omega_1}(\Delta w+k^2w)\bar{w}\,dx=-|w|^2_{1,\Omega_1}+k^2\|w\|^2_{0,\Omega_1}-\left(ik+\frac{1}{2R}\right)\|w\|^2_{0,\Gamma_R}.$$

It implies that

$$-ik\|w\|^2_{0,\Gamma_R}=0.$$

So

$$w|_{\Gamma_R}=0,\quad \frac{\partial w}{\partial n}\Big|_{\Gamma_R}=\frac{\partial w}{\partial r}\Big|_{\Gamma}=-\left(ik+\frac{1}{2R}\right)w\Big|_{\Gamma_R}=0.$$

In order to get $w=0$ in Ω_1 we utilize the usual duality technique

$$\|w\|_{0,\Omega_1}=\sup_{\forall\varphi\in C_0^\infty(\Omega_1)}\frac{(w,\varphi)_{\Omega_1}}{\|\varphi\|_{0,\Omega_1}}=\sup_{\forall\varphi\in C_0^\infty(\Omega_1)}\frac{(w,\Delta\Phi+k^2\Phi)_{\Omega_1}}{\|\varphi\|_{0,\Omega_1}},$$

where Φ is a solution of (2.1) with $f=\varphi$. Applying integration by parts and the boundary conditions of Φ, w and $\frac{\partial w}{\partial n}$, we see that

$$\|w\|_{0,\Omega_1}=\sup_{\forall\varphi\in C_0^\infty(\Omega_1)}\frac{\int_{\Omega_1}w(\Delta\bar{\Phi}+k^2\bar{\Phi})dx}{\|\varphi\|_{0,\Omega_1}}=\sup_{\forall\varphi\in C_0^\infty(\Omega_1)}\frac{\int_{\Omega_1}(\Delta w+k^2w)\bar{\Phi}dx}{\|\varphi\|_{0,\Omega_1}}=0.$$

So $w=0$ in Ω_1. Similarly, the uniqueness of the solution of (2.11) can be proved. From the results proved in [23] there exists a unique solution $w\in H^m(\Omega_1)$ of (2.9) which satisfies (a). The remainder of this theorem follows from [24].

§ 3. Error Estimates for the Asymptotic Radiation Condition

In this section we give errors between the solutions of (2.9) and (2.1). For simplicity, let C stand for various constants in various places unless specially stated. We begin by

Lemma 3.1. *Suppose that $u\in C^\infty(\Omega^C)$ is a solution of (2.1), $B\subset\Omega_1$ is a fixed and bounded subdomain, and $\Gamma\subset\partial B$ and* supp $(f)\subset B$. *Then for $r=|x|$ sufficiently large, we have*

(a) $\left|\frac{\partial u}{\partial r}+\left(ik+\frac{1}{2R}\right)u\right|^2\leqslant\frac{C}{r^5}\|f\|^2_{0,B}$,

(b) $|u(x)|\leqslant\frac{C}{\sqrt{r}}\|f\|_{0,B}$,

(c) $\left|\frac{\partial u}{\partial r}\right|\leqslant\frac{C}{\sqrt{r}}\|f\|_{0,B}$,

where the constant C is independent of f and r.

Proof. We may use (2.1) and integration by parts to see that

$$u(x)=\int_\Gamma \frac{\partial u}{\partial n}H_0^{(2)}(k\rho)ds_{x'}-\int_{\Omega^0} fH_0^{(2)}(k\rho)dx', \tag{3.1}$$

where $\rho=|x-x'|$ and $H_0^{(2)}$ is a Hankel function. Set $x=(r,\theta)$, $x'=(r',\theta')$. It implies that

$$\rho=r\sqrt{1-\frac{2r'}{r}\cos\alpha+(r'/r)^2},$$

where α is the angle between the rays joining x and x' with the origin. Choosing r sufficiently large, we have the asymptotic expansion

$$H_0^{(2)}(k\rho)=C_0\sqrt{\frac{1}{\rho}}\,e^{-ik\rho}\left[1+\frac{C_1}{\rho}\right]+O((k\rho)^{-\frac{5}{2}}), \tag{3.2}$$

where $C_0=\frac{\sqrt{2}}{\pi k}e^{i\pi/4}$ and $C_1=\frac{1}{2ik}\Gamma(3/2)/\Gamma(-1/2)$.

It is clear that we may differentiate under the integral sign in (3.1). Hence (a) follows readily from (3.1), (3.2), the Schwarz inequality and Lemma 2.1. (b) and (c) can be obtained similarly. Q.E.D.

Remark 3.1. If we adopt the method of [14], e.g. directly using the boundary condition $\frac{\partial u}{\partial r}+iku=0$ of (2.1) at infinity on Γ_R, we only get $\left|\frac{\partial u}{\partial r}+iku\right|^2=O\left(\frac{1}{r^3}\right)$. Hence $\frac{\partial u}{\partial r}+\left(ik+\frac{1}{2R}\right)u=0$ is clearly a better boundary condition on Γ_R.

Remark 3.2. If we want to obtain a better boundary condition than that in Lemma 3.1, we must employ the asymptotic radiation conditions of higher order see [1] for details. We shall estimate errors between the solutions of (2.1) and (2.9) in the following.

Suppose that u and w are the solutions of (2.1) and (2.9), respectively. Set $e_R=u-W$ for $x\in\Omega_1$, and $E=\frac{\partial u}{\partial r}+\left(ik+\frac{1}{2R}\right)u$. Thus we have the following error bound.

Theorem 3.1. *Suppose that $B\subset\Omega_1$ is a fixed and bounded subdomain, $\Gamma\subset\partial B$ and* $\operatorname{supp}(f)\subset B$. *Then we have*

(a) $\|e_R\|_{0,B}\leqslant\frac{C}{R^2}\|f\|_{0,B}$,

(b) $\|e_R\|_{0,\Gamma_2}\leqslant\frac{C}{R^2}\|f\|_{0,B}$,

(c) $\|e_R\|_{1,B}\leqslant\frac{C}{R^2}\|f\|_{0,B}$,

where the constant C is independent of R and f.

Proof. On Γ_R we see that

$$\frac{\partial e_R}{\partial n}=\frac{\partial e_R}{\partial r}=\frac{\partial u}{\partial r}-\frac{\partial w}{\partial r}=\frac{\partial u}{\partial r}+\left(ik+\frac{1}{2R}\right)u-\left(ik+\frac{1}{2R}\right)u+\left(ik+\frac{1}{2R}\right)w$$
$$=E-\left(ik+\frac{1}{2R}\right)e_R.$$

So e_R satisfies

$$\begin{cases}\Delta e_R+k^2e_R=0, & x\in\Omega_1,\\ e_R=0, & x\in\Gamma,\\ \frac{\partial e_R}{\partial r}=E-\left(ik+\frac{1}{2R}\right)e_R, & x\in\Gamma_R.\end{cases}\tag{3.3}$$

Using (3.3) and integrating by parts, it is easy to see that

$$-|e_R|^2_{1,\Omega_1}+k^2\|e_R\|^2_{0,\Omega_1}+\int_{\Gamma_R}E\bar{e}_R ds_x-\left(ik+\frac{1}{2R}\right)\|e_R\|^2_{0,\Gamma_R}=0.$$

Hence

$$\operatorname{Im}\int_{\Gamma_R}E\bar{e}_R ds_x=k\|e_R\|^2_{0,\Gamma_R}.\tag{3.4}$$

It readily follows from the Schwarz inequality, (3.4) and Lemma 3.1 that

$$\|e_R\|_{0,\Gamma_R}=\frac{1}{k\|e_R\|_{0,\Gamma_R}}\operatorname{Im}\int_{\Gamma_R}E\bar{e}_R ds_x\leqslant\frac{1}{k}\|E\|_{0,\Gamma_R}$$
$$=\frac{1}{k}\left(\int_{\Gamma_R}|E|^2ds_x\right)^{\frac{1}{2}}\leqslant\frac{C\|f\|_{0,B}}{R^{5/2}}\sqrt{\int_{\Gamma_R}ds_x}\leqslant\frac{C}{R^2}\|f\|_{0,B}.\tag{3.5}$$

This shows (b) is valid. We apply a usual duality argument to prove (a). Denote the inner product of $L^2(\Omega_1)$ by $(\cdot,\cdot)_{\Omega_1}$, For $\forall\tilde{f}\in C_0^\infty(B)$, we extend $\tilde{f}$ to Ω^C by

$$f=\begin{cases}\tilde{f}, & \text{as } x\in B,\\ 0, & \text{as } x\in\Omega^C\backslash B.\end{cases}$$

Then $f\in C_0^\infty(\Omega^C)$ and $f\in C_0^\infty(\Omega_1)$.

It follows from Lemma 2.1 that there exists an unique $u'\in C^\infty(\Omega^C)$ satisfying (2.1), and hence

$$\|e_R\|_{0,B}=\sup_{\tilde{f}\in C_0^\infty(B)}\frac{|(e_R,\tilde{f})_B|}{\|\tilde{f}\|_{0,B}}=\sup_{\tilde{f}\in C_0^\infty(B)}\frac{|(e_R,f)_{\Omega_1}|}{\|\tilde{f}\|_{0,B}}=\sup_{\tilde{f}\in C_0^\infty(B)}\frac{|(e_R,\Delta u'+k^2u')_{\Omega_1}|}{\|\tilde{f}\|_{0,B}}.$$

Using integration by parts, (3.3), the Schwarz inequality, Lemma 3.1 and (3.5), we have

$$\|e_R\|_{0,B}=\sup_{\tilde{f}\in C_0^\infty(B)}\frac{\left|\int_{\Omega_1}(\Delta e_R+k^2e_R)\bar{u}'dx-\int_{\Gamma_R}\frac{\partial e_R}{\partial r}\bar{u}'\,ds_x+\int_{\Gamma_R}\frac{\partial\bar{u}'}{\partial n}e_R\,ds_x\right|}{\|\tilde{f}\|_{0,B}}$$
$$=\sup_{\tilde{f}\in C_0^\infty(B)}\left|\left(ik+\frac{1}{2R}\right)\int_{\Gamma_R}e_R\bar{u}'\,ds_x-\int_{\Gamma_R}E\bar{u}'ds_x+\int_{\Gamma_R}\frac{\partial\bar{u}'}{\partial n}e_R\,ds_x\right|\Big/\|\tilde{f}\|_{0,B}$$

$$\leqslant \sup_{\tilde{f}\in C_0^\infty(B)} \left\{C(\|e_R\|_{0,\Gamma_R}+\|E\|_{0,\Gamma_R})\|u'\|_{0,\Gamma_R}+\left\|\frac{\partial u'}{\partial n}\right\|_{0,\Gamma_R}\|e_R\|_{0,\Gamma_R}\right\}\Big/\|\tilde{f}\|_{0,B}$$

$$\leqslant \sup_{\tilde{f}\in C_0^\infty(B)} C(\|e_R\|_{0,\Gamma_R}\|+E\|_{0,\Gamma_R})\|f\|_{0,B}/\|\tilde{f}\|_{0,B}$$

$$\leqslant C(\|e_R\|_{0,\Gamma_R}+\|E\|_{0,\Gamma_R})\leqslant\frac{C}{R^2}\|f\|_{0,B},$$

where the constant C is independent of R and f.

Applying the definition of $\|\cdot\|_{1,B}$, integration by parts and (3.3), we see that

$$\|e_R\|_{1,B}=\sup_{\psi\in C_0^\infty(B)}\{(e_R,\psi)_B-(\Delta e_R,\psi)_B\}/\|\psi\|_{1,B}$$

$$=\sup_{\psi\in C_0^\infty(B)}(e_R+k^2e_R,\psi)_B/\|\psi\|_{1,B}$$

$$\leqslant C\sup_{\psi\in C_0^\infty(B)}\frac{\|e_R\|_{0,B}\|\psi\|_{0,B}}{\|\psi\|_{1,B}}\leqslant C\|e_R\|_{0,B}\leqslant\frac{C}{R^2}\|f\|_{0,B}. \qquad \text{Q.E.D.}$$

With the help of Green's formulation we obtain the variational problem of (2.9)

$$\begin{cases}\text{Find } w\in H_E^1(\Omega_1) \text{ such that}\\ B(w,v)=(f,v)_{\Omega_1},\quad \forall v\in H_E^1(\Omega_1),\\ B(w,v)=k^2\int_{\Omega_1}w\bar{v}\,dx-\int_{\Omega_1}\nabla w\nabla\bar{v}\,dx-\left(ik+\frac{1}{2R}\right)\int_{\Gamma_R}w\bar{v}\,ds_x.\end{cases} \tag{3.6}$$

We are now able to prove the existence, uniqueness and convergence of the solution of the finite element approximation of (3.6), and give error estimates.

§ 4. Error Estimates for the Finite Element Approximation

Suppose that $v_h\subset H_E^1(\Omega_1)$ is a finite-dimensional space dependent on parameter h such that

(H_1) $\forall v\in H^m(\Omega_1)$ there exists a $\pi_h v\in V_h$ such that

$$\|v-\pi_h v\|_{l,\Omega_1}\leqslant Ch^{m-l}\|v\|_{m,\Omega_1}$$

(for $\forall 0\leqslant l<m\leqslant N$, l, m and N are integals).

We shall later see that (H_1) is vital to the convergence of the finite element solution.

The finite element subspaces, V_h, are typically obtained by subdividing Ω_1 into simple subsets, $T_h=\{T\}$, of diameter $O(h)$. V_h may then be defined as the subspace of $H_E^1(\Omega_1)$ consisting of all continuous functions, v_h, vanishing on Γ, such that the restriction of v_h to each element T is a polynomial of degree less than or equal to some integer N. It is not necessary to impose any boundary condition on the function in v_h (note that the asymptotic radiation condition on Γ_R is a natural boundary condition). Then V_h constructed by the above method satisfies (H_1). See [23, 25, 26] for the details of constructing the finite element subspaces.

The discrete forms of (3.6) and (2.10) are

$$\begin{cases} \text{Find } w_h \in v_h \text{ such that} \\ B(w_h, v_h) = (f, v_h) \quad \text{for} \quad \forall v_h \in V, \end{cases} \tag{4.1}$$

$$w_h(x) = \frac{1}{4i} \int_{\Gamma_2} \frac{\partial H_0^{(2)}(k\rho)}{\partial n} w_h(x') ds_{x'} + \frac{1}{4i} \int_{\Gamma_2} \left(ik + \frac{1}{2R}\right) w_h(x') H_0^{(2)}(k, \rho) ds_{x'} \quad \forall x \in \Omega_2, \tag{4.2}$$

where $\rho = |x - x'|$ and $B(\cdot, \cdot)$ is defined as in (3.6). (4.2) is in fact a formulation for finding the approximate values of $u(x)$ in Ω_2. We shall give error estimates later. First we show that (4.1) is also well-posed.

Theorem 4.1. *For h sufficiently small, there exists a unique solution $w_h \in V_h$ of (4.1). Furthermore, for $\forall u_h \in V_h$, there exists a $v_h \in V_h$ such that*

(a) $|B(u_h, v_h)| \geq C \|u_h\|_{1,\Omega_1} \|v_h\|_{1,\Omega_1}$,

where the constant C is independent of h, but dependent on R.

Proof. For $\forall u, v \in H_E^1(\Omega_1)$, it is easy to see that

$$B(u, v) \leq C \|u\|_{1,\Omega_1} \|v\|_{1,\Omega_1}.$$

So $B(\cdot, \cdot)$ is a bilinear continuous function on $V_h \times V_h$. In addition, let z be a solution of problem (2.11) with f replaced by $-(k^2+1)u_h$ with $u_h \in V_h \subset H_E^1(\Omega_1)$. Then it follows from Theorem 2.1 that

$$\|z\|_{3,\Omega_1} \leq C \|u_h\|_{1,\Omega_1}. \tag{4.3}$$

Observe that

$$|B(u_h, u_h + z)| = |Q(u_h, u_h) + B(u_h, z) + R(u_h, u_h)|, \tag{4.4}$$

where

$$Q(u_h, u_h) = -\|u_h\|_{1,\Omega_1}^2 - \left(ik + \frac{1}{2R}\right) \|u_h\|_{0,\Gamma_2}^2,$$

$$R(u_h, u_h) = (k^2+1) \int_{\Omega_1} u_h \bar{v}_h \, dx.$$

Integrating by parts and with z being a solution of (2.11) with f replaced by $-(k^2+1)u_h$, we obtain

$$B(u_h, z) = -R(u_h, u_h) \tag{4.5}$$

and we readily see that

$$|Q(u_h, u_h)| \geq \frac{1}{2} |\mathrm{Re}\, Q(u_h, u_h)| + |\mathrm{Im}\, Q(u_h, u_h)| \geq \frac{1}{2} \left(\|u_h\|_{1,\Omega_1}^2 + \frac{1}{2R} \|u_h\|_{0,\Gamma_2}^2 + k \|u_h\|_{0,\Gamma_2}^2 \right) \geq C \|u_h\|_{1,\Omega_1}^2. \tag{4.6}$$

Combining (4.4) with (4.5) and (4.6), we have

$$|B(u_h, u_h + z)| \geq C \|u_h\|_{1,\Omega_1}^2. \tag{4.7}$$

Since $z \in H^3(\Omega_1) \cap H_E^1(\Omega_1)$, from ($H_1$) there exists a $\pi_h z \in V_h$ such that

$$\|z - \pi_h z\|_{1,\Omega_1} \leq C h^2 \|z\|_{3,\Omega_1}. \tag{4.8}$$

Applying (4.8) and (4.3) gives

$$\|\pi_h z\|_{1,\Omega_1} \leq C \|z\|_{3,\Omega_1} \leq C \|u_h\|_{1,\Omega_1}.$$

Hence

$$\|u_h + \pi_h z\|_{1,\Omega_1} \leq C \|u_h\|_{1,\Omega_1}. \tag{4.9}$$

From (4.3), (4.7), (4.8) and (4.9), it is asserted that

$$\begin{aligned}|B(u_h, u_h+\pi_h z)| &\geqslant |B(u_h, u_h+z)| - |B(u_h, z-\pi_h z)| \\ &\geqslant C\|u_h\|^2_{1,\Omega_1} - C\|u_h\|_{1,\Omega_1}\|z-\pi_h z\|_{1,\Omega_1} \\ &\geqslant C\|u_h\|_{1,\Omega_1}(\|u_h\|_{1,\Omega_1} - Ch^2\|z\|_{3,\Omega_1}) \geqslant C\|u_h\|^2_{1,\Omega_1}(1-Ch^2) \\ &\geqslant C\|u_h\|_{1,\Omega_1}\|u_h+\pi_h z\|_{1,\Omega_1},\end{aligned}$$

where the constant C is independent of h. Thus (a) holds and (a) verifies

$$\inf_{\substack{u_h\in V_h\\ \|u_h\|=1}} \sup_{\substack{v_h\in V_h\\ \|v_h\|=1}} B(u_h, v_h) \geqslant C > 0.$$

By reversing the role of u_h and v_h in the above argument we obtain the inequality

$$\sup_{u_h\in v_h} |B(u_h, v_h)| \geqslant C\|u_h\|_{1,\Omega_1}\|v_h\|_{1,\Omega_1}.$$

Then the remainder of this theorem follows from the Lax-Milgram theorem. Q.E.D.

Remark 4.1. From the above argument we see that the condition of h being sufficiently small will guarantee $1-Ch^2 \geqslant C_0 > 0$ such that $B(\cdot, \cdot)$ is weakly coercive. The condition, however, may be deleted when the wave number k is small enough and R is not large enough (i.e. kR may be sufficiently small and in this case the finite element mesh may be coarse). In the case of k being sufficiently small, using the Poincaré inequality, we see that

$$\begin{aligned}|B(u_h, v_h)| &\geqslant \operatorname{Re} B(u_h, u_h)| = \left|k^2\|u_h\|^2_{0,\Omega_1} - |u_h|^2_{1,\Omega} - \frac{1}{2R}\|u_h\|^2_{0,\Gamma_R}\right| \\ &\geqslant |u_h|^2_{1,\Omega_1} + \frac{1}{2R}\|u_h\|^2_{0,\Gamma_R} - k^2\|u_h\|^2_{0,\Omega_1} \geqslant |u_h|^2_{1,\Omega_1} - k^2\|u_h\|^2_{0,\Omega_1} \\ &\geqslant |u_h|^2_{1,\Omega_1} - k^2R^2|u_h|^2_{1,\Omega_1} \geqslant (1-(kR)^2)|u_h|^2_{1,\Omega_1} \\ &\geqslant C_0|u_h|^2_{1,\Omega_1} \geqslant \frac{C}{R^2}\|u_h\|^2_{1,\Omega_1},\end{aligned}$$

where the constant C is independent of R and h. So $B(\cdot,\cdot)$ is weakly coercive. Thus Theorem 4.1 holds, and the following theorems remain valid.

Theorem 4.2. *Suppose that w and w_h are respectively solutions of* (2.1) *and* (4.1). *Then*

$$\|w-w_h\|_{l,\Omega_1} \leqslant Ch^{m-l}\|w\|_{m,\Omega_1}, \quad l=0, 1,$$

where the constant C is independent of h.

Proof. Obviously

$$B(w, w_h) = (f, v_h) = B(w_h, v_h), \quad \forall v_h \in V_h.$$

From (a) of Theorem 4.1 we have

$$\begin{aligned}\|w_h - v_h\|_{1,\Omega_1} &\leqslant C \sup_{s_h\in V_h} B(w_h - v_h, s_h)/\|s_h\|_{1,\Omega_1} \\ &= C \sup_{s_h\in V_h} B(w-v_h, s_h)/\|s_h\|_{1,\Omega_1} \leqslant C\|w-v_h\|_{1,\Omega_1}, \quad \forall v_h\in V_h.\end{aligned}$$

So

$$\|w-w_h\|_{1,\Omega_1} \leqslant Ch^{m-1}\|w\|_{m,\Omega_1}.$$

In order to obtain an error estimate in $L^2(\Omega)$-norm we make use of the well-known Nitche's duality argument.

Let $B(\cdot,\cdot)$ defined by (3.6) be a bilinear continuous function from $H_E^1\times H_E^1$ to R. The duality problem corresponding to problem (3.6) is formulated as

$$\begin{cases}\text{Find } \varphi\in H_E^1(\Omega_1) \text{ such that}\\ B(v,\varphi)=(g,v)_{\Omega_1}, \quad \forall v\in H_E^1(\Omega_1).\end{cases} \tag{4.10}$$

(4.10) is in fact the variational form of the following problem

$$\begin{cases}\Delta\varphi+k^2\varphi=\bar g, \quad \text{in } \Omega_1,\\ \varphi|_\Gamma=0,\\ \dfrac{\partial\varphi}{\partial r}+\left(-ik+\dfrac{1}{2R}\right)\varphi\Big|_{\Gamma_R}=0.\end{cases}$$

Theorem 2.1 ensures that $\forall f,\ g\in L^2(\Omega_1)$ there exists respectively a uniqueness solution $w,\ \varphi\in H^2(\Omega_1)\cap H_E^1(\Omega_1)$ of (3.6) and (4.10) such that

$$\|\varphi\|_{2,\Omega_1}\leqslant C\|g\|_{0,\Omega_1}.$$

In particular, take $g=w-w_h\in L^2(\Omega_1)$. From (3.6), (4.1) and (4.10) we obtain

$$B(w-w_h,\ v_h)=0, \quad \forall v_h\in V_h$$

and

$$B(w-w_h,\ \varphi)=\|w-w_h\|^2_{0,\Omega_1}.$$

Hence

$$\begin{cases}\|w-w_h\|^2_{0,\Omega_1}=B(w-w_h,\varphi)=B(w-w_h,\varphi-v_h)\leqslant C\|w-w_h\|_{1,\Omega_1}\|\varphi-v_h\|_{1,\Omega_1},\\ \inf\limits_{v_h\in V_h}\|\varphi-v_h\|_{1,\Omega_1}\leqslant\|\varphi-\pi_h\varphi\|_{1,\Omega_1}\leqslant Ch\|\varphi\|_{2,\Omega_1}\leqslant Ch\|w-w_h\|_{0,\Omega_1}.\end{cases} \tag{4.11}$$

Combining the above inequality with (4.11) implies that

$$\|w-w_h\|_{0,\Omega_1}\leqslant Ch\|w-w_h\|_{1,\Omega_1}\leqslant Ch^m\|w\|_{m,\Omega_1}. \qquad \text{Q.E.D.}$$

The following theorem gives the convergence and error estimates between the solutions of (2.1) and (4.1). To begin with, we verify some useful inequalities about the Hankel function. We know from 6.13 (3) of [27] that, for any real $x>0$, we have

$$H_n^{(2)}(x)=\left(\frac{2}{\pi x}\right)^{\frac12}\frac{e^{-i\left(x-\frac{n\pi}{2}-\frac{\pi}{4}\right)}}{\Gamma\left(n+\frac12\right)}\int_0^\infty e^{-t}\,t^{n-\frac12}\left(1-\frac{it}{2x}\right)^{n-\frac12}dt, \tag{4.12}$$

where n is a natural number.

In terms of (4.12) and definition of Γ-function we obtain

$$\begin{aligned}H_0^{(2)}(x)&\leqslant\left(\frac{2}{\pi x}\right)^{\frac12}\left(\Gamma\left(\frac12\right)\right)^{-1}\int_0^\infty e^{-t}\,t^{-\frac12}\left|\left(1-\frac{it}{2x}\right)^{-\frac12}\right|dt\\ &\leqslant\left(\frac{2}{\pi x}\right)^{\frac12}\left(\Gamma\left(\frac12\right)\right)^{-1}\int_0^\infty e^{-t}t^{-\frac12}\left(1+\frac{t^2}{4x^2}\right)^{-\frac14}dt\\ &\leqslant\left(\frac{2}{\pi x}\right)^{\frac12}\left(\Gamma\left(\frac12\right)\right)^{-1}\int_0^\infty e^{-t}t^{-\frac12}dt\leqslant\left(\frac{2}{\pi x}\right)^{\frac12}.\end{aligned} \tag{4.13}$$

Similarly,

$$H_1^{(2)}(x)\leqslant C(x^{-\frac12}+x^{-1}). \tag{4.14}$$

From Sect. 2 we know on Γ_R

$$\frac{\partial H_0^{(2)}(k\rho)}{\partial n}=\frac{\partial H_0^{(2)}(k\rho)}{\partial r'}=-kH_1^{(2)}\frac{\partial\rho}{\partial r'}. \tag{4.15}$$

Let $x=(r,\theta)\in\Omega_2$ satisfy $\inf_{y\in\Gamma_R}|x-y|\geqslant\beta>0$ (where $\beta>0$ is an arbitrarily given real number), i.e. $r-R\geqslant\beta>0$. Let $x'=(R,\theta')\in\Gamma_R$. Then

$$\left.\frac{\partial\rho}{\partial r'}\right|_{\Gamma_R}=\frac{-\cos(\theta-\theta')+R/r}{\left(1-\frac{2R}{r}\cos(\theta-\theta')+\left(\frac{R}{r}\right)^2\right)^{\frac{1}{2}}}$$

$\left(\text{since } \rho=|x-x'|=r\left(1-\frac{2r'}{r}\cos(\theta-\theta')+\left(\frac{r'}{r}\right)^2\right)^{\frac{1}{2}}\right)$. Clearly

$$\left(\left.\frac{\partial\rho}{\partial r'}\right|_{\Gamma_R}\right)^2\leqslant 1.$$

Combining (4.14) with (4.15), we get

$$\left\|\frac{\partial H_0^{(2)}(k\rho)}{\partial n}\right\|_{0,\Gamma_R}\leqslant C\left(\int_{\Gamma_R}\left(\frac{1}{\rho}+\rho^{-\frac{3}{2}}+\rho^{-2}\right)ds_{x'}\right)^{\frac{1}{2}}. \tag{4.16}$$

Employing coordinate transformation on Γ_R,

$$x_1'=R\cos\theta',\quad x_2'=R\sin\theta'$$

gives

(A) $$\int_{\Gamma_R}\frac{ds_{x'}}{\rho}=\int_0^{2\pi}\frac{R}{r}\left(1-\frac{2R}{r}\cos(\theta-\theta')+\left(\frac{R}{r}\right)^2\right)^{-\frac{1}{2}}d\theta$$
$$\leqslant\frac{R}{r}\int_0^{2\pi}\left(1-\frac{R}{r}\right)^{-1}d\theta'\leqslant C\frac{R}{r-R}\leqslant C\frac{R}{\beta}.$$

Similarly,

(B) $$\int_{\Gamma_R}\frac{ds_{x'}}{\rho^2}\leqslant C\frac{R}{\beta^2},$$

(C) $$\int_{\Gamma_R}\frac{ds_{x'}}{\rho^{\frac{3}{2}}}\leqslant C\frac{R}{\beta^{\frac{3}{2}}}.$$

Hence we deduce

$$\left\|\frac{\partial H_0^{(2)}(k\rho)}{\partial n}\right\|_{0,\Gamma_R}\leqslant C\frac{R^{\frac{1}{2}}}{\beta}(1+\beta^{-\frac{1}{2}}+\beta^{-1})^{\frac{1}{2}} \tag{4.17}$$

(it follows from (A), (B), (C) and (4.16)).

Applying the same arguments as the above, we can show

$$\|H_0^{(2)}(k\rho)\|_{0,\Gamma_R}\leqslant C(R/\beta)^{\frac{1}{2}}. \tag{4.18}$$

With the help of (4.17) and (4.18), it is easy to prove

Theorem 4.3. *Suppose that u, w and w_h satisfy* (2.1), (2.9) *and* (4.1), *respectively* (*or w_h is the solution of* (4.2) *as $x\in\Omega_2$*). *Then*

(a) $\|u-w_h\|_{0,B}\leqslant C(R^{-2}+C_0h^m)(\|f\|_{0,B}+\|w\|_{m,\Omega_1})$,

(b) $\|u-w_h\|_{0,\Gamma_R}\leqslant C(R^{-2}+C_0h^{m-1})(\|f\|_{0,B}+\|w\|_{m,\Omega_1})$,

(c) $\|u-w_h\|_{1,B}\leqslant C(R^{-2}+C_0h^{m-1})(\|f\|_{0,B}+\|w\|_{m,\Omega_1})$,

(d) $|u(x)-w_h(x)|\leqslant CR^{\frac{1}{2}}(\beta^{-1}+\beta^{-\frac{3}{2}}+\beta^{-2})^{\frac{1}{2}}\cdot(R^{-2}+C_0h^{m-1})(\|f\|_{0,B}+\|w\|_{m,\Omega_1})$,

where $\beta>0$ is an arbitrarily given real number. $x=(r,\theta)\in\Omega_2$ satisfies

$$\inf_{y\in\Gamma_R}|x-y|\geqslant\beta>0.$$

The constant C in (a), (b), (c) *and* (d) *is independent of R and h, and the constant C_0 is independent of h.*

Proof. It follows from Theorems 3.1 and 4.2, and the triangle inequality that

$$\|u-w_h\|_{0,B}\leqslant C(R^{-2}+C_0h^m)(\|f\|_{0,B}+\|w\|_{m,\Omega_1}).$$

We can similarly prove (b) and (c). Let (2.5) subtract (4.2). This gives

$$\begin{aligned}|u(x)-w_h(x)|&=\left|\frac{1}{4i}\left(\int_{\Gamma_R}\left[\frac{\partial H_0^{(2)}(k\rho)}{\partial n}(u-w_h)\right.\right.\right.\\&\quad\left.\left.\left.-\left(\frac{\partial u}{\partial n}+\left(ik+\frac{1}{2R}\right)w_h\right)H_0^{(2)}(k\rho)\right]ds_{x'}\right)\right|\\&\leqslant C\left(\left\|\frac{\partial H_0^{(2)}}{\partial n}\right\|_{0,\Gamma_R}\|u-w_h\|_{0,\Gamma_R}\right.\\&\quad\left.+\left\|\frac{\partial u}{\partial n}+\left(ik+\frac{1}{2R}\right)w_h\right\|_{0,\Gamma_R}\|H_0^{(2)}\|_{0,\Gamma_R}.\right.\end{aligned}\tag{4.19}$$

Since $\frac{\partial u}{\partial n}+\left(ik+\frac{1}{2R}\right)w_h=E+\left(ik+\frac{1}{2R}\right)\cdot(u-w_h)$, then

$$\left\|\frac{\partial u}{\partial n}+\left(ik+\frac{1}{2R}\right)w_h\right\|_{0,\Gamma_R}\leqslant C(\|E\|_{0,\Gamma_R}+\|u-w_h\|_{0,\Gamma_R}).$$

Hence

$$\begin{aligned}|u(x)-w_h(x)|&\leqslant C\left\{\left(\left\|\frac{\partial H_0^{(2)}}{\partial n}\right\|_{0,\Gamma_R}+\|H_0^{(2)}\|_{0,\Gamma_R}\right)\cdot\|u-w_h\|_{0,\Gamma_R}\right.\\&\quad\left.+\|H_0^{(2)}(k\rho)\|_{0,\Gamma_R}\cdot\|E\|_{0,\Gamma_R}\right\}\\&\leqslant C\left[\frac{R}{\beta}\left(1+\frac{1}{\sqrt{\beta}}+\frac{1}{\beta}\right)\right]^{\frac{1}{2}}\left(\frac{1}{R^2}+C_0h^{m-1}\right)(\|f\|_{0,B}+\|w\|_{m,\Omega_1})\end{aligned}$$

(It follows from (b) of this theorem, Lemma 3.1, (4.17) and (4.18)). Q.E.D.

Remark 4.2. We conclude from (a) that $w_h\to u$ (in L^2 norm) if $m\geqslant1$ and $h\to0$ and $R\to\infty$ (note that linear interpolation provides $N=m=1$). In the same manner, from (c), $w_h\to u$ (in H^1 norm) if $m\geqslant2$ and $h\to0$ and $R\to\infty$. It verifies the convergence of this method.

We conclude this section by pointing out that for the non-homogeneous Helmholtz equation,

$$\begin{cases}\Delta u+k^2u=f,\\u|_\Gamma=g,\\\frac{\partial u}{\partial r}+iku=O\left(\frac{1}{\sqrt{r}}\right),\quad r=|x|\to\infty,\end{cases}\tag{4.20}$$

where $f\in C^\infty(\Omega^C)$ with a bounded support, $g\in C^\infty(\partial\Omega)$. From [28], g can be extended to a smooth function G with a bounded support in Ω^C. If u is the solution of (2.1) with f replaced by $F=f+(\Delta+k^2)G$, then it immediately follows that $u_1=u+G$ is the solution of (4.20).

Therefore the method for (2.1) presented in the paper can be applied to problem (4.20) without essential modifications.

§ 5. Numerical Example

Take

$$\Omega=\{x=(x_1, x_2)\in R^2 \,|\, |x|=(x_1^2+x_2^2)^{\frac{1}{2}}<2\},$$
$$\Omega_1=\{x=(x_1, x_2)\in R^2 \,|\, 2<|x|<10\}.$$

By employing polar coordinates (r, θ) defined by $x=r\cos\theta$ and $y=r\sin\theta$ problem (4.20) is transformed into

$$\begin{cases} \dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}+k^2u=f, & x=(r, \theta)\in\Omega^c, \\ u|_{r=2}=g, & x=(r, \theta)\in\Gamma, \\ \dfrac{\partial u}{\partial r}+iku=o\left(\dfrac{1}{\sqrt{r}}\right), & r=|x|\to\infty. \end{cases} \tag{5.1}$$

Specifically we take $f=0$, $g=H_{\frac{1}{2}}^{(2)}(2k)e^{\frac{i}{2}\theta}$ and the wave number $k=0.1$. Then, from the nature of the Hankel function we know that the solution of (5.1) is

$$u(r, \theta)=H_{\frac{1}{2}}^{(2)}(kr)e^{\frac{i}{2}\theta}, \quad 2\leqslant r\leqslant\infty,\ 0\leqslant\theta\leqslant 2\pi.$$

One thirty sixth of the computation domain and the finite element mesh are showed in Fig. 2. The refined mesh is showed in Fig. 3. Figs. 4 and 5 are results of numerical computation.

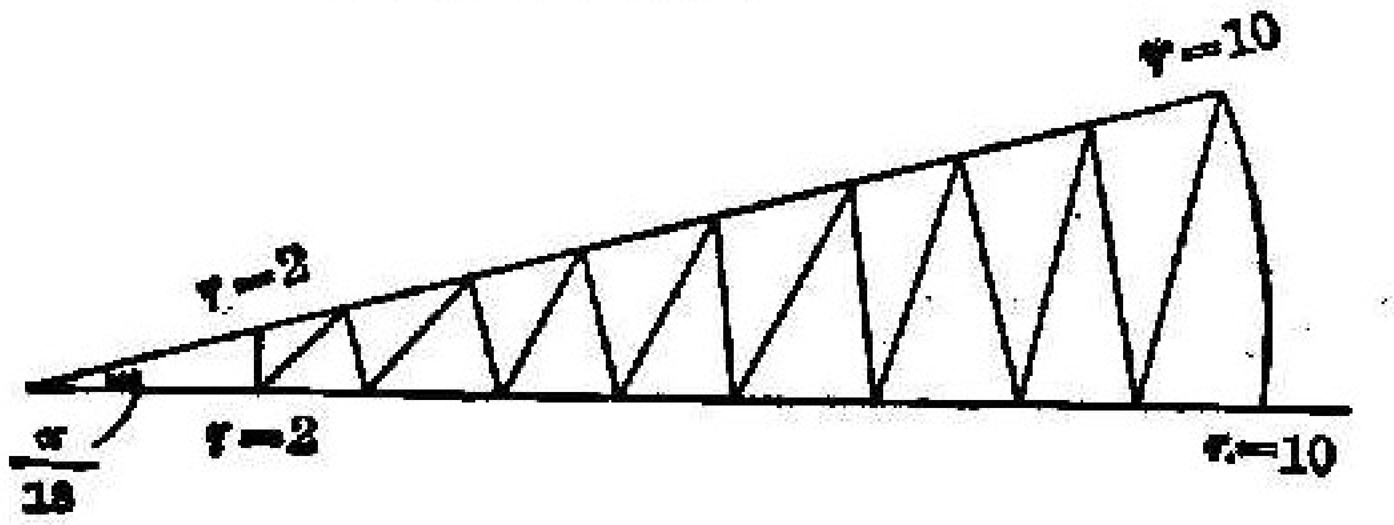

Fig. 2

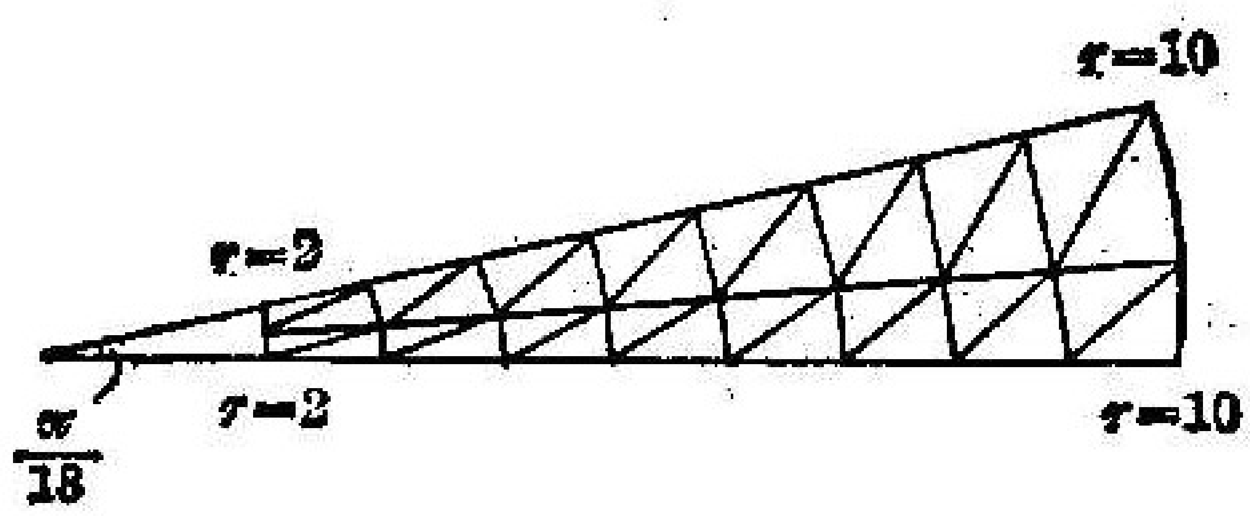

Fig. 3

From Figs. 4, 5 we see that the accuracy of the finite element solution is still high although the mesh is very coarse (note $h>1$), and that with the refinement of the mesh, the computational accuracy increases. Hence, the method in this paper is efficient.

Since the asymptotic radiation condition (F_p), $p\geqslant 2$, of high order in (1.1) contains terms of partial differential to θ, it is difficult to treat. So, for (F_p), $p\geqslant 2$, results similar to Sect. 3 and Sect. 4 are not obtained in this paper.

Finally, for the three-dimensional exterior Helmholtz problem

$$\begin{cases} \Delta u+k^2u=f \quad \text{in } \Omega^c\subset R^3, \\ u|_\Gamma=0, \\ u=O(r^{-2}), \\ \dfrac{\partial u}{\partial r}+iku=O(r^{-1}), \quad r\to\infty, \end{cases} \tag{5.2}$$

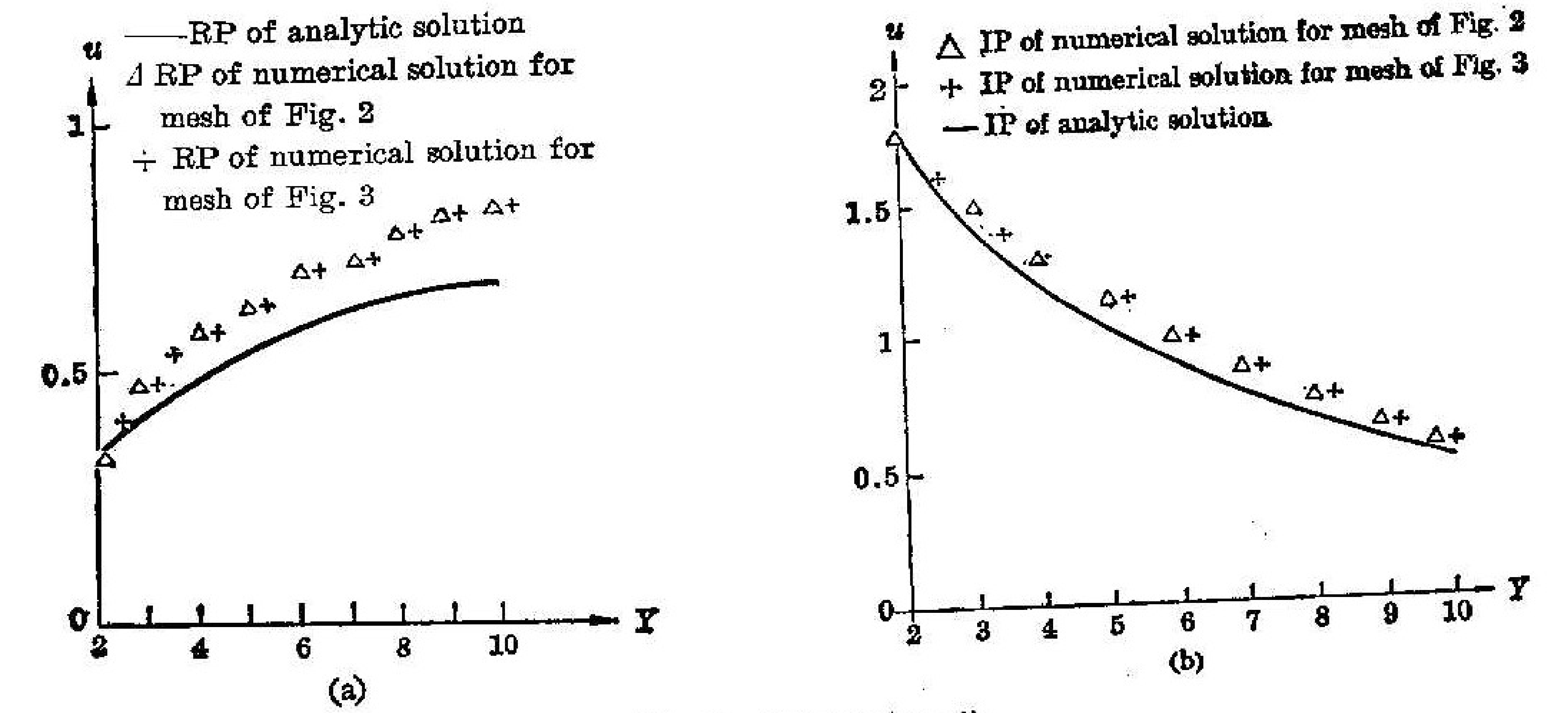

Fig. 4 (for fixed $\theta=0$)

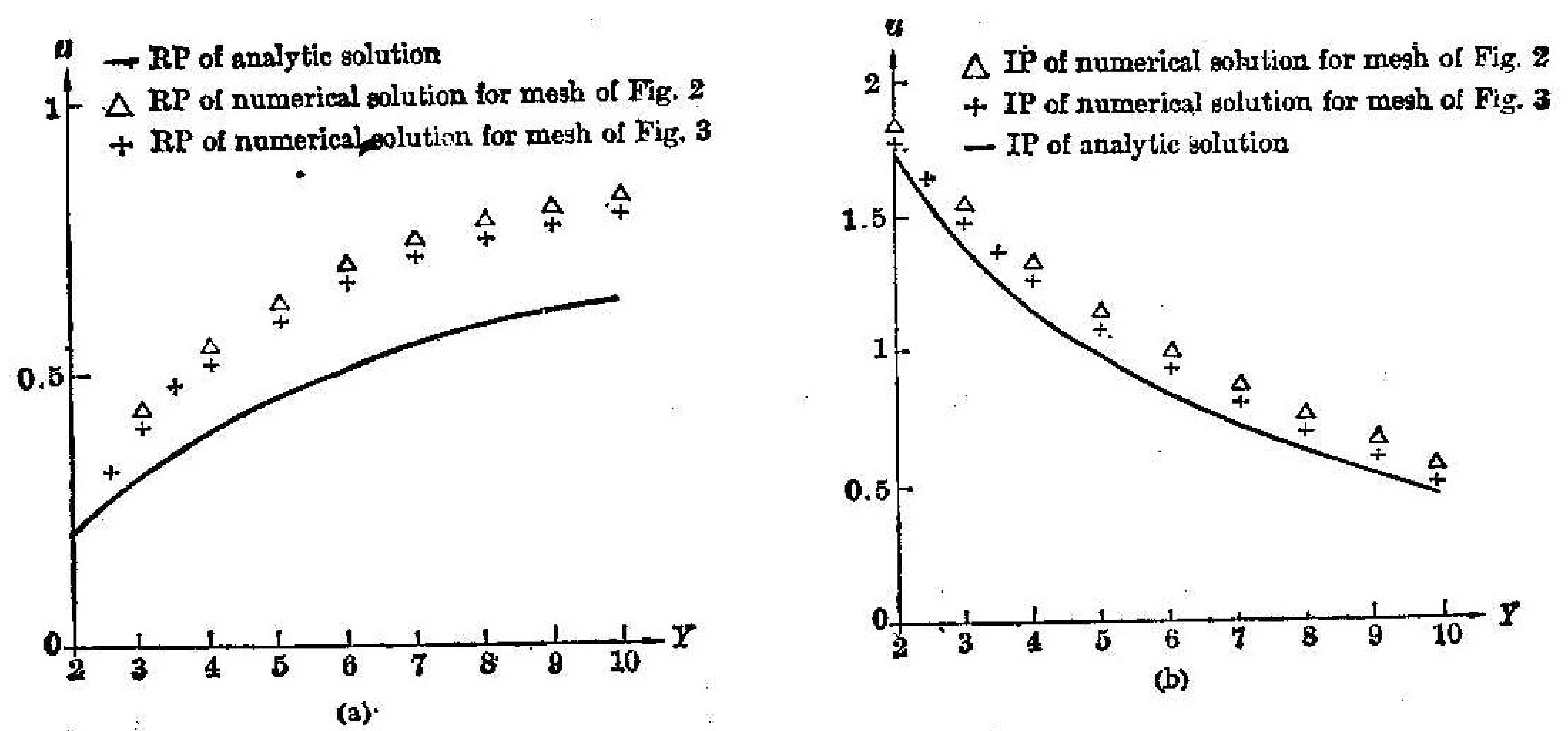

Fig. 5 (for fixed $\theta=\pi/18$)

by condition (F_1^*) in (1.2), the results of this paper hold valid without any essential changes in the arguments.

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