THE THEORY OF FILLED FUNCTION METHOD FOR FINDING GLOBAL MINIMIZERS OF NONLINEARLY CONSTRAINED MINIMIZATION PROBLEMS*

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Abstract

This paper is an extension of [1]. In this paper the descent and ascent segments are introduced to replace respectively the descent and ascent directions in [1] and are used to extend the concepts of S-basin and basin of a minimizer of a function. Lemmas and theorems similar to those in [1] are proved for the filled function

$$P(x, r, \rho)=\frac{1}{r+F(x)}\exp(-\|x-x_1^*\|^2/\rho^2), \tag{0.1}$$

which is the same as that in [1], where x_1^* is a constrained local minimizer of the problem (0.3) below and

$$F(x)=f(x)+\sum_{i=1}^{m'}\mu_i|c_i(x)|+\sum_{i=m'+1}^{m}\mu_i\max(0, -c_i(x)) \tag{0.2}$$

is the exact penalty function for the constrained minimization problem

$$\min_x f(x),$$

subject to

$$\begin{aligned} &c_i(x)=0, \quad i=1, 2, \cdots, m', \\ &c_i(x)\geqslant 0, \quad i=m'+1, \cdots, m, \end{aligned} \tag{0.3}$$

where $\mu_i>0$ $(i=1, 2, \cdots, m)$ are sufficiently large. When x_1^* has been located, a saddle point or a minimizer $\hat{x}$ of $P(x, r, \rho)$ can be located by using the nonsmooth minimization method with some special termination principles. The $\hat{x}$ is proved to be in a basin of a lower minimizer x_2^* of $F(x)$, provided that the ratio $\rho^2/[r+F(x_1^*)]$ is appropriately small. Thus, starting with $\hat{x}$ to minimize $F(x)$, one can locate x_2^*. In this way a constrained global minimizer of (0.3) can finally be found and termination will happen.

§ 1. Introduction

Many existing methods for constrained minimization problems are only used for finding a constrained local minimizer. This paper develops a method which can be used to find the constrained global minimizer. This method is based on two existing methods. One is the nonsmooth exact penalty function method, which transforms a constrained minimization problem

$$\min_x f(x),$$

subject to

$$\begin{aligned} &c_i(x)=0, \quad i=1, 2, \cdots, m', \\ &c_i(x)\geqslant 0, \quad i=m'+1, \cdots, m \end{aligned} \tag{1.1}$$

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into a nonsmooth unconstrained minimization problem[2]

$$\min_x F(x)=f(x)+\sum_{i=1}^{m'}\mu_i|c_i(x)|+\sum_{i=m'+1}^{m}\mu_i\max[0,\ -c_i(x)], \tag{1.2}$$

where $\mu_i>0$ ($i=1, 2, \cdots, m$) are sufficiently large and $F(x)$ is not differentiable on the surfaces

$$c_i(x)=0, \quad i=1, 2, \cdots, m. \tag{1.3}$$

The other is the filled function method for finding global minimizers of a continuously differentiable function[1], which can be extended to the nonsmooth minimization case. Here the nonsmoothness means that the objective function has no continuous gradient.

Thus, in Section 2 an extension of the filled function method to the nonsmooth case is developed. Section 3 gives an algorithm which uses the extension above to solve the constrained global minimization problems. Section 4 is a brief discussion.

This paper is closely related to [1]. Therefore it is not necessary to repeat the same context and only the different points will be mentioned in this paper.

By the way, it is still assumed that the functions $f(x)$ and $c_i(x)$ in (1.2) and (1.3) are all twice continuously differentiable.

§ 2. The Global Minimization of a Nonsmooth Function

This section is concerned with the problem of finding a global minimizer of a nonsmooth function. The definitions 1 and 2 in [1] should be changed in the following way.

Definition 1. *A segment x_1-x_2 is said to be a descent segment of $F(x)$ if the inequality*

$$F(x_1)<F(\alpha x_2+(1-\alpha)x_1)<F(\beta x_2+(1-\beta)x_1)<F(x_2) \tag{2.1}$$

holds for $x_1\neq x_2$ and any α, β satisfying

$$0<\alpha<\beta<1. \tag{2.2}$$

If the inequality opposite to (2.1) holds, then x_1-x_2 is said to be an ascent segment of $F(x)$.

Definition 2. *Suppose x_1^* is a minimizer of $F(x)$, where $F(x)$ is a nonsmooth function. The S-basin of $F(x)$ at x_1^* is a connected set S_1^*, which contains x_1^* and in which for any point x the segment x_1^*-x is a descent segment of $F(x)$, provided $x\neq x_1^*$.*

Definition 3. *Suppose x_1^* is a minimizer of $F(x)$. The basin of $F(x)$ at x_1^* is a connected set B_1^* which contains the S-basin S_1^* at x_1^* and for any $x\in B_1^*$ there exist a finite number of points $x_i\in B_1^*$ ($i=1, 2, \cdots, m$) (it is allowed that $x=x_i$) but $x_m\in S_1^*$, such that the inequalities*

$$F(x_i)<F(\alpha_i x_{i-1}+(1-\alpha_i)x_i)<F(\beta_i x_{i-1}+(1-\beta_i)x_i)<F(x_{i-1}) \tag{2.3}$$

hold for $i=1, 2, \cdots, m$, $x_0=x$, $0<\alpha_i<\beta_i<1$. If x_1^* is a maximizer of $G(x)$, then the hill of $G(x)$ at x_1^* is the basin of the function $-G(x)$.

These definitions clearly imply the following lemmas.

Lemma 1. *There exists a descent route which leads any $x\in B_1^*$ to descend to x_1^*.*

Lemma 2. *Suppose B_1^* is the basin of $F(x)$ at x_1^*. Then the inequality*

$$F(x)>F(x_1^*) \tag{2.4}$$

holds for any point $x\in B_1^*$ and $x\neq x_1^*$.

Lemma 3. *The smallest radius of the S-basin of $F(x)$ at x_1^* is not zero if x_1^* is an isolated minimizer of $F(x)$.*

Suppose $F(x)$ is a continuous function from R^n to R but nonsmooth as defined before and

$$F(x)\to+\infty \quad \text{as} \quad \|x\|\to+\infty. \tag{2.5}$$

This assumption allows us only to investigate a closed bounded domain Ω, which contains all global minimizers of $F(x)$. It is apparent that there exists a constant M_b such that

$$F(x)\geqslant M_b \quad \text{for} \quad x\in\Omega. \tag{2.6}$$

Assume a minimizer x_1^* of $F(x)$ has been located. Now the task is to find another minimizer x_2^* of $F(x)$, which satisfies the inequality

$$F(x_2^*)\leqslant F(x_1^*), \tag{2.7}$$

or to decide whether x_1^* is already a global minimizer.

Definition 4[1]. *A function $P(x, r, \rho)$ is called a filled function of $F(x)$ at x_1^* if $P(x, r, \rho)$ has the following properties:*

(i) x_1^* is a maximizer of $P(x, r, \rho)$ and the whole basin of $F(x)$ at x_1^* becomes a part of a hill of $P(x, r, \rho)$;

(ii) $P(x, r, \rho)$ does have minimizers or saddle points in the basins of some lower minimizers of $F(x)$, but has no minimizers or saddle points in the basins of higher minimizers of $F(x)$ for some parameters r and ρ.

It will be proved that the function

$$P(x, r, \rho)=\frac{1}{r+F(x)}\exp(-\|x-x_1^*\|^2/\rho^2) \tag{2.8}$$

is a filled function of $F(x)$ at x_1^* under some mild conditions on parameters r and ρ and the function $F(x)$ as well.

Theorem 1. *Suppose x_1^* is a minimizer of $F(x)$ and $P(x, r, \rho)$ is defined by (2.8), where r is a constant but perhaps a negative one, such that*

$$r+F(x_1^*)>0. \tag{2.9}$$

Then x_1^* is a maximizer of $P(x, r, \rho)$ and the whole S-basin of S_1^* is a part of an S-hill of $P(x, r, \rho)$ regardless of ρ.

Proof. By Lemma 2, from $F(x)>F(x_1^*)$ for $x\in B_1^*$, $x\neq x_1^*$, it follows that the inequality

$$P(x, r, \rho)=\frac{1}{r+F(x)}\exp(-\|x-x_1^*\|^2/\rho^2)<\frac{1}{r+F(x_1^*)}=P(x_1^*, r, \rho) \tag{2.10}$$

holds for $x\in B_1^*$, $x\neq x_1^*$. This proves that x_1^* is a maximizer of $P(x, r, \rho)$.

Assume x is in S_1^*. Then

$$0<r+F(x_1^*)<r+F(\alpha x+(1-\alpha)x_1^*)<r+F(\beta x+(1-\beta)x_1^*) \tag{2.11}$$

holds for $0<\alpha<\beta<1$ and therefore,

$$\frac{1}{r+F(\alpha x+(1-\alpha)x_1^*)}>\frac{1}{r+F(\beta x+(1-\beta)x_1^*)}>0. \tag{2.12}$$

On the other hand, since

$$\alpha x+(1-\alpha)x_1^*-x_1^*=\alpha(x-x_1^*), \tag{2.13}$$

hence

$$\exp(-\alpha^2\|x-x_1^*\|^2/\rho^2)>\exp(-\beta^2\|x-x_1^*\|^2/\rho^2). \tag{2.14}$$

Thus,

$$P(\alpha x+(1-\alpha)x_1^*, r, \rho)>P(\beta x+(1-\beta)x_1^*, r, \rho) \tag{2.15}$$

i.e.

$$-P(\alpha x+(1-\alpha)x_1^*, r, \rho)<-P(\beta x+(1-\beta)x_1^*, r, \rho). \tag{2.16}$$

This proves that the S-basin S_1^* of $F(x)$ is a part of an S-hill of $P(x, r, \rho)$.

To prove that the whole basin B_1^* of $F(x)$ is a hill of $P(x, r, \rho)$ we first prove the following elementary lemma.

Lemma 4. *The function of one variable*

$$\varphi(\alpha)=\|\alpha(x_1-x_2)+(x_2-x_1^*)\|^2 \tag{2.17}$$

is monotonically increasing for $\alpha>0$ if

$$(x_1-x_2)^T(x_2-x_1^*)\geqslant 0. \tag{2.18}$$

Proof. It is apparent that

$$\varphi'(\alpha)=2\alpha\|x_1-x_2\|^2+2(x_1-x_2)^T(x_2-x_1^*)>0 \tag{2.19}$$

is implied by (2.18).

Now we have the following theorem.

Theorem 2. *Suppose x_1-x_2 is an ascent segment of $F(x)$ and*

$$(x_1-x_2)^T(x_2-x_1^*)\geqslant 0, \tag{2.20}$$

$$r+F(x_2)>0. \tag{2.21}$$

Then x_1-x_2 is a downhill segment of $P(x, r, \rho)$ regardless of ρ.

Proof. That x_1-x_2 is an ascent segment of $F(x)$ means that for $0<\alpha<\beta<1$ one has

$$F(x_2)<F(\alpha x_1+(1-\alpha)x_2)<F(\beta x_1+(1-\beta)x_2)<F(x_1). \tag{2.22}$$

Thus,

$$\frac{1}{r+F(\alpha x_1+(1-\alpha)x_2)}>\frac{1}{r+F(\beta x_1+(1-\beta)x_2)}. \tag{2.23}$$

Furthermore, it follows from the identity

$$\alpha x_1+(1-\alpha)x_2-x_1^*=\alpha(x_1-x_2)+(x_2-x_1^*) \tag{2.24}$$

and Lemma 4 that when (2.20) holds one has

$$\exp(-\|\alpha x_1+(1-\alpha)x_2-x_1^*\|^2/\rho^2)>\exp(-\|\beta x_1+(1-\beta)x_2-x_1^*\|^2/\rho^2). \tag{2.25}$$

Therefore, from (2.23) and (2.25) it follows that

$$P(\alpha x_1+(1-\alpha)x-x_1^*, r, \rho)>P(\beta x_1+(1-\beta)x_2-x_1^*, r, \rho), \tag{2.26}$$

that is, x_1-x_2 is a downhill segment of $P(x, r, \rho)$.

Even some downhill segments of $F(x)$ can be downhill segments of $P(x, r, \rho)$, provided the ratio $\rho^2/[r+F(x_1^*)]$ is appropriately small as shown in the following theorem.

Theorem 3. *Suppose the segment x_2-x_1 is a downhill segment of $F(x)$, then the inequality*

$$F(x_2)<F(\alpha x_1+(1-\alpha)x_2)<F(\beta x_1+(1-\beta)x_2)<F(x_1) \tag{2.27}$$

holds for $0<\alpha<\beta<1$, *and*

$$(x_1-x_2)^T(x_1-x_1^*)<0 \quad and \quad \frac{|(x_1-x_2)^T(x_1-x_1^*)|}{\|x_1-x_2\|\,\|x_1-x_1^*\|}\geqslant c_1>0, \tag{2.28}$$

where c_1 *is a fixed constant. Then the segment of* x_2-x_1 *is also a downhill segment of* $P(x, r, \rho)$, *provided*

$$\frac{\rho^2}{r+F(x_1^*)}\leqslant\frac{2Dc_1}{L}, \tag{2.29}$$

where

$$D=\min_{x\in S_1^*}\|x-x_1^*\|, \quad \|\partial F(x)\|\leqslant L, \quad x\in\Omega, \tag{2.30}$$

and

$$F(x_2)\geqslant F(x_1^*). \tag{2.31}$$

Proof. Note that

$$\alpha x_1+(1-\alpha)x_2-x_1^*=\alpha(x_1-x_2)+(x_2-x_1^*). \tag{2.32}$$

It is required that

$$\frac{1}{r+F(\alpha x_1+(1-\alpha)x_2)}\exp(-\|\alpha x_1+(1-\alpha)x_2-x_1^*\|^2/\rho^2)$$
$$<\frac{1}{r+F(\beta x_1+(1-\beta)x_2)}\exp(-\|\beta x_1+(1-\beta)x_2-x_1^*\|^2/\rho^2). \tag{2.33}$$

It is easy to see that

$$\frac{F(\beta x_1+(1-\beta)x_2)-F(\alpha x_1+(1-\alpha)x_2)}{r+F(\alpha x_1+(1-\alpha)x_2)}$$
$$<\exp\left[\frac{(\alpha-\beta)(x_1-x_2)^T[(\alpha+\beta)(x_1-x_2)+2(x_2-x_1^*)]}{\rho^2}\right]-1. \tag{2.34}$$

The exponential variable is always positive provided (2.28) holds, because

$$(\alpha-\beta)(x_1-x_2)^T[(\alpha+\beta)(x_1-x_2)+2(x_2-x_1^*)]$$
$$=(\alpha-\beta)(x_1-x_2)^T[(\alpha+\beta)(x_1-x_2)+2(x_2-x_1+x_1-x_1^*)]$$
$$=(\alpha-\beta)(\alpha+\beta-2)\|x_1-x_2\|^2+2(\alpha-\beta)(x_1-x_2)^T(x_1-x_1^*)>0, \tag{2.35}$$

since $\alpha-\beta<0$, $\alpha+\beta-2<0$ and $(x_1-x_2)^T(x_1-x_1^*)<0$. Therefore inequality (2.34) is implied by the inequality

$$\frac{\rho^2}{r+F(\alpha x_1+(1-\alpha)x_2)}<\frac{(\alpha-\beta)(x_1-x_2)^T[(\alpha+\beta)(x_1-x_2)+2(x_2-x_1^*)]}{F(\beta x_1+(1-\beta)x_2)-F(\alpha x_1+(1-\alpha)x_2)}. \tag{2.36}$$

Note that in nonsmooth analysis we have

$$\lim_{\beta\to\alpha}\frac{F(\beta x_1+(1-\beta)x_2)-F(\alpha x_1+(1-\alpha)x_2)}{\beta-\alpha}=\max_{g\in\partial F(x)}(x_1-x_2)^Tg, \tag{2.37}$$

where $\partial F(x)$ denotes the subdifferential of $F(x)$ at x and g is an element of $\partial F(x)$, $x=\alpha(x_1-x_2)+x_2$ (see [2]). Thus, the right hand side of (2.36) has the limit case

$$\frac{\rho^2}{r+F(x)}\leqslant\frac{-2(x_1-x_2)^T(x-x_1^*)}{\max_{g\in\partial F(x)}(x_1-x_2)^Tg}. \tag{2.38}$$

Since x_2-x_1 is a descent segment of $F(x)$, hence x_1-x_2 is a ascent segment of $F(x)$ and so it follows from (2.37) that

$$\max_{g\in\partial F(x)}(x_1-x_2)^Tg>0. \tag{2.39}$$

From (2.28) it follows that

$$\begin{aligned}(x_1-x_2)^T(x-x_1^*) &= (x_1-x_2)^T[\alpha x_1+(1-\alpha)x_2-x_1^*] \\ &= \alpha\|x_1-x_2\|^2+(x_1-x_2)^T(x_2-x_1^*) \\ &= \alpha\|x_1-x_2\|^2+(x_2-x_1+x_1-x_1^*) \\ &= (\alpha-1)\|x_1-x_2\|^2+(x_1-x_2)^T(x_1-x_1^*)<0 \end{aligned} \tag{2.40}$$

and (2.40) gives

$$\begin{aligned}-\|x_1-x_2\|^2+(x_1-x_2)^T(x_1-x_1^*) &< (x_1-x_2)^T(x-x_1^*) \\ &< (x_1-x_2)^T(x_1-x_1^*)<0.\end{aligned} \tag{2.41}$$

Thus,

$$-2(x_1-x_2)^T(x-x_1^*)>-2(x_1-x_2)(x_1-x_1^*)>0. \tag{2.42}$$

From (2.28) and (2.30) it follows that

$$|(x_1-x_2)^T(x_1-x_1^*)|\geqslant\|x_1-x_2\|Dc_1 \tag{2.43}$$

and

$$0<\max_{g\in\partial F(x)}(x_1-x_2)^Tg\leqslant\|x_1-x_2\|L, \tag{2.44}$$

where $\|\partial F(x)\|$ is defined as

$$\|\partial F(x)\|=\max_{g\in\partial F(x)}\|g\|. \tag{2.45}$$

Then (2.38) and (2.36) are implied by

$$\frac{\rho^2}{r+F(x_1^*)}\leqslant\frac{\rho^2}{r+F(x)}\leqslant\frac{2Dc_1}{L}. \tag{2.46}$$

Note that here $2Dc_1/L$ is a fixed number regardless of x_1, x_2, x_1^*. One can choose ρ^2 and r to make (2.46) hold and therefore (2.36) and (2.38) hold.

Theorem 3 is proved now.

Theorem 4. *Under the assumptions in the above theorems, $P(x, r, \rho)$ cannot have any minimizer or saddle point in the basin of x_1^* provided inequality (2.29) holds. Furthermore, suppose x_2^* is a minimizer of $F(x)$ and*

$$F(x_2^*)>F(x_1^*). \tag{2.47}$$

If (2.29) and (2.9) hold, then there are no minimizers or saddle points of $P(x, r, \rho)$ in the basin of $F(x)$ at x_2^*.

Proof. Theorem 1 shows that the whole S-basin S_1^* of $F(x)$ at x_1^* is a part of S-hill of $P(x, r, \rho)$ and there are no minimizers or saddle points in S_1^*. In the sets $B_1^*\backslash S_1^*$ and B_2^* the inequality

$$F(x)>F(x_1^*) \tag{2.48}$$

holds. Thus, (2.29) and (2.9) imply that $P(x, r, \rho)$ has no minimizers at least along the direction $x-x_1^*$ and so $P(x, r, \rho)$ cannot have any minimizers or saddle points in both $B_1^*\backslash S_1^*$ and B_2^* according to Theorems 2 and 3. The proof is complete.

Theorem 4 shows that one cannot locate the minimizers of $F(x)$, at which the values of $F(x)$ are larger than $F(x_1^*)$ by using the filled function method with the filled function $P(x, r, \rho)$, provided (2.29) and (2.9) hold.

However, can one find the minimizers of $F(x)$, at which the values of $F(x)$ are not greater than $F(x_1^*)$? This is equivalent to whether there do exist minimizers or saddle points of $P(x, r, \rho)$ in the basins of some lower minimizers of $F(x)$. The following theorem shows that it is possible under some conditions.

Theorem 5. *Suppose x_2 and x_1 are such that (2.28) holds and x_2-x_1 is a downhill segment of $F(x)$, that is, (2.27) holds. Then*

(i) *If*

$$r+F(\alpha x_1+(1-\alpha)x_2)>0 \tag{2.49}$$

and

$$\frac{r+F(\beta x_1+(1-\beta)x_2)}{r+F(\alpha x_1+(1-\alpha)x_2)}>\exp\left(-\frac{(\beta-\alpha)(x_1-x_2)^T[(\beta+\alpha)(x_1-x_2)+2(x_2-x_1^*)]}{\rho^2}\right), \tag{2.50}$$

then x_2-x_1 is an uphill segment of $P(x, r, \rho)$.

(ii) *If*

$$r+F(x)<0, \tag{2.51}$$

then

$$P(x, r, \rho)<0. \tag{2.52}$$

(iii) *If*

$$r+F(x)\to 0^+, \tag{2.53}$$

then

$$P(x, r, \rho)\to+\infty. \tag{2.54}$$

Proof. (i) To make

$$P(\alpha x_1+(1-\alpha)x_2, r, \rho)>P(\beta x_1+(1-\beta)x_2, r, \rho), \tag{2.55}$$

i.e.

$$\frac{1}{r+F(\alpha x_1+(1-\alpha)x_2)}\exp\left(-\frac{\|\alpha x_1+(1-\alpha)x_2-x_1^*\|^2}{\rho^2}\right) > \frac{1}{r+F(\beta x_1+(1-\beta)x_2)}\exp\left(-\frac{\|\beta x_1+(1-\beta)x_2-x_1^*\|^2}{\rho^2}\right) \tag{2.56}$$

is equivalent to requiring

$$\frac{r+F(\beta x_1+(1-\beta)x_2)}{r+F(\alpha x_1+(1-\alpha)x_2)}>\exp\left(-\frac{(\beta-\alpha)(x_1-x_2)^T[(\beta+\alpha)(x_1-x_2)+2(x_2-x_1^*)]}{\rho^2}\right). \tag{2.57}$$

(ii) and (iii) are obvious by the definition of $P(x, r, \rho)$.

$P(x, r, \rho)$ is downhill in the direction $x-x_1^*$ for such x that

$$F(x)>F(x_1^*) \tag{2.58}$$

provided the ratio $\rho^2/[r+F(x_1^*)]$ is appropriately small, while $P(x, r, \rho)$ becomes uphill in the direction $x-x_1^*$ when $F(x)<F(x_1^*)$ and $r+F(x)\to 0^+$. Therefore it is certain that there exists a minimizer or a saddle point $\hat{x}$ of $P(x, r, \rho)$ in the basin of a lower minimizer of $F(x)$. The $\hat{x}$ can be used as an initial point to minimize $F(x)$ again to obtain a lower minimizer x_2^* of $F(x)$. If $F(x)$ has only a finite number of minimizers, then one can finally find a global minimizer of $F(x)$ in this way.

As in the smooth case, when the ratio $\rho^2/[r+F(x_1^*)]$ is too small, even a global minimizer of $F(x)$ might be lost as stated in the following theorem. Its proof is the same as the proof of Theorem 3 except that x and x^* should be used to replace x_1 and x_2 respectively.

Theorem 6. *If the choice of the parameters r and ρ makes*

$$0<\frac{\rho^2}{r+F(x^*)}<\frac{(\alpha-\beta)(x-x^*)^T[(\alpha+\beta)(x-x^*)+2(x^*-x_1^*)]}{F(\beta x+(1-\beta)x^*)-F(\alpha x+(1-\alpha)x^*)}, \tag{2.59}$$

where x^* is a global minimizer of $F(x)$, then x^*-x is always a descent segment, where x is an arbitrary point on the segment $x^*-x_1^*$. Therefore $P(x, r, \rho)$ has no minimizers or saddle points at all even in the basin of x^*.

The theorems in this section give an algorithm for finding a global minimizer of a nonsmooth function $F(x)$ and so an algorithm for finding a constrained global minimizer of a constrained minimization problem. They also tell us how to adjust the parameters r and ρ. The detailed algorithm and some practical consideration will be described in the next section.

§ 3. The Algorithm and Practical Consideration

First we describe the algorithm and then make some explanation. The algorithm is as follows.

1. Minimize $F(x)$ from an initial point x_1 in Ω by the nonsmooth minimization method. Denote the minimizer of $F(x)$ by x_1^*. Set $MS=0$.

2. Construct the filled function

$$P(x, r, \rho)=\frac{1}{r+F(x)}\exp(-\|x-x_1^*\|^2/\rho^2) \tag{3.1}$$

and minimize it from an initial point near x_1^*, for instance

$$x_1=x_1^*+\delta e_1, \quad \delta>0, \tag{3.2}$$

where e_1 is the first coordinate and δ is a preset number.

3. If the minimization sequence of $P(x, r, \rho)$ goes out of Ω, then change an initial point, for instance

$$x_1=x_1^*-\delta e_1, \quad \delta>0 \tag{3.3}$$

and if this happens again and again, then change initial points

$$x_1=x_1^*\pm\delta e_1, \quad i=2, 3, \cdots, n, \delta>0 \tag{3.4}$$

in turn. If all the x_1 in (3.2) to (3.4) have been used but all the sequences go out of Ω, go to 9.

4. Try the following criteria for terminating the minimization of $P(x, r, \rho)$:

a. $P(x_k, r, \rho)\geqslant P(x_{k-1}, r, \rho)$, or

b. $P(x_k, r, \rho)<0$, then take x_k as $\hat{x}$.

c. Minimize $P(\alpha x_k+(1-\alpha)x_{k-1}, r, \rho)$ with respect to α and denote the minimizer as $\alpha_{\min}$. If $\alpha_{\min}<1$, take x_k as $\hat{x}$; if $\alpha_{\min}=1$, do next k.

d. Minimize $P(\alpha x_k+(1-\alpha)\hat{x}_{k-1}, r, \rho)$ with respect to α and denote the minimizer as $\alpha_{\min}$. If $\alpha_{\min}<1$, take x_k as $\hat{x}$; if $\alpha_{\min}=1$, do next k. Here it is set that

$$\hat{x}_{k-1}=[1-(x_k-x_{k-1})^T(x_k-x_1^*)](x_1-x_1^*)+x_1^*. \tag{3.5}$$

e. $F(x_k)<F(x_1^*)$.

5. Minimize $F(x)$ from the initial point $\hat{x}$ to obtain another minimizer x_2^* of $F(x)$.

6. If $F(x_2^*)\leqslant F(x_1^*)$, set $x_1^*\leftarrow x_2^*$ and go to 2.

7. If $F(x_2^*)>F(x_1^*)$ and $MS<NS$, a preset integer, enlarge ρ^2 and $r+F(x_2^*)$ but make the ratio $\rho^2/[r+F(x_2^*)]$ smaller than before. Set $x_1^*\leftarrow x_2^*$, go to 2, but use the first initial point

$$x = x_2^* + \delta(x_2^* - x_1^*)/\|x_2^* - x_1^*\|. \tag{3.6}$$

8. If $F(x_2^*) > F(x_1^*)$ and $MS \geqslant NS$, stop.

9. Decrease $r + F(x_1^*)$ to increase the ratio $\rho^2/[r + F(x_1^*)]$, set $MS \leftarrow MS + 1$, go to 2.

By the analysis in Section 2, a stationary point $\hat{x}$ of $P(x, r, \rho)$ is a minimizer of $P(x, r, \rho)$ at least along the radius direction $x_2^* - x_1^*$, and $P(x, r, \rho)$ will ascend only when x has been in a basin of a lower minimizer of $F(x)$. This can be expressed by criteria a, c and d. Since the fact $P(x, r, \rho) < 0$ implies $r + F(x) < 0$, that is, $F(x) < F(x_1^*)$, it is the criterion b.

That all the sequences of minimizing $P(x, r, \rho)$ go out of the domain Ω is caused by either a too small ratio $\rho^2/[r + F(x_1^*)]$ or by the fact that x_1^* is already a global minimizer of $F(x)$. In the first case, for a little larger ratio $\rho^2/[r + F(x_1^*)]$ another lower minimizer of $F(x)$ can be found. In the second case, for a little larger ratio $\rho^2/[r + F(x_1^*)]$, a higher minimizer of $F(x)$ can be found. Therefore, when all sequences of minimizing $P(x, r, \rho)$ go out of Ω, this ratio should be increased a little bit by decreasing $r + F(x_1^*)$. We do not increase ρ^2 to increase the ratio $\rho^2/[r + F(x_1^*)]$ because Theorem 5 shows that a smaller positive $r + F(x_1^*)$ is preferred to a larger one when $F(x_1^*)$ is close to a global minimum of $F(x)$. This argument is used in steps 7, 8 and 9.

§ 4. Conclusion

The purpose of this paper is to find a method for finding global minimizers of a nonlinearly constrained minimization problem. We have seen from the above analysis that the filled function method for finding global minimizers of a smooth function of several variables can be extended to the case of finding global minimizers of a nonsmooth function. Since a nonlinearly constrained global minimization problem can be transformed to a nonsmooth minimization problem via the exact penalty function method, the filled function method can also be used to solve nonlinearly constrained global minimization problems.

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