USING A PREDICTOR–CORRECTOR SCHEME TO COMPUTE NAVIER–STOKES EQUATIONS IN THREE–DIMENSIONAL SPHERICAL COORDINATES*

CHANG QIAN-SHUN (常谦顺)
(Institute of Applied Mathematics, Academia Sinica, Beijing, China)

Abstract

A new predictor–corrector difference scheme is described for solving the time-dependent Navier-Stokes equations in three-dimensional spherical coordinates. A boundary condition for the pressure is deduced by auxiliary velocity. A multigrid algorithm is employed in solving equations of the pressure. An example of application of this scheme is computed and its results are presented.

§1. Introduction

In a general numerical scheme for Navier-Stokes equations, velocities are advanced explicitly in time. In such explicit schemes, the time step is restricted by stability conditions. This is more stringent for a smaller Reynolds number and, especially, in spherical coordinates. A predictor–corrector scheme is given in this paper. It saves computational time and keeps properties of the differential equations $(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{u})=0$. The pressure is computed by a Poisson's equation. A boundary condition for the equation is given by means of auxiliary velocities. In order to reduce computational time, a multigrid algorithm is employed in solving Poisson's equation.

In this paper h_1, h_2 and h_3 are step sizes in r, θ and φ direction, respectively. The following notations for difference operators are used:

$$\delta_r f^n_{i,j,k}=\frac{1}{2h_1}(f^n_{i+1,j,k}-f^n_{i-1,j,k}),\qquad \delta_\theta f^n_{i,j,k}=\frac{1}{2r_i h_2}(f^n_{i,j+1,k}-f^n_{i,j-1,k}),$$

$$\delta_\varphi f^n_{i,j,k}=\frac{1}{2r_i\cdot\sin\theta_j\cdot h_3}(f^n_{i,j,k+1}-f^n_{i,j,k-1}),$$

$$\delta^2_r f^n_{i,j,k}=\frac{1}{r_i^2\cdot h_1^2}[r^2_{i+1/2}(f^n_{i+1,j,k}-f^n_{i,j,k})-r^2_{i-1/2}(f^n_{i,j,k}-f^n_{i-1,j,k})],$$

$$\delta^2_\theta f^n_{i,j,k}=\frac{1}{r_i^2\cdot\sin\theta_j\cdot h_2^2}[\sin\theta_{j+1/2}(f^n_{i,j+1,k}-f^n_{i,j,k})-\sin\theta_{j-1/2}(f^n_{i,j,k}-f^n_{i,j-1,k})],$$

$$\delta^2_\varphi f^n_{i,j,k}=\frac{1}{r_i^2\sin^2\theta_j\cdot h_3^2}[f^n_{i,j,k+1}-2f^n_{i,j,k}+f^n_{i,j,k-1}],$$

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$$\Delta_\theta f^n_{i,j,k}=\frac{1}{2r_i\cdot\sin\theta_j\cdot h_2}[\sin\theta_{j+1/2}(f^n_{i,j+1,k}+f^n_{i,j,k})-\sin\theta_{j-1/2}(f^n_{i,j,k}+f^n_{i,j-1,k})].$$

§ 2. Differential Equations and Basic Algorithm

We consider the Navier-Stokes problem in spherical coordinates

$$\begin{aligned}\frac{\partial u}{\partial t}&+\left[u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial\theta}+\frac{w}{r\cdot\sin\theta}\frac{\partial u}{\partial\varphi}-\frac{v^2+w^2}{r}\right]\\&-\nu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\cdot\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\cdot\sin^2\theta}\frac{\partial^2u}{\partial\varphi^2}\right.\\&\left.-\frac{2}{r^2}\left(u+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cdot v)+\frac{1}{\sin\theta}\frac{\partial w}{\partial\varphi}\right)\right]+\frac{\partial p}{\partial r}=F_1,\end{aligned}\tag{2.1}$$

$$\begin{aligned}\frac{\partial v}{\partial t}&+\left[u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial\theta}+\frac{w}{r\cdot\sin\theta}\frac{\partial v}{\partial\varphi}+\frac{uv}{r}-\frac{w^2}{r}\operatorname{ctg}\theta\right]\\&-\nu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right)+\frac{1}{r^2\cdot\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cdot\frac{\partial v}{\partial\theta}\right)+\frac{1}{r^2\cdot\sin^2\theta}\frac{\partial^2v}{\partial\varphi^2}\right.\\&\left.+\frac{2}{r^2}\left(\frac{\partial u}{\partial\theta}-\frac{v}{2\sin^2\theta}-\frac{\cos\theta}{\sin^2\theta}\frac{\partial w}{\partial\varphi}\right)\right]+\frac{1}{r}\frac{\partial p}{\partial\theta}=F_2,\end{aligned}\tag{2.2}$$

$$\begin{aligned}\frac{\partial w}{\partial t}&+\left[u\frac{\partial w}{\partial r}+\frac{v}{r}\frac{\partial w}{\partial\theta}+\frac{w}{r\cdot\sin\theta}\frac{\partial w}{\partial\varphi}+\frac{u\cdot w}{r}+\frac{vw}{r}\operatorname{ctg}\theta\right]\\&-\nu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial w}{\partial r}\right)+\frac{1}{r^2\cdot\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial w}{\partial\theta}\right)+\frac{1}{r^2\cdot\sin^2\theta}\frac{\partial^2w}{\partial\varphi^2}\right.\\&\left.+\frac{2}{r^2\cdot\sin\theta}\left(\frac{\partial u}{\partial\varphi}+\operatorname{ctg}\theta\frac{\partial v}{\partial\varphi}-\frac{w}{2\cdot\sin\theta}\right)\right]+\frac{1}{r\cdot\sin\theta}\frac{\partial p}{\partial\varphi}=F_3,\end{aligned}\tag{2.3}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2u)+\frac{1}{r\cdot\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cdot v)+\frac{1}{r\cdot\sin\theta}\frac{\partial w}{\partial\varphi}=0,\tag{2.4}$$

$$\boldsymbol{u}|_\Gamma=0,\tag{2.5}$$

$$\boldsymbol{u}|_{t=0}=\boldsymbol{u}_0(r\cdot\theta\cdot\varphi),\tag{2.6}$$

where $\boldsymbol{u}=(u, v, w)$ is the velocity vector, p is the ratio of the pressure to constant density (for brevity, we refer to p simply as pressure) and ν is a kinematic viscosity coefficient.

In order to deduce the basic splitting scheme and the boundary condition of the pressure, we write the Navier-Stokes problem in vector form:

$$\frac{\partial\boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\nabla p=\nu\Delta\boldsymbol{u}+\boldsymbol{F},\tag{2.7}$$

$$\nabla\cdot\boldsymbol{u}=0,\tag{2.8}$$

$$\boldsymbol{u}|_\Gamma=0,\tag{2.9}$$

$$\boldsymbol{u}|_{t=0}=\boldsymbol{u}_0(r\cdot\theta\cdot\varphi).\tag{2.10}$$

First, equations (2.7)—(2.10) are written in difference form only for time:

$$\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^n}{\tau}=-\boldsymbol{u}^n\cdot\nabla\boldsymbol{u}^n-\nabla p^{n+1}+\nu\Delta\boldsymbol{u}^n+\boldsymbol{F}^n,\tag{2.11}$$

$$\nabla\cdot\boldsymbol{u}^{n+1}=0, \tag{2.12}$$

$$\boldsymbol{u}|_{\Gamma}=0. \tag{2.13}$$

Let the auxiliary velocity

$$\overline{\boldsymbol{u}}=\boldsymbol{u}^n+\tau[-\boldsymbol{u}^n\cdot\nabla\boldsymbol{u}^n+\nu\Delta\boldsymbol{u}^n+\boldsymbol{F}^n]. \tag{2.14}$$

It follows from (2.11) that

$$\boldsymbol{u}^{n+1}=\overline{\boldsymbol{u}}-\tau\nabla p^{n+1}. \tag{2.15}$$

In view of (2.12), we have Poisson's equation for the pressure

$$\Delta p^{n+1}=\frac{1}{\tau}\nabla\cdot\overline{\boldsymbol{u}}. \tag{2.16}$$

Assume $\boldsymbol{\xi}$ is a unit vector in the out ward normal direction on boundary Γ. Then the boundary condition for the pressure is given from (2.13) and (2.15) as

$$\left.\frac{\partial p^{n+1}}{\partial\xi}\right|_{\Gamma}=\frac{1}{\tau}\overline{\boldsymbol{u}}\cdot\boldsymbol{\xi}|_{\Gamma}. \tag{2.17}$$

After equation (2.16) is written in difference form, the term $\overline{\boldsymbol{u}}\cdot\boldsymbol{\xi}|_{\Gamma}$ of equation (2.16) and the boundary condition (2.17) will be deleted in the rigid wall. Thus, the equation for the pressure has a simple form in the rigid wall. Therefore, the boundary condition for the pressure (2.17) is simple and useful.

Equations (2.14)—(2.16) and the boundary conditions (2.13), (2.17) are a splitting scheme for Navier-Stokes equations, which can be used in two or three dimensions and in any coordinates. In computation, we compute auxiliary velocity $\overline{\boldsymbol{u}}$ from (2.14) firstly. Then Poisson's equation for the pressure is solved. Finally, using the pressure values, the auxiliary velocity is modified to velocity $\boldsymbol{u}^{n+1}$ in new time.

§ 3. Predictor-corrector Difference Scheme and Boundary Conditions

We use the non-staggered grid, because equations (2.1)—(2.4) in $r=0$, $\theta=0$ and $\theta=\pi$ are singular. All velocities and pressures in the difference scheme are located at cell centers. Thus, singularities are avoided in computation. The difference scheme has the following form:

$$\overline{\boldsymbol{u}}'_{i,j,k}-\frac{\tau\nu}{2}\delta_r^2\overline{\boldsymbol{u}}'_{i,j,k}=\boldsymbol{u}^n_{i,j,k}+\frac{\tau\nu}{2}(\delta_\theta^2\boldsymbol{u}^n_{i,j,k}+\delta_\varphi^2\boldsymbol{u}^n_{i,j,k})+\tau\overline{\boldsymbol{F}}_{i,j,k}, \tag{3.1}$$

$$\overline{\boldsymbol{u}}''_{i,j,k}-\frac{\tau\nu}{2}\delta_\theta^2\overline{\boldsymbol{u}}''=\boldsymbol{u}^n_{i,j,k}+\frac{\tau\nu}{2}(\delta_r^2\overline{\boldsymbol{u}}'_{i,j,k}+\delta_\varphi^2\boldsymbol{u}^n_{i,j,k})+\tau\overline{\boldsymbol{F}}_{i,j,k}, \tag{3.2}$$

$$\overline{\boldsymbol{u}}_{i,j,k}-\frac{\tau\nu}{2}\delta_\varphi^2\overline{\boldsymbol{u}}=\boldsymbol{u}^n_{i,j,k}+\frac{\tau\nu}{2}(\delta_r^2\overline{\boldsymbol{u}}'_{i,j,k}+\delta_\theta^2\overline{\boldsymbol{u}}'')+\tau\overline{\boldsymbol{F}}_{i,j,k}, \tag{3.3}$$

$$\delta_r^2p^{n+1}_{i,j,k}+\delta_\theta^2p^{n+1}_{i,j,k}+\delta_\varphi^2p^{n+1}_{i,j,k}=\frac{1}{2\tau r_i^2h_1}\left[r^2_{i+1/2}(\bar{u}_{i+1,j,k}+\bar{u}_{i,j,k})-r^2_{i-1/2}(\bar{u}_{i,j,k}+\bar{u}_{i-1,j,k})\right]$$

$$+\frac{1}{\tau}\Delta_\theta\bar{v}_{i,j,k}+\frac{1}{\tau}\delta_\varphi\bar{w}_{i,j,k}, \tag{3.4}$$

$$u_{i,j,k}^{n+1}=\bar{u}_{i,j,k}-\tau\delta_r p_{i,j,k}^{n+1}, \tag{3.5}$$

$$v_{i,j,k}^{n+1}=\bar{v}_{i,j,k}-\tau\delta_\theta p_{i,j,k}^{n+1}, \tag{3.6}$$

$$w_{i,j,k}^{n+1}=\bar{w}_{i,j,k}-\tau\delta_\varphi p_{i,j,k}^{n+1}, \tag{3.7}$$

where

$$\bar{\boldsymbol{u}}=(\bar{u},\ \bar{v},\ \bar{w}),\quad \bar{\boldsymbol{F}}=(\bar{F}_1,\ \bar{F}_2,\ \bar{F}_3),$$

$$\begin{aligned}\bar{F}_{1,i,j,k}\equiv F_{1,i,j,k}&-[u_{i,j,k}^n\cdot\delta_r u_{i,j,k}^n+v_{i,j,k}^n\cdot\delta_\theta u_{i,j,k}^n+w_{i,j,k}^n\cdot\delta_\varphi u_{i,j,k}^n]\\&+\frac{1}{r_i}[(v_{i,j,k}^n)^2+(w_{i,j,k}^n)^2]+\nu\Big[\frac{1}{2}\delta_r^2u_{i,j,k}^n+\frac{1}{2}\delta_\theta^2u_{i,j,k}^n\\&+\frac{1}{2}\delta_\varphi^2u_{i,j,k}^n-\frac{2}{r_i^2}u_{i,j,k}^n-\frac{2}{r_i}\Delta_\theta v_{i,j,k}^n-\frac{2}{r_i}\delta_\varphi w_{i,j,k}^n\Big],\end{aligned}$$

$$\begin{aligned}\bar{F}_{2,i,j,k}\equiv F_{2,i,j,k}&-[u_{i,j,k}^n\cdot\delta_r v_{i,j,k}^n+v_{i,j,k}^n\cdot\delta_\theta v_{i,j,k}^n+w_{i,j,k}^n\cdot\delta_\varphi v_{i,j,k}^n]\\&-\frac{1}{r_i}\Big[u_{i,j,k}^n\cdot v_{i,j,k}^n-(w_{i,j,k}^n)^2\frac{\cos\theta_i}{\sin\theta_j}\Big]+\nu\Big[\frac{1}{2}\delta_r^2v_{i,j,k}^n+\frac{1}{2}\delta_\theta^2v_{i,j,k}^n+\frac{1}{2}\delta_\varphi^2v_{i,j,k}^n\\&+\frac{2}{r_i}\delta_\theta u_{i,j,k}^n-\frac{v_{i,j,k}^n}{r_i^2\cdot\sin^2\theta_j}-\frac{2\cos\theta_j}{r_i\cdot\sin\theta_j}\cdot\delta_\varphi w_{i,j,k}\Big],\end{aligned}$$

$$\begin{aligned}\bar{F}_{3,i,j,k}\equiv F_{3,i,j,k}&-[u_{i,j,k}^n\cdot\delta_r w_{i,j,k}^n+v_{i,j,k}^n\cdot\delta_\theta w_{i,j,k}^n+w_{i,j,k}^n\cdot\delta_\varphi w_{i,j,k}^n]\\&-\frac{w_{i,j,k}^n}{r_i}\Big[u_{i,j,k}^n+v_{i,j,k}^n\frac{\cos\theta_j}{\sin\theta_j}\Big]+\nu\Big[\frac{1}{2}\delta_r^2w_{i,j,k}^n+\frac{1}{2}\delta_\theta^2w_{i,j,k}^n+\frac{1}{2}\delta_\varphi^2w_{i,j,k}^n\\&+\frac{2}{r_i}\delta_\varphi u_{i,j,k}^n+\frac{2\cos\theta_j}{r_i\cdot\sin\theta_j}\delta_\varphi v_{i,j,k}^n-\frac{w_{i,j,k}^n}{r_i^2\cdot\sin^2\theta_j}\Big],\end{aligned}$$

$1\leqslant i\leqslant I_B$, $1\leqslant j\leqslant J_B$; $1\leqslant k\leqslant K_B$; I_B, J_B and K_B are total numbers of points in r, θ and φ direction, respectively.

We consider a unit ball centered at $r=0$ as a model problem. The no-slip rigid wall is located in $r=1$. The boundary conditions for the difference scheme in $r=1$ have the following forms

$$u_{I_B+1,j,k}^n=-u_{I_B,j,k}^n,\quad v_{I_B+1,j,k}^n=-v_{I_B,j,k}^n,\quad w_{I_B+1,j,k}^n=-w_{I_B,j,k}^n,$$

$$\frac{1}{h_1}(p_{I_B+1,j,k}-p_{I_B,j,k})=\frac{1}{\tau}\bar{u}_{I_B+1/2,j,k},$$

where the pressure boundary condition is given in view of taking ξ as r and the boundary Γ as $r=1$ in condition (2.17). The boundary conditions in $r=0$, $\theta=0$ and $\theta=\pi$ are given by the property of spherical coordinates:

$$u_{0,j,k}^n=-u_{1,J_B+1-j,k\pm K_B/2}^n,\qquad v_{0,j,k}^n=v_{1,J_B+1-j,k\pm K_B/2}^n,$$

$$w_{0,j,k}^n=-w_{1,J_B+1-j\ k\pm K_B/2}^n,\qquad p_{0,j,k}^n=p_{1,J_B+1-j,k\pm K_B/2}^n;$$

$$u_{i,0,k}^n=u_{i,1,k\pm K_B/2}^n,\qquad v_{i,0,k}^n=-v_{i,1,k\pm K_B/2}^n,$$

$$w^n_{i,0,k}=-w^n_{i,1,k\pm K_B/2},\qquad p^n_{i,0,k}=p^n_{i,1,k\pm K_B/2};$$

$$u^n_{i,J_B+1,k}=u^n_{i,J_B,k\pm K_B/2},\qquad v^n_{i,J_B+1,k}=-v^n_{i,J_B,k\pm K_B/2},$$

$$w^n_{i,J_B+1,k}=-w^n_{i,J_B,k\pm K_B/2},\qquad p^n_{i,J_B+1,k}=p^n_{i,J_B,k\pm K_B/2};$$

where in the term $k\pm K_B/2$ plus is taken if $k\pm K_B/2$ is less than K_B; otherwise minus is taken. The solution satisfies periodic conditions in the boundary of variable φ:

$$u^n_{i,j,0}=u^n_{i,j,K_B},\qquad v^n_{i,j,0}=v^n_{i,j,K_B},\qquad w^n_{i,j,0}=w^n_{i,j,K_B},\qquad p^n_{i,j,0}=p^n_{i,j,K_B};$$

$$u^n_{i,j,K_B+1}=u^n_{i,j,1},\qquad v^n_{i,j,K_B+1}=v^n_{i,j,1},\qquad w^n_{i,j,K_B+1}=w^n_{i,j,1},\qquad p^n_{i,j,K_B+1}=p^n_{i,j,1}.$$

We know the property $(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\ \boldsymbol{u})=0$ of the Navier-Stokes equations. The corresponding property is kept in the above-mentioned difference scheme and boundary conditions, since

$$\begin{aligned}\sum_{i,j,k}\Big\{&[u^n_{i,j,k}\cdot\delta_r u^n_{i,j,k}+v^n_{i,j,k}\cdot\delta_\theta u^n_{i,j,k}+w^n_{i,j,k}\cdot\delta_\varphi u^n_{i,j,k}]\cdot u^n_{i,j,k}+\frac{1}{r_i}[(v^n_{i,j,k})^2+(w^n_{i,j,k})^2]\cdot u^n_{i,j,k}\\&+[u^n_{i,j,k}\cdot\delta_r v^n_{i,j,k}+v^n_{i,j,k}\cdot\delta_\theta v^n_{i,j,k}+w^n_{i,j,k}\cdot\delta_\varphi v^n_{i,j,k}]\cdot v^n_{i,j,k}\\&-\frac{1}{r_i}\left[u^n_{i,j,k}\cdot v^n_{i,j,k}-(w^n_{i,j,k})^2\frac{\cos\theta_j}{\sin\theta_j}\right]\cdot v^n_{i,j,k}+[u^n_{i,j,k}\cdot\delta_r w^n_{i,j,k}+v^n_{i,j,k}\cdot\delta_\theta w^n_{i,j,k}\\&+w^n_{i,j,k}\cdot\delta_\varphi w^n_{i,j,k}]\cdot w^n_{i,j,k}-\frac{w^n_{i,j,k}}{r_i}\left[u^n_{i,j,k}+v^n_{i,j,k}\frac{\cos\theta_j}{\sin\theta_j}\right]\cdot w^n_{i,j,k}\Big\}\cdot r_i^2\cdot\sin\theta_j\cdot h_1\cdot h_2\cdot h_3=0.\end{aligned}$$

In order to obtain auxiliary velocities $\overline{\boldsymbol{u}}'$ and $\overline{\boldsymbol{u}}''$ from the difference formulas (3.1), (3.2) and the corresponding boundary conditions, we need to solve linear algebraic equations with tridiagonal matrix, and they can be solved quickly using two sweeps. The auxiliary velocity $\overline{\boldsymbol{u}}$ satisfies linear equations with almost tridiagonal matrix, which can be solved easily by the method of triangular decomposition.

§ 4. Application of the Multigrid Method

Equations (3.4) satisfying the pressure are complicated, and are generally solved by the iterative method, though in an expensive way. The multigrid method is efficient for solving equations (3.4) in a finer grid.

In order that this paper may be reasonably self-contained, we consider a brief description of the multigrid method applied to the equation

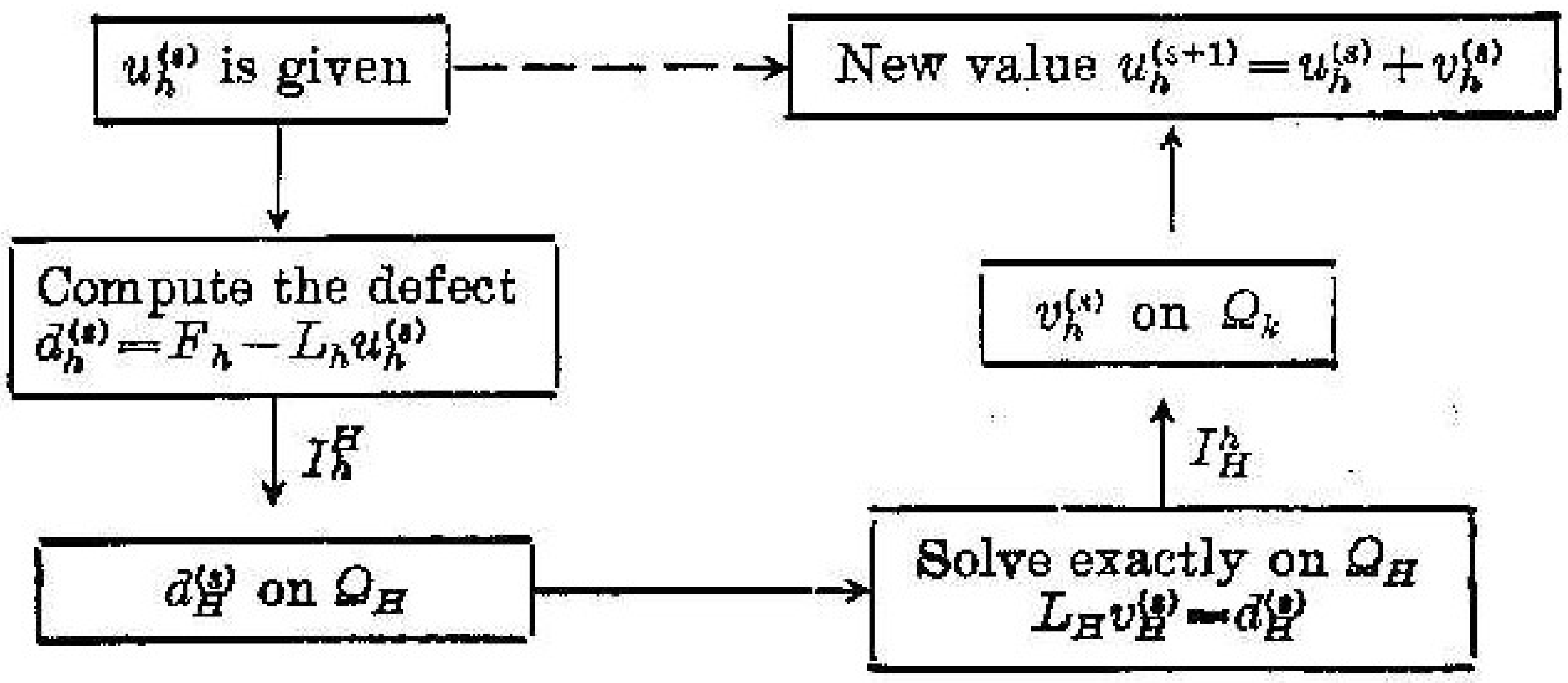
$$Lu=F \text{ in } \Omega.$$

Its approximate solution u_h on the fine grid Ω_h satisfies

$$L_h u_h=F_h \text{ in } \Omega_h.$$

The operator L_H is an appropriate approximation of the operator L_h on a coarse grid Ω_H. The operator I_h^H is a restriction operator from the fine grid Ω_h to the coarse grid Ω_H. The operator I_H^h is an interpolation operator from the coarse grid

Ω_H to the fine grid Ω_h. An approximate value $u_h^{(s)}$ of the equation $L_h u_h = F_h$ is given. In order to find a new approximate value $u_h^{(s+1)}$, the process of the multigrid method should be:



Of course, this basic principle can be used recursively employing coarser and coarser grids.

In our computation, the ratio of the step sizes of the coarse grid Ω_H to the fine grid Ω_h equals 2. The restriction operator is the average of neighboring points with the same weight

$$p^H_{i,j,k} = \frac{1}{8}\left(p^h_{2i-1,2j-1,2k-1} + p^h_{2i-1,2j-1,2k} + p^h_{2i-1,2j,2k-1} + p^h_{2i-1,2j,2k} \right.$$
$$\left. + p^h_{2i,2j-1,2k-1} + p^h_{2i,2j-1,2k} + p^h_{2i,2j,2k-1} + p^h_{2i,2j,2k} \right).$$

Linear interpolation in three dimensions is employed as the interpolation operator

$$p^h_{2i,2j,2k} = \frac{1}{64}\left(27 p^H_{i,j,k} + 9 p^H_{i+1,j,k} + 9 p^H_{i,j+1,k} + 9 p^H_{i,j,k+1} + 3 p^H_{i,j+1,k+1} \right.$$
$$\left. + 3 p^H_{i+1,j,k+1} + 3 p^H_{i+1,j+1,k} + p^H_{i+1,j+1,k+1} \right).$$

§ 5. Numerical Stability

The basic condition of stability is that fluid must not be permitted to flow across more than one computational cell in one time step. This restricts the time increment to

$$\tau < \min_{i,j,k} \left(\frac{h_1}{|u_{i,j,k}|}, \ \frac{r_i \cdot h_2}{|v_{i,j,k}|}, \ \frac{r_i \cdot \sin\theta_j \cdot h_3}{|w_{i,j,k}|} \right). \tag{5.1}$$

This condition is also necessary for accuracy, because the convective flux approximations in the difference scheme (3.1)—(3.7) assume exchanges between adjacent cells only.

The heuristic stability theory proposed by Hirt[8] is important for analysis of stability. Making Taylor's expansion for the difference scheme at a point, the lowest-order terms in the expansion must represent an approximate differential equation. If we ignore the terms with order $O(\tau^2 + h^2)$ only, another differential equation is obtained, which is called the first differential approximation in the Soviet Union. The correctly posed conditions of the first differential approximation are conditions of stability for the difference scheme. This method of determining stability is rigorous for a linear equation with constant coefficient. Thomée in [9] gave a theorem: If there exists a w_p stable difference operator, then the initial value problem of a consistent differential equation is correctly posed in w_p. The difference scheme is consistent not only with the prime differential equation but also with its first differential approximation. Therefore, Thomée's conditions are necessary conditions of stability of the difference scheme. This method is not rigorous for general difference schemes, and is extremely useful and effective for a number of complicated difference schemes.

Now, we use the method to deduce stability conditions. Making Taylor's expansion at the point (i, j, k, n), the following first differential approximation for u is obtained

$$\begin{aligned}
&\frac{\partial u}{\partial t}+\left[u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial \theta}+\frac{w}{r\cdot\sin\theta}\frac{\partial u}{\partial \varphi}-\frac{v^2+w^2}{r}\right]-\nu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)\right.\\
&\quad+\frac{1}{r^2\cdot\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\cdot\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}\\
&\quad\left.-\frac{2}{r^2}\left(u+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cdot v)+\frac{1}{\sin\theta}\frac{\partial w}{\partial\varphi}\right)\right]+\frac{\partial p}{\partial r}\\
&=F_1-\frac{\tau}{2}\frac{\partial^2 u}{\partial t^2}+\frac{\tau\nu}{2}\frac{\partial}{\partial t}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial t}\right)+\frac{1}{r^2\cdot\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)\right.\\
&\quad\left.+\frac{1}{r^2\cdot\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}\right]+O(\tau^2+h^2).
\end{aligned}$$

Differentiating equation (2.1) and eliminating $\frac{\partial^2 u}{\partial t^2}$ from the above formula, we have

$$\begin{aligned}
&\frac{\partial u}{\partial t}+\left[u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial \theta}+\frac{w}{r\cdot\sin\theta}\frac{\partial u}{\partial \varphi}-\frac{v^2+w^2}{r}\right]+\frac{2\nu}{r}\left[u+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cdot v)\right.\\
&\quad\left.+\frac{1}{\sin\theta}\frac{\partial w}{\partial\varphi}\right)\Big]+\frac{\partial p}{\partial r}\\
&=F_1+\nu\left(\frac{2}{r}\frac{\partial u}{\partial r}+\frac{\cos\theta}{r^2\cdot\sin\theta}\frac{\partial u}{\partial\theta}\right)+\frac{\partial^2 u}{\partial r^2}\left[\nu+\frac{\nu\tau}{2}\left(\frac{\partial u}{\partial r}+\frac{2\nu}{r^2}\right)-\frac{\tau}{2}u^2\right]\\
&\quad+\frac{\partial^2 u}{r^2\cdot\partial\theta^2}\left[\nu+\frac{\nu\tau}{2}\left(\frac{\partial u}{\partial r}+\frac{2\nu}{r^2}\right)-\frac{\tau}{2}v^2\right]+\frac{1}{r^2\cdot\sin\theta}\frac{\partial^2 u}{\partial\varphi^2}\\
&\quad\times\left[\nu+\frac{\nu\tau}{2}\left(\frac{\partial u}{\partial r}+\frac{2\nu}{r^2}\right)-\frac{\tau}{2}w^2\right]-2uv\frac{\partial^2 u}{r\,\partial r\,\partial\theta}-2wu\frac{1}{r\cdot\sin\theta}\frac{\partial^2 u}{\partial r\,\partial\varphi}\\
&\quad-2vw\frac{1}{r^2\cdot\sin\theta}\frac{\partial^2 u}{\partial\theta\cdot\partial\varphi}+\tau\cdot D(u,\ v,\ w)+O(\tau^2+h^2),
\end{aligned}$$

where $D(u, v, w)$ denotes several terms excluding second-order partial derivatives of u. The correctly posed necessary conditions of the above equation are the nonnegativeness of the diffusion coefficients i.e.,

$$\nu+\frac{\nu\tau}{2}\left(\frac{\partial u}{\partial r}+\frac{2\nu}{r^2}\right)>\frac{\tau}{2}u^2,\ \nu+\frac{\nu\tau}{2}\left(\frac{\partial u}{\partial r}+\frac{2\nu}{r^2}\right)>\frac{\tau}{2}v^2$$

$$\nu+\frac{\nu\tau}{2}\left(\frac{\partial u}{\partial r}+\frac{2\nu}{r^2}\right)>\frac{\tau}{2}w^2.$$

Since τ is a small quantity, we have

$$\nu>\max_{i,j,k}\left(\frac{\tau}{2}u_{i,j,k}^2,\ \frac{\tau}{2}v_{i,j,k}^2,\ \frac{\tau}{2}w_{i,j,k}^2\right). \tag{5.2}$$

The same stability condition (5.2) is obtained from the first approximation for v and w. It is better and is often employed as a necessary condition. This condition can be explained as avoiding the occurrence of truncation errors that have the form of negative diffusion. Therefore, this is an important condition. Sometimes, the conditions for stability are more stringent than conditions (5.1) and (5.2) in computation, because they are not sufficient for stability.

The truncation errors are unavoidable in finite difference approximations, and they do influence the accuracy of calculaton. In order that the effects of ν are not be obscured by truncation errors, the following condition is necessary: $\mathrm{Re}<4(I_B)^2$, where Re is the flow Reynolds number.

§ 6. Computational Examples

To demonstrate the capacity of the three-dimensional predictor-corrector difference scheme, two examples have been run on Prime-750 Computer and the computational results of this scheme are compared with those of an explicit scheme. In the first example we take

$$F_1^{(1)}=-2H^2r[1-\sin^2\theta\cdot\cos\varphi\cdot\sin\varphi-\sin\theta\cdot\cos\theta\cdot(\sin\varphi+\cos\varphi)],$$

$$F_2^{(1)}=(e^{-r^2}-e^{-1})r(\cos\varphi-\sin\varphi)+H^2r[2\sin\varphi\cdot\cos\varphi\cdot\cos\theta\cdot\sin\theta + (\cos^2\theta-\sin^2\theta)(\sin\varphi+\cos\varphi)]-\nu(\cos\varphi-\sin\varphi)(t+1)(4r^2-10)\cdot r\cdot e^{-r^2},$$

$$F_3^{(1)}=(e^{-r^2}-e^{-1})r[\sin\theta-\cos\theta(\sin\varphi+\cos\varphi)]+H^2r[\sin\theta(\cos^2\varphi-\sin^2\varphi) +\cos\theta(\cos\varphi-\sin\varphi)]-\nu(t+1)(4r^2-10)e^{-r^2}[\sin\theta-\cos\theta(\sin\varphi+\cos\varphi)]\cdot r,$$

where $H=(e^{-r^2}-e^{-1})(t+1)$. The exact solution to the example is

$$\boldsymbol{u}^{(1)}=Hr\begin{pmatrix}0\\ \cos\varphi-\sin\varphi\\ \sin\theta-\cos\theta(\sin\varphi+\cos\varphi)\end{pmatrix},\quad p^{(1)}=0.$$

The right hand side of the Navier-Stokes equation in the second example is given as

$$F_1^{(2)}=F_1^{(1)}+H\cdot\sin\varphi-2r^2e^{-r^2}(t+1)\sin\varphi,$$

$$F_2^{(2)}=F_2^{(1)},\quad F_3^{(2)}=F_3^{(1)}+H\cdot\frac{\cos\varphi}{\sin\theta}.$$

Its exact solution is

$$\boldsymbol{u}^{(2)}=\boldsymbol{u}^{(1)}=Hr\begin{pmatrix}0\\ \cos\varphi-\sin\varphi\\ \sin\theta-\cos\theta(\sin\varphi+\cos\varphi)\end{pmatrix},\quad p^{(2)}=Hr\sin\varphi.$$

The two examples have the same exact solution of velocity $\boldsymbol{u}$, which is written in the Cartesian coordinates (x, y, z) as

$$\boldsymbol{u}_e=\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}=(e^{-(x^2+y^2+z^2)}-e^{-1})\cdot(t+1)\cdot\begin{pmatrix} z-y \\ x-z \\ y-x \end{pmatrix}.$$

For comparison, the explicit difference scheme for the problem (2.1)—(2.6) is

$$\overline{\boldsymbol{u}}_{i,j,k}=\boldsymbol{u}^n_{i,j,k}+\frac{\tau\nu}{2}(\delta^2_r\boldsymbol{u}^n_{i,j,k}+\delta^2_\theta\boldsymbol{u}^n_{i,j,k}+\delta^2_\varphi\boldsymbol{u}^n_{i,j,k})+\tau\overline{\boldsymbol{F}}_{i,j,k}.$$

The quantities p^{n+1}, u^{n+1}, v^{n+1} and w^{n+1} are computed by (3.4), (3.5), (3.6) and (3.7), respectively.

We consider the first example and compare the implicit scheme with the explicit in the coarse grid $I_B=4$, $J_B=8$, $K_B=16$ and the fine grid $I_B=8$, $J_B=16$, $K_B=32$. The computational results are listed in Tables 1 and 2. Since a bigger step size of time is used, the implicit scheme saves much computational time. The results computed by the multigrid algorithm are compared with those of the iterative algorithm in Table 3 for the grid $I_B=8$, $J_B=16$, $K_B=32$ and $\nu=0.01$. In the tables, T denotes that the problem (2.1)—(2.6) is computed from $t=0$ to $t=T$,

$$\mathrm{EEU}\equiv\max_{i,j,k}|u^d_{i,j,k}-u^e_{i,j,k}|,$$

where $u^d_{i,j,k}$ and $u^e_{i,j,k}$ are the difference solution and exact solution of velocity u, respectively. EEV, EEW and EEP are the maximum errors for v, w and p, similar to EEU, and

$$\mathrm{AEV}\equiv\frac{1}{N}\sum_{\substack{i,j,k\\ v^e\neq 0}}\left|\frac{v^d_{i,j,k}-v^e_{i,j,k}}{v^e_{i,j,k}}\right|$$

where N is the total number of the points, in which $v^e_{i,j,k}\neq 0$. The relative average error AEW and AEP are similar to AEV.

In Table 2, the multigrid algorithm has saved only little computational time

Table 1 Results of the first example for $I_B=4$, $J_B=8$, $K_B=16$

ν		0.01		0.1		1.0	
Scheme		Explicit	Implicit	Explicit	Implicit	Explicit	Implicit
T		2	2	1	1	1	1
CPU time (Second)		1968	127	7756	722	90000	7089
Error in $t=T$	EEU	0.0387	0.0291	0.0058	0.0065		0.0013
	EEV	0.0462	0.0423	0.0349	0.0343		0.0342
	EEW	0.0499	0.0444	0.0336	0.0326		0.0377
	EEP	0.0556	0.0345	0.0743	0.0578		0.3288
	AEV	6.40%	5.77%	7.80%	7.79%	8.5%	8.19%
	AEW	5.00%	4.63%	6.23%	6.10%	7.0%	6.93%

because the grid employed in the example is too coarse. If we compute in a finer grid, then the algorithm is much more efficient.

For the second example, we computed only in the case $I_B=4$, $J_B=8$, $K_B=16$. Its results are given in Table 4. The CPU time is deleted because its computation is made on an IBM-Computer. In view of the computational results, we know that the implicit scheme saves much computational time and is efficient when the pressure gradient is not equal to zero.

Table 2 Results of the first example for $I_B=8$, $J_B=16$, $K_B=32$, $\nu=0.01$

Scheme	T	CPU time (Second)	Error in $t=T$					
			EEU	EEV	EEW	EEP	AEV	AEW
Explicit	0.001	2567	0.0004	0.0019	0.0022	0.0091	0.31%	0.46%
Implicit	1	12710	0.0023	0.0073	0.0053	0.0072	1.59%	1.01%

Table 3 Results of the first example for $I_B=8$, $J_B=16$, $K_B=32$, $\nu=0.01$

Method of Solving the pressure	T	CPU time (Second)
Iterative Algorithm	1	12710
Multigrid Algorithm	1	11924

Table 4 Results of the second example for $I_B=4$, $J_B=8$, $K_B=16$

ν	Scheme	T	Error in $t=T$						
			EEU	EEV	EEW	EEP	AEV	AEW	AEP
0.01	Explicit	2	0.0467	0.0497	0.0521	0.0586	6.51%	5.32%	6.73%
	Implicit	2	0.0325	0.0448	0.0475	0.0428	5.93%	4.82%	5.85%
0.1	Explicit	1	0.0082	0.0362	0.0408	0.0767	7.97%	6.84%	8.05%
	Implicit	1	0.0071	0.0357	0.0386	0.0609	7.95%	6.59%	7.16%

From the computational process, we know that the errors near the singular places $r=0$, $\theta=0$ or $\theta=\pi$ are much bigger than inside. Especially, ratio of the errors at $r=\frac{1}{2}h_1$, $\theta=\frac{1}{2}h_2$ $\left(\text{or } r=\frac{1}{2}h_1,\ \theta_B=\left(J_B-\frac{1}{2}\right)h_2\right)$ to inside is about ten. The reason is easy to see in the difference scheme (3.1)—(3.3) and the formulas describing $\overline{F}$, in which there are terms $\frac{1}{r_i^2\cdot\sin^2\theta_j}v_{i,j,k}^n$, $\frac{1}{r_i^2\cdot\sin^2\theta_j}w_{i,j,k}^n$ and so on. Further, we should consider special techniques near $r=0$, $\theta=0$ and $\theta=\pi$.

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References

[1] Harlow, F. H.; Welch, J. E.: Numerical calculation of time-dependent viscous incompressible flow of

fluid with free surface, *Phys. Fluids*, **8** (1965), 2182—2189.

[2] Amsden, A. A.; Harlow, F. H.: The SMAC Method, LA-4370, 1970.

[3] Chorin A. J.: Numerical solution of Navier-Stokes equations, *Math. Comput.*, **22** (1968), 745—762.

[4] Белоцерковский, О. М.; Гущин, В. А.; Щенников, В. В.: Use of the splitting method to solve problems of the dynamics of a viscous incompressible fluid, *Жу. Вычисл. мат. и мат Физ*, **15**(1975), 197—207.

[5] Гущин В. А.: The flow of a viscous fluid past three-dimensional bodies, *Жу. Вычисл. мат. и мат. физ.*, **16** (1976), 529—534.

[6] Hirt C.W.; Cook, J. L.; Calculating three-dimensional flows around structures and over rough terrain, *J. Comput. Phys.*, **10** (1972), 324—340.

[7] Katuhiko Goda: A multistep technique with implicit difference scheme for calculating two-or three-dimensional cavity flows, *J. Comput. Phys.*, **30** (1979), 76—95.

[8] Hirt, C. W.: Heuristic stability theory for finite-difference equations, *J. Comput. Phys.*, **2** (1968), 339—355.

[9] Thomee, V.: Stability theory for partial difference operator, *SIAM Review*, **11** (1969), 152—195.

[10] Brandt, A.: Multigrid Techniques: 1984 Guide with Applications to Fluid Dynamics, GMD-Studien No. 85, 1984.