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THE EXISTENCE OF THE SOLUTION AND THE GLOBALLY CONVERGENT SHOOTING METHOD FOR A CLASS OF TWO-POINT BOUNDARY VALUE PROBLEMS*

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Abstract

A class of two-point boundary value problems are studied. A new existence theorem of solution is constructively proved and a globally convergent shooting method for it is given.

§1. Introduction

In this paper we give a new existence theorem of solution for the two-point boundary value problem

$$u_k'=g_k(t,\ u_1,\ u_2,\ \cdots,\ u_n),\quad k=1,\ 2,\ \cdots,\ n,\ 0\leqslant t\leqslant 1, \tag{1.1}$$

$$u_i(0)=c_i(i=i_1,\ \cdots,\ i_{n-r}),\quad u_j(1)=d_j(j=j_1,\ \cdots,\ j_r), \tag{1.2}$$

$$\{i_1,\ \cdots,\ i_{n-r}\}\cap\{j_1,\ \cdots,\ j_r\}=\emptyset,$$

where $g_k(t,\ u_1,\ \cdots,\ u_n)$ $(k=1,\ \cdots,\ n)$ are Lipschitz continuous on $[0,\ t_0]*R^n$ $(t_0>1)$ and $\emptyset$ is an empty set. We prove that if in a proper arrangement of $g_1,\ \cdots,\ g_n$ and $u_1,\ \cdots,\ u_n,\ g_1,\ \cdots,\ g_n$ satisfy

$$\varlimsup_{\substack{\|U\|\to\infty\\ |u_j|<q}}|g_i(t,\ u_1,\ \cdots,\ u_n)|/|u_i|<+\infty \text{ for any } g>0,\quad i=1,\ 2,\ \cdots,\ n$$

then (1.1)—(1.2) has at least one solution.

Our result is proved by using a shooting function and the generalized Newton homotopy.

Consider the related initial value problem

$$u_k'=\lambda g_k(t,\ u_1,\ \cdots,\ u_n),\quad k=1,\ \cdots,\ n, \tag{1.3}$$

$$u_i(0)=c_i(i=i_1,\ \cdots,\ i_{n-r}),\quad u_{j_p}(0)=x_p, \tag{1.4}$$

$$p=1,\ \cdots,\ r,\ j_p\in\{1,\ \cdots,\ n\}\backslash\{i_1,\ \cdots,\ i_{n-r}\}$$

and denote the solution of (1.3), (1.4) by $u_k=u_k(t,\ x,\ \lambda)(k=1,\ \cdots,\ n)$. Then we seek x such that

$$\begin{aligned}F(x,\ 1)&=(f_1(x,\ 1),\ \cdots,\ f_r(x,\ 1))^T\\&=(u_{j_1}(1,\ x,\ 1),\ \cdots,\ u_{j_r}(1,\ x,\ 1))^T-(d_{j_1},\ \cdots,\ d_{j_r})^T=0.\end{aligned} \tag{1.5}$$

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If $x=x^*$ is a root of (1.5), then $u_s(t)=u_s(t, x, 1)$ is a solution of (1.1)—(1.2). Obviously for any solution $u_s(t, x)$ of (1.1)—(1.2) the value $x_p=u_{j_p}(0)$, $j_p\in\{j_1, \cdots, j_r\}$, is a root of (1.5). Thus solving (1.1)—(1.2) is equivalent to solving equations $F(x, 1)=0$. It is clear that the latter is much easier than the former. The shooting methods are just to solve $F(x, 1)=0$ by some iteration. But most of the shooting methods have only local convergence, i.e. only when the initial shooting point x is chosen in the neighborhood of the real shooting point x^*, can the methods be successful (see [1], [2]). Therefore an open problem, which is also difficult in shooting methods, is to enlarge the domain in which initial points can be chosen. Any globally convergent shooting method is thus very significant. In this paper we prove constructively the existence of the solution of $F(x, 1)=0$ and thus give a globally convergent shooting method for solving numerically (1.1)—(1.2).

For more detailed discussion on globally convergent shooting methods, we refer to Watson[3] and Zhang[4].

§ 2. Existence Theorem and Its Proof

Consider the boundary value problem (1.1)—(1.2). Set $E_1=(e_{i_1}^T, \cdots, e_{i_{n-r}}^T)^T$, $\widetilde{E}_1=(e_{k_1}^T, \cdots, e_{k_r}^T)^T$, $\{i_1, \cdots, i_{n-r}\}\cap\{k_1, \cdots, k_r\}=\emptyset$ and $E_2=(e_{j_1}^T, \cdots, e_{j_r}^T)^T$, $\widetilde{E}_2=(e_{l_1}^T, \cdots, e_{l_{n-r}}^T)^T$, $\{j_1, \cdots, j_r\}\cap\{l_1, \cdots, l_{n-r}\}=\emptyset$. Then (1.1)—(1.2) and (1.3), (1.4) can be written as

$$U'(t)=G(t, U(t)),\ G=(g_1, \cdots, g_n)^T,\ U=(u_1, \cdots, u_n)^T, \tag{2.1}$$

$$E_1U(0)=C,\ E_2U(1)=D,\ C=(c_{i_1}, \cdots, c_{i_{n-r}})^T,\ D=(d_{j_1}, \cdots, d_{j_r})^T, \tag{2.2}$$

and

$$U'(t)=\lambda G(t, U(t)),\quad 0\leqslant\lambda\leqslant1, \tag{2.3}$$

$$E_1U(0)=C,\quad E_1U(0)=x. \tag{2.4}$$

By the definition of E_1, $\widetilde{E}_1$, E_2, $\widetilde{E}_2$,

$$E_1^TE_1+\widetilde{E}_1^T\widetilde{E}_1=E_2^TE_2+\widetilde{E}_2^T\widetilde{E}_2=I.$$

Throughout we will assume that $G(t, U(t))$ is a Lipschitz continuous function on $[0, t_0]*R^n(t_0>1)$, $D=0$ and $\|x\|=(x_1^2+, \cdots, +x_n^2)^{\frac{1}{2}}$.

Definition 1. *Matrix $B=(b_{ij})$ is called an Ind (Indication) matrix of G if B is defined as follows:*

Given g, if indexes $j_1, \cdots, j_p\in\{1, \cdots, n\}$ satisfy that for any $g>0$ there exists $\alpha(q)$ such that

$$\left.\begin{array}{c}\overline{\lim\limits_{\|U\|\to\infty}}|g_i|/|u_i|=\alpha(q)<+\infty,\\ \|u_j\|\leqslant q,\\ j=j_1, \cdots, j_p,\end{array}\right. \tag{2.5}$$

then let

$$b_{ij}=\begin{cases}1, & j=j_1, \cdots, j_p,\\ 0, & j\neq j_1, \cdots, j_p.\end{cases}$$

By (2.5), $b_{ij}=0$ $(i\neq j)$ implies that g_i is bounded with repect to variable u_j and $b_{ii}=0$ implies that

$$\overline{\lim_{|u_i|\to\infty}}\ |g_i|/|u_i|<+\infty.$$

Remark 1. Since G is continuous on $[0,\ 1]*R^n$, indexes $j_1,\ \cdots,\ j_p$ in Definition 1 exist. Thus the Ind matrices of G are well defined but may not be unique, e.g. function

$$G=\begin{pmatrix} e^{u_1}u_2 \\ u_2\sin u_1 \end{pmatrix}$$

has the following two Ind matrices

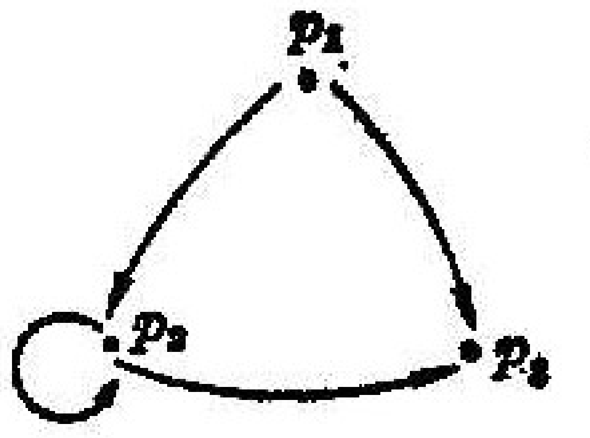
$$B_1=\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},\quad B_2=\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Definition 2. *The directed graph Γ_0 of matrix B consists of n points $p_1,\ p_2,\ \cdots,\ p_n$ and some paths. The paths are defined as follows:*

If $b_{ij}\neq 0$, then $\overrightarrow{p_ip_j}$ is a path directed from p_i to p_j·p_i is named as the starting point and p_j the end point. e.g.

$$B=\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

Γ_B:



Here p_1 is a starting point, p_3 is an end point and p_2 is a starting point as well as an end point.

Definition 3. *G is called a B-T (Boundedness-Transmission) function if there is an Ind matrix B of G such that its directed graph does not include any closed chain, that is, there are no such indexes $i_1,\ \cdots,\ i_p$ that $b_{i_1i_2}=b_{i_2i_3}=\cdots=b_{i_pi_1}=1$.*

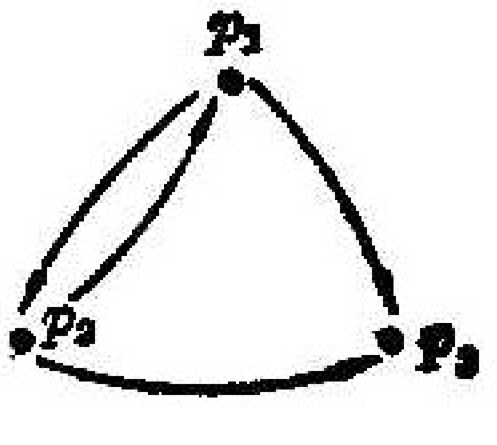
For instance, function

$$G=\begin{pmatrix} g_1(t,\ u_1,\ u_2,\ u_3) \\ g_2(t,\ u_1,\ u_2,\ u_3) \\ g_3(t,\ u_1,\ u_2,\ u_3) \end{pmatrix}=\begin{pmatrix} u_3\sin u_1+u_2+t \\ u_1+u_3^2+5t-\sin 2t \\ \sin t+\cos u_2 \end{pmatrix}$$

has the following Ind matrix B and the directed graph

$$B=\begin{array}{c} \\ \\ \\ \end{array}\begin{array}{ccc|c} u_1 & u_2 & u_3 & \\ 0 & 1 & 1 & g_1 \\ 1 & 0 & 1 & g_1 \\ 0 & 0 & 0 & g_2 \end{array},$$

Γ_B:

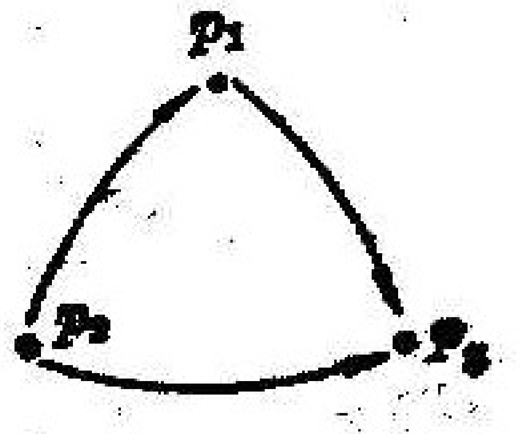


In Γ_B there exists a closed chain $\overrightarrow{p_1p_2}\ \overrightarrow{p_2p_1}$. It is easy to prove that every Ind matrix of G has at least a closed chain; thus G is not a B-T function. But function

$$G=\begin{pmatrix} g_1(t,\ u_1,\ u_2,\ u_3) \\ g_2(t,\ u_1,\ u_2,\ u_3) \\ g_3(t,\ u_1,\ u_2,\ u_3) \end{pmatrix}=\begin{pmatrix} \sin u_1+u_3+t \\ u_1+u_3^2+5t-\sin 2t \\ \sin t+\cos u_2 \end{pmatrix}$$

is a B-T function. One Ind matrices of G and its directed graph are

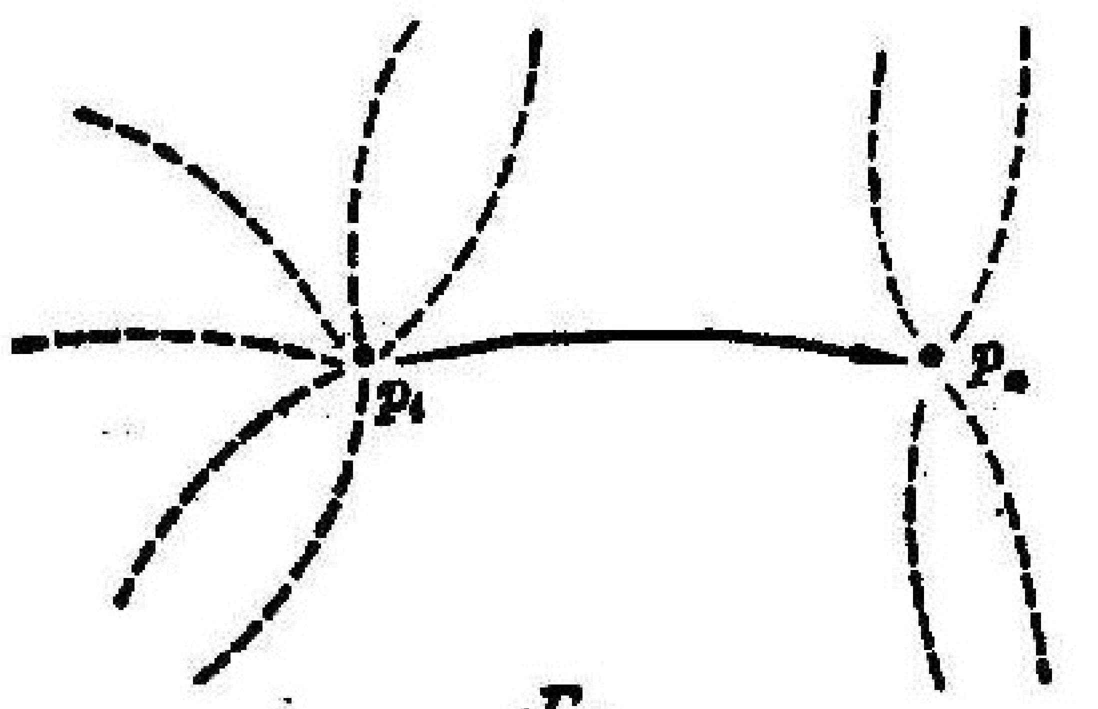
$$B=\begin{array}{c c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} & \\ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & \begin{array}{c} g_1 \\ g_2 \\ g_3 \end{array} \end{array}$$



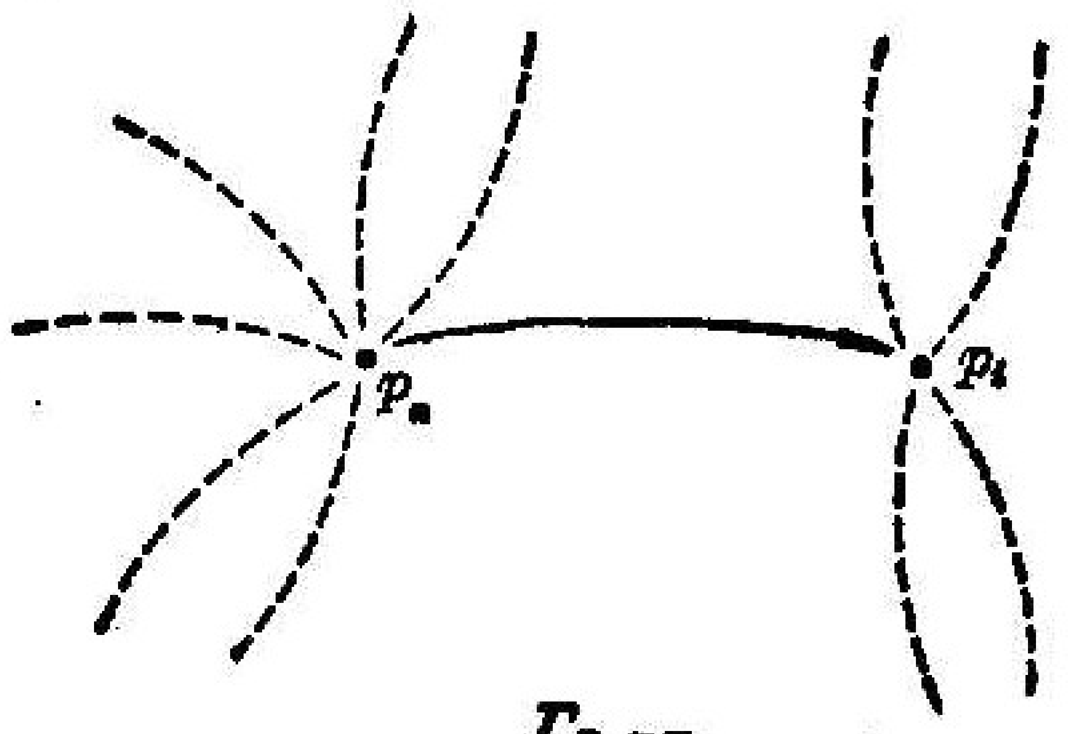
About the Ind matrix we have

Lemma 1. *G is a B-T function if and only if there are an Ind matrix B of G and a product P of some permutation matrices such that PBP^T is a strictly upper matrix.*

Proof. Let G be a B–T function and B be its Ind matrix whose directed graph Γ_B has no closed chain. Then in Γ_B there is at least one point which is only an end point (if not, since the number of points in Γ_B is finite, there must exist a closed chain). Thus in B there exists at least one 0 row, say, the i–th row. Exchanging this row with the last row and the i–th column with the last column, we obtain P_0BP_0 (P_0 is the corresponding permutation matrix). Since the difference between Γ_B and $\Gamma_{P_0BP_0}$ is only in the position of two points P_i and P_n, see the figure, the directed graph of P_0BP_0 has no closed chain.



Γ_B



$\Gamma_{P_0BP_0}$

Let B_1 be the first $(n-1)*(n-1)$ main submatrix of P_0BP_0. Then Γ_{B_1} has no closed chain; thus B_1 has at least one 0 row. Exchanging this row with the last row and the same column with the last column, we get $P_1B_1P_1$ (P_1 is the corresponding permutation matrix). Let B_2 be the first $(n-2)*(n-2)$ main submatrix of $P_1B_1P_1$. In recurrence, we have permutation matrices P_0, P_1, $\cdots$, P_{n-2} such that PBP^T is a strictly upper triangular matrix, where

$$P=\begin{pmatrix} P_{n-2} & 0 \\ 0 & I_{n-2} \end{pmatrix}\cdot\begin{pmatrix} P_{n-3} & 0 \\ 0 & I_{n-3} \end{pmatrix}\cdots\begin{pmatrix} P_1 & 0 \\ 0 & 1 \end{pmatrix}P_0$$

The sufficiency is obvious.

Lemma 1 shows: If G is a B–T function, by exchanging the position of g_i and u_i we can get a new function $\bar{G}=(\bar{g}_1, \cdots, \bar{g}_n)^T$ whose one Ind matrix is a strictly upper triangular matrix. Thus for any q there exist $\alpha_i(q)$ and P such that

$$|\bar{g}_i|<\alpha_i|u_i| \text{ when } \|U\|>p,\ |u_j|\leqslant q,\quad j=i+1,\ \cdots,\ n,\ 0\leqslant t\leqslant t_0.$$

Let $M_i(q)=\max\limits_{|u|<p}|\bar{g}_i(t,\ u_1,\ \cdots,\ u_n)|$. Then

$$|\bar{g}_i|<M_i(q)+\alpha_i(q)\,|u_i|,\ \text{for } |u_j|<q,\ j=i+1,\ \cdots,\ n,\ 0\leqslant t\leqslant t_0,\ i=1,\cdots,\ n. \quad (2.6)$$

If G is a C^1 function on $[0,\ t_0]*R^n$ and its derivative satisfies

$$\overline{\lim_{|u_j|\to\infty}}|g_i'|/|u_j|=0,\quad j<i,$$

$$\overline{\lim_{|u_i|\to\infty}}|g_i'|/|u_i|=+\infty,$$

then G is a B-T function.

Lemma 2. *If G is a B–T function, then for any $\lambda\in[0,\ 1]$ and any initial value $C=(c_1,\ \cdots,\ c_n)$, the initial value problem*

$$U'(t)=\lambda G(t,\ U(t)), \tag{2.7}$$

$$U(0)=C \tag{2.8}$$

has a unique solution on $[0,\ 1]$.

Proof. Since G is Lipschitz continuous, there is $b>0$ such that (2.7)—(2.8) has a unique solution on $[0,\ b)$. To prove that $[0,\ b)$ is not the maximum interval of solution for any $b<t_0$, we need only to prove that $U(t)$ is bounded on $[0,\ b)$. By the assumption that F is a B-T function, we may assume by Lemma 1 that there exists an Ind matrix of G which is a strictly upper triangular matrix. Thus there exist from (2.6) $M_n>0$ and $\alpha_n>0$ such that $|g_n|<M_n+\alpha_n|u_n|$. From

$$|u_n(\lambda,\ t)|<|u_n(\lambda,\ 0)|+\lambda\int_0^t|g_n|ds<\|C\|+M_n+\alpha_n\int_0^t|u_n|ds$$

we have by Bellman's inequality

$$|u_n|<(\|C\|+M_n)e^{\alpha_n b}\triangleq q_n,\quad 0\leqslant\lambda\leqslant1,\ t<b.$$

Since G is a B-T function, from (2.6) there are $M_{n-1}(q_n)>0$ and $\alpha_{n-1}(q_n)>0$ such that $|g_{n-1}|<M_{n-1}(q_n)+\alpha_{n-1}(q_n)|u_{n-1}|$. Using Bellman's inequality once more, we have

$$|u_{n-1}|<(\|C\|+M_{n-1})e^{\alpha_{n-1}b}\triangleq q_n',\quad 0\leqslant\lambda\leqslant1,\ t<b. \tag{2.9}$$

Let $q_{n-1}=\max\{q_n,\ q_n'\}$. In recurrence, we obtain the boundedness of U on $[0,\ b)$. By the extension theorem of solution the maximum interval of the solution of (2.7)—(2.8) is not $[0,\ b)$ for any $0\leqslant b\leqslant t_0$. Thus by $t_0>1$ the conclusion of Lemma 2 is true.

Let $x=(x_1,\ x_2,\ \cdots,\ x_n)^T$ and $U(t,\ \lambda,\ x)$ be the solution of the initial value problem

$$U'(t)=\lambda G(t,\ U(t)),$$

$$E_1U(0)=C,\quad \widetilde{E}_1U(0)=x.$$

Define the generalized shooting function F: $R^r*[0,\ 1]\to R^r$ by

$$F(x,\ \lambda)=EU(1,\ \lambda,\ x). \tag{2.10}$$

Due to Lemma 2, $F(x,\ \lambda)$ is well defined on $R^r*[0,\ 1]$.

Lemma 3. *If G is a B–T function and $E_1=\widetilde{E}_2$, then the shooting function $F(x,\ \lambda)$ has the following property:*

If MR^r is bounded, then

$$M^{-1}=\{x;\ F(x,\ \lambda)\in M,\ \lambda\in[0,\ 1]\}$$

is also bounded.

Proof. Let the set M be bounded and $N>0$ such that $\|F(x,\ \lambda)\|<N$ when

$F(x, \lambda) \in M$, and $\|C\| < N$. We may assume by Lemma 1 that B being one Ind matrix of G is a strictly triangular matrix. Let $M_n > 0$ and $\alpha_n > 0$ such that $|g_n| < \alpha_n |u_n| + M_n$. Then from

$$u_n(t, \lambda, x) = u_n(0, \lambda, x) + \lambda \int_0^t g_n \, ds,$$

$$u_n(t, \lambda, x) = u_n(1, \lambda, x) + \lambda \int_1^t g_n \, ds$$

and $E_1 = \tilde{E}_2$ which implies either $u_n(0, \lambda, x) = (e_{i_{n-r}}, U(0)) = c_{i_{n-r}}$ or $u_n(1, \lambda, x) = (e_r, F(x, \lambda))$, we have

$$u_n(t, \lambda, x) < N + \alpha_n \int_0^t |u_n| \, ds + M_n$$

or

$$u_n(t, \lambda, x) < N + \alpha_n \int_1^t |u_n| \, ds + M_n.$$

By Bellman's inequality,

$$u_n(t, \lambda, x) < (N + M_n) e^{\alpha_n} \triangleq q_n, \quad 0 \leqslant t \leqslant 1. \tag{2.11}$$

Since B is a strictly triangular matrix, there exist $M_{n-1}(q_n)$ and $\alpha_{n-1}(q_n)$ such that $|g_{n-1}| < M_{n-1}(q_n) + \alpha_{n-1}(q_n) |u_{n-1}|$. Thus

$$|u_{n-1}(t, \lambda, x)| < (N + M_{n-1}) e^{\alpha_{n-1}} \triangleq q_n'.$$

Set $q_{n-1} = \max\{q_n, q_n'\}$. In recurrence, we have constants $M_i(q_{i+1})$ and $\alpha_i(q_{i+1})$ such that

$$|u_i(t, \lambda, x)| < (N + M_i(q_{i+1}) e^{\alpha_i(q_{i+1})}, \quad i = 1, \cdots, n-1. \tag{2.12}$$

Thus, combining (2.11) and (2.12) we have

$$\|U(t, \lambda, x)\| < \sqrt{n} (N + K) e^{a}, \; K = \max_{1 \leqslant i \leqslant n-1} \{M_i(q_{i+1}), M_n\},$$

$$d = \max_{1 \leqslant i \leqslant n-1} \{\alpha_i(q_{i+1}), \alpha_n\}.$$

Particularly,

$$\|x\| < \|U(0, \lambda, x)\| < \sqrt{n} (N + K) e^{a}.$$

Therefore, M^{-1} is bounded.

Theorem 1. *Assume G is a B-T function and $E_1 = \tilde{E}_2$. Then the boundary value problem* (1.1)—(1.2) *has at least one solution on* $[0, 1]$. *Moreover, if G is a C^2 function, then for almost all $x^0 \in R^n$, there is a C^1 one-dimensional manifold $(x(s), \lambda(s))$ such that*

(i) $H(x(s), x^0, \lambda(s) = F(x(s), \lambda(s)) - (1 - \lambda(s)) F(x^0, 0) = 0$,
Rank $H'(x(s), x^0, \lambda(s)) = r$, $H' = (H_x', H_\lambda')$;

(ii) $(x(0), \lambda(0)) = (x^0, 0)$ *and there is* $(x^*, 1) \in \overline{\{(x(s), \lambda(s));\ 0 \leqslant (s) < 1\}}$ *such that* $F(x^*, 1) = 0$;

(iii) *if* $\det F_x'(x^*, 1) = 0$, *then* $(x(s), \lambda(s))$ *is finitely long.*

Proof. We give the proof for $G \in C^2([0, t_0] * R^n, R^n)$. Consider homotopy

$$H(x, x^0, \lambda) = F(x, \lambda) - (1 - \lambda) F(x^0, 0). \tag{2.13}$$

Differentiating (2.13) with respect to x^0, we obtain

$$H'_{x^0}(x, x^0, \lambda) = -(1-\lambda)F'_{x^0}(x^0, 0) = -(1-\lambda)[E_2U(1, 0, x^0)]'_{x^0} = -(1-\lambda)I. \quad (2.14)$$

By the generalized Sard's Theorem[5] for almost all $x^0 \leftarrow R$ there is a C' one-dimensional manifold $(x(s), \lambda(s))$ with $(x(0), \lambda(0)) = (x^0, 0)$ which satisfies (i). Now, we turn to (ii) and (iii).

Since $(x(s), \lambda(s))$ satisfies $F(x(s), \lambda(s)) = (1-\lambda)F(x^0, 0)$, by Lemma 3 there is $R>0$ such that $x(s) \in B(R/2) = \{x;\ \|x\| < R/2\}$ as $\lambda(s) \in [0, 1)$. Consider the limit points of $(x(s), \lambda(s))$ in the closed set $\overline{B(R)*[0, 1]}$. Since $F(x, \lambda) = E_2U(1, \lambda, x) = E_2U(0, \lambda, x) + E_2\lambda\int_0^t G\, ds = x + \lambda E_2\int_0^t G\, ds$, $(x(s), \lambda(s))$ intersects $R^r*\{0\}$ at only one point by $F(x, 0) = x$ and $(x(s), \lambda(s))$ is not a homeomorphism of a circle by $F'_x(x, 0) = I$. Thus there exist other limit points of $(x(s), \lambda(s))$ in $B(R)*(0, 1]$. By $x(s) \in B(R/2)$ the points do not lie on $\partial B(R)*[0, 1]$. Since Rank $H'_x(x(s), \lambda(s)) = r$ and by the implicit function theorem the limit points also do not lie on $B(R)*(0, 1)$, they are on $B(R)*\{1\}$. Denoting them by $(x^*, 1)$ we have from (2.13) $F(x^*, 1) = H(x^*, 1) = 0$.

To prove (iii) we take the arc length as parameter s. If some connected component of $(x(s), \lambda(s))$ with $(x(0), \lambda(0)) = (x^0, 0)$ is not finitely long on $R^r*[0, 1)$, then $\lim\limits_{s\to\infty}|\dot\lambda(s)| = 0$. If not, there are $\alpha>0$ and $s_0>0$ such that $|\dot\lambda(s)| > \alpha$ as $s>s_0$; thus $|\lambda(s+3/\alpha)| \geqslant \alpha*3/\alpha - \lambda(s) \geqslant 3-1=2$. It contradicts $|\lambda(s)| \leqslant 1$. But from [6] the nonsingularity of $F'_x(x^*, 1)$ implies $\lim\limits_{s\to\infty}|\dot\lambda(s)| \neq 0$. A contradiction. Therefore (iii) follows.

For the multi-point boundary value problem

$$U'(t) = G(t, U(t)), \quad (2.15)$$

$$E_1U(a_1) = C,\ E_2U(a_2) = C_2,\ \cdots,\ E_mU(a_m) = C_m, \quad (2.16)$$

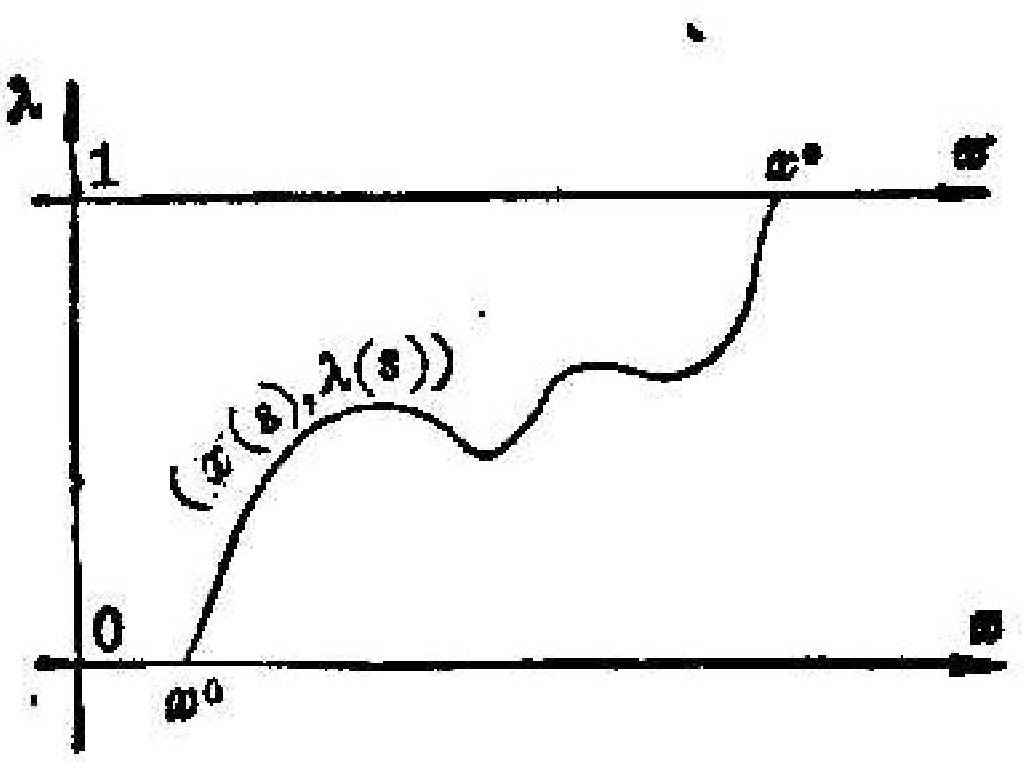
let $F(x, \lambda) = (E_2U(a_2, x, \lambda), \cdots, E_mU(a_m, x, \lambda))^T - (C_2, \cdots, C_m)^T$, where $U(t, x, \lambda)$ is the solution of (2.3)—(2.4). Then we have

Theorem 2. *If G is a B-T function and* Rank$(E_1^T, \cdots, E_m^T) = n$, *then for* (2.15)—(2.16) *all conclusions in Theorem* 1 *are valid.*

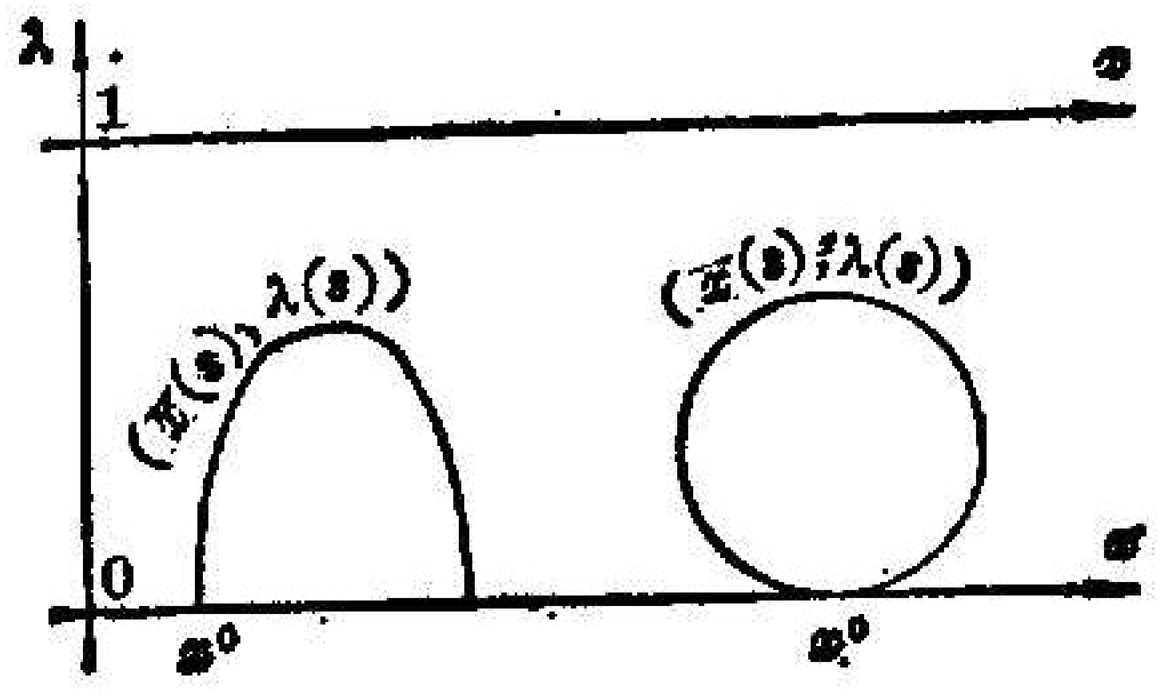
The proof of Theorem 2 is similar to that of Theorem 1.

The reason we introduce parameter λ in (2.10) is to guarantee that $(x(s), \lambda(s))$ does not go to the initial level $R^r*\{0\}$ and does not form a closed path. See the figures.

With parameter



Without parameter



§ 3. The Numerical Algorithm for Following Curve $(x(s), \lambda(s))$ and Examples

Let $(x(s), \lambda(s))$ satisfy (i), (ii) of Theorem 1. Differentiating $H(x(s), x^0, \lambda(s))=0$ with respect to s, we have

$$H'_x(x(s), x^0, \lambda(s))\dot{x}(s)+H'_\lambda(x(s), x^0, \lambda(s))\dot{\lambda}(s)=0, \tag{3.1}$$

$$(x(0), \lambda(0))=(x^0, 0). \tag{3.2}$$

Assume s is the arc length of $(x(s), \lambda(s))$ starting from $(x^0, 0)$. Then (3.1)—(3.2) can be written as

$$(x(s), \lambda(s))=T(x(s), \lambda(s)), \tag{3.3}$$

$$(x(0), \lambda(0))=(x^0, 0), \tag{3.4}$$

where $T(x(s), \lambda(s))$ is obtained as follows:

$$\begin{gathered}(H'_x(x, \lambda), H'(x, \lambda))T=0, \ \|T\|=1, \\ \mathrm{Sign}\left(\det\begin{pmatrix} H'_x(x, \lambda)H'_\lambda(x, \lambda) \\ T(x, \lambda)^T \end{pmatrix}\right)=\mathrm{Sign}\left(\det\begin{pmatrix} H'_x(x^0, 0)H'_\lambda(x^0, 0) \\ T(x^0, 0)^T \end{pmatrix}\right) \\ e^T_{n+1}T(x^0, 0)>0. \end{gathered} \tag{3.5}$$

There is a neighborhood of $(x(s), \lambda(s))$ on which $T(x, \lambda)$ is Lipschitz continuous. See [7].

Now, we can obtain $T(x, \lambda)$ by solving (3.5). While H'_x and H'_λ are obtained as follows:

$F(x, \lambda)=E_2U(1, x, \lambda)$, where $U(t, x, \lambda)$ solves (2.3)—(2.4),

$F'_x(x, \lambda)=E_2Y(1, x, \lambda)$, where $Y(t, x, \lambda)$ solves the initial problem,

$$Y'(t)=\lambda G'\mathrm{U}(t, U(t, x, \lambda))Y(t), \tag{3.6}$$

$$E_1Y(0)=0, \ \bar{E}_1Y(0)=I, \tag{3.7}$$

$F'_\lambda(x, \lambda)=E_2y(1, x, \lambda)$, where $y(t, x, \lambda)$ solves the initial problem,

$$y'(t)=G(t, U(t, x, \lambda))+\lambda G'\mathrm{U}(t, U(t, x, \lambda))y(t), \tag{3.8}$$

$$y(0)=0, \tag{3.9}$$

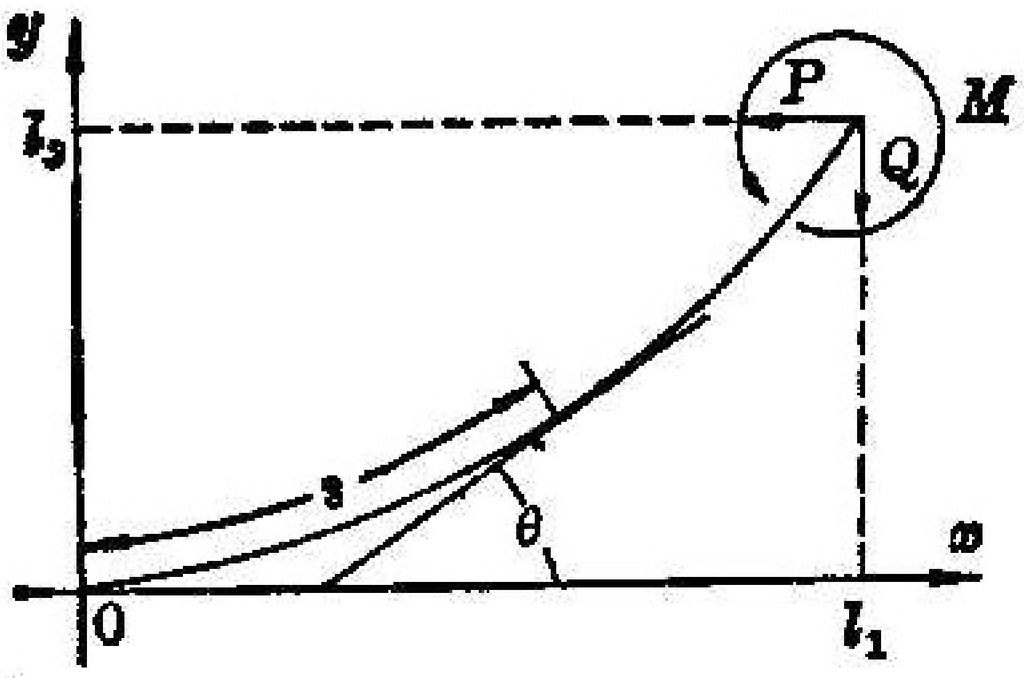
where $U(t, x, \lambda)$ in (3.6)—(3.9) is the solution of (2.3)—(2.4). By the definition of H, we have

$$H'_x(x, x^0, \lambda)=F'_x(x, \lambda), \ H'_\lambda(x, x^0, \lambda)=F'_\lambda(x, \lambda)+F(x^0, 0). \tag{3.10}$$

The discussion on the computation of T and the numerical algorithm for following curve $(x(s), \lambda(s))$ can be found in [3], [7].

Example 1. Consider a thin incompressible elastic rod clamped at the origin and acted on by forces Q, P and a torque M on (l_1, l_2). See the following figure. The governing nonlinear equations are

$$Z'(s)=G(s, Z)=\begin{pmatrix} g_1(s, Z) \\ g_2(s, Z) \\ g_3(s, Z) \end{pmatrix}=\begin{pmatrix} (Q(1-x)-p(1-y)+M)/EI \\ \cos\theta \\ \sin\theta \end{pmatrix}, \tag{3.11}$$



$$x(0)=y(0)=0,\ \theta(1)=\alpha(Q,\ P,\ M). \tag{3.12}$$

Here $\alpha(P,\ Q,\ M)$ is a prescribed function, $Z(s)=(\theta(s),\ x(s),\ y(s))^T$, EI is the flexural rigidity. In this problem the Ind matrix of G

$$B=\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is a strictly upper triangular matrix. Thus G is a B-T function.

Let

$$E_1=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ E_2=(1\quad 0\quad 0).$$

Then

$$E_1Z(0)=E_1\begin{pmatrix} \theta(0) \\ x(0) \\ y(0) \end{pmatrix}=\begin{pmatrix} x(0) \\ y(0) \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$E_2Z(1)=E_2\begin{pmatrix} \theta(1) \\ x(1) \\ y(1) \end{pmatrix}=\theta(1)=\alpha(Q,\ P,\ M).$$

Thus, we know by $\dot{E}_1=\widehat{E}_2$ that the boundary value problem (3.11)—(3.12) satisfies the condition of Theorem 1.

Example 2. Consider a high order equation

$$u^{(n)}=f(t,\ u,\ u',\ \cdots,\ u^{(n-1)}), \tag{3.13}$$

$$u^{(i)}(0)=c_i,\ u^{(j)}(1)=d_j,\ i=i_1,\ \cdots,\ i_{n-r},\ j=j_1,\ \cdots,\ j_r. \tag{3.14}$$

Let $u_{i+1}=u^{(i)}$, $i=0,\ 1,\ \cdots,\ n-1$. Then (3.13) can be written as

$$U'(t)=G(t,\ U)=\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}U(t)+\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ f \end{pmatrix}.$$

If f is a bounded function, then $G(t,\ U)$ is a B-T function. More discussion on (3.13)—(3.14) will be given in another paper.

Example 3. In mechanics the following boundary value problem often occurs

$$u''(x)=f'_x(u(x))=f'_u(u(x))u'(x),$$

$$u^{(i)}(0)=c,\ u^{(j)}(1)=d,\quad 0\leqslant i,\ j\leqslant 1.$$

Let $u'=v$, then $G=(v,\ f'_x)^T$ is a B-T function when f'_u is bounded.

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