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A FOURTH ORDER FINITE DIFFERENCE APPROXIMATION TO THE EIGENVALUES OF A CLAMPED PLATE*

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Abstract

In a 21-point finite difference scheme, assign suitable interpolation values to the fictitious node points. The numerical eigenvalues are then of $O(h^2)$ precision. But the corrected value $\hat{\lambda}_h=\lambda_h+\frac{h^2}{6}\lambda_h^{3/2}$ and extrapolation $\hat{\lambda}_h=\frac{4}{3}\lambda_{\frac{h}{2}}-\frac{1}{3}\lambda_h$ can be proved to have $O(h^4)$ precision.

§ 1. Introduction

Consider the following eigenvalue problem of a clamped plate

$$\begin{cases} \Delta^2 u-\lambda u=0, & (x, y)\in\Omega, \\ u=\dfrac{\partial u}{\partial n}=0, & (x, y)\in\partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open area in the X–Y plane, $\partial\Omega$ is the boundary of Ω, and $\frac{\partial}{\partial n}$ denotes the outward normal derivatives.

Let

$$S_h=\{(mh, nh)\,|\,m, n \text{ integer}\},$$
$$\Omega_h=\Omega\cap S_h,\quad \partial\Omega_h=\partial\Omega\cap S_h.$$

Let Δ_h and $\Delta_h^\times$ be the well-known 5-point and skewed 5-point difference operators respectively.

In dealing with (1.1) by numerical methods, usually Δ^2 will be approximated by Δ_h^2, the so called 13-point scheme. Thomée[1] proved λ_h is of $O(h^{1/2})$ precision, where λ_h satisfies:

$$\begin{cases} \Delta_h^2 u_h-\lambda_h u_h=0, & (x, y)\in\Omega_h, \\ u_h=0, & (x, y)\in S_h\backslash\Omega_h. \end{cases} \tag{1.2}$$

Using the operator Δ_h^2 to approximate Δ^2 in irregular interior points[2], Kuttler[3] obtained $O(h^2)$ and $O(h^2|\ln h|^{1/2})$ approximations to the eigenvalues and eigenvectors of (1.1) respectively.

In this paper, the biharmonic operator is approximated using the 21-point stencil

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$$M_h u=\frac{1}{3h^4}\begin{bmatrix} & 1 & 1 & 1 & \\ 1 & -2 & -10 & -2 & 1 \\ 1 & -10 & 36 & -10 & 1 \\ 1 & -2 & -10 & -2 & 1 \\ & 1 & 1 & 1 & \end{bmatrix} u.$$

It is easy to see that

$$M_h=\frac{1}{3}\Delta_h^2+\frac{2}{3}\Delta_h\Delta_h^\times.$$

If $u\in C^6(R^2)$, by direct evaluation,

$$(M_h-\Delta^2)u=\frac{h^2}{6}\Delta^3u+O(h^4). \tag{1.3}$$

Lu et al.[4] applied the 21-point scheme to the biharmonic boundary value problem and showed that the error is $O(h^4)$. Here, we generalize his result to the eigenvalue problem. First, we should point out that a biharmonic operator satisfying (1.1) with $u\in C^4$ on $\partial\Omega$ is positive definite, and hence its square operator $\sqrt{\Delta^2}=-\Delta$ exists and is also positive definite. If $u\in D(\Delta^3)$ is an eigenvector of (1.1), then

$$\Delta^3u=\lambda\Delta u=-\lambda^{3/2}u$$

and (1.3) becomes

$$(M_h-\Delta^2)u=-\frac{h^2}{6}\lambda^{3/2}u+O(h^4). \tag{1.4}$$

Applications of the correction method to the eigenproblems were first introduced by Kuttler[5], who corrected the 9-point scheme of a Laplace operator. In 1984, one of the authors made some extension of the method[6].

§ 2. Correction Method of the 21-point Scheme

Let

$$\Omega_h'=\{P\in\Omega_h \mid |Q-P|\leqslant\sqrt{5}\,h \text{ implies } Q\in\Omega_h\},$$
$$\Omega_h^*=\Omega_h/\Omega_h'.$$

Ω_h' is the set of regular points and Ω_h^* is the set of irregular points.

Suppose $P\in\Omega_h^*$. In order to evaluate $M_hu_h(P)$, values of u_h on some points outside Ω have to be defined. This can be done by interpolation. The simplest way is to interpolate along the grid lines. For example, if $P_1\in\Omega_h^*$ is an irregular point, $P_{-1}\overline{\in}\Omega_h$, $P_0\in\partial\Omega_h$, $P_i\in\Omega_h$ $(i=2, 3, 4)$, $P_i=P_0+ihe_h$, where e_h is the unit vector of the inward normal, then

$$I_hu_h(P_{-1})=10u_h(P_1)-5u_h(P_2)+\frac{5}{3}u_h(P_3)-\frac{1}{4}u_h(P_4) \tag{2.1}$$

has $O(h^6)$ precision. Similarly,

$$\hat{I}_hu_h(P_{-1})=6u_h(P_1)-2u_h(P_2)+\frac{1}{3}u_h(P_3) \tag{2.2}$$

is easily proved to have $O(h^5)$ precision. Of course, unequally spaced interpolation formulae on a smooth region are also available.

Now, for $P\in\Omega_h^*$, $M_hu_h(P)$ is well-defined, and let this be denoted by $\overline{M}_hu_h(P)$, and the finite difference approximation of (1.1) is

$$\begin{cases} M_hu_h(P)=\lambda_hu_h(P), & P\in\Omega_h', \\ \overline{M}_hu_h(P)=\lambda_hu_h(P), & P\in\Omega_h^*. \end{cases} \tag{2.3}$$

Let (λ_h, u_h) be the solution of (2.3), and let (λ, u) be the solution of (1.1). We have the following result.

Theorem. *If* $u\in C^6(\Omega)$, *by using* (2.1), *then*

$$\lambda-\lambda_h-\frac{h^2}{6}\lambda^{3/2}=O(h^4). \tag{2.4}$$

§ 3. Proof

Let X_h be the functional space on Ω_h and let

$$M=\begin{bmatrix} M_h \\ \overline{M}_h \end{bmatrix}.$$

Define the inner product as follows:

$$(u_h, v_h)_h=h^2\sum_{P\in\Omega_h}u_h(P)\bar{v}_h(P). \tag{3.1}$$

Then (2.3) can be simplified to

$$Mu_h(P)=\lambda_hu_h(P), \quad P\in\Omega_h. \tag{3.2}$$

Let v_h be the eigenvector of M^*(conjugate of M) corresponding to $\bar{\lambda}_h$. First normalize u such that $(u, u)_h=1$ and then v_h and u_h such that

$$(u, v_h)_h=1 \tag{3.3}$$

and

$$(u_h, v_h)_h=1. \tag{3.4}$$

Without loss of generality, we may assume both λ and λ_h are simple. For $P\in\Omega_h'$, we have

$$\begin{aligned}(M_h-\lambda_h)u(P)&=(M_h-\Delta^2)u(P)+(\lambda-\lambda_h)u(P)\\ &=\frac{h^2}{6}\Delta^3u(P)+(\lambda-\lambda_h)u(P)+O(h^4)\\ &=\left(\lambda-\lambda_h-\frac{h^2}{6}\lambda^{3/2}\right)u(P)+O(h^4).\end{aligned} \tag{3.5}$$

Further, if we assume that u can be extended to a neighbourhood of Ω, then the 21-point scheme M_h can be formally extended to the irregular points $P\in\Omega_h^*$ and we have

$$\begin{aligned}(\overline{M}_h-\lambda_h)u(P)&=(\overline{M}_h-M_h)u(P)+(M_h-\Delta^2)u(P)+(\lambda-\lambda_h)u(P)\\ &=(\overline{M}_h-M_h)u(P)+\left(\lambda-\lambda_h-\frac{h^2}{6}\lambda^{3/2}\right)u(P)+O(h^4).\end{aligned} \tag{3.6}$$

From the definition of v_h, it follows that

$$\begin{aligned}0&=((M-\lambda_h)u, v_h)_h\\ &=\sum_{P\in\Omega_h'}h^2(M_h-\lambda_h)u\bar{v}_h+\sum_{P\in\Omega_h^*}h^2(\overline{M}_h-\lambda_h)u\bar{v}_h\end{aligned}$$

$$=h^2\sum_{P\in\Omega_h}\left(\lambda-\lambda_h-\frac{h^2}{6}\lambda^{3/2}\right)u\bar{v}_h+h^2\sum_{P\in\Omega_h^*}(\overline{M}_h-M_h)u\bar{v}_h+O(h^4)$$

$$=\lambda-\lambda_h-\frac{h^2}{6}\lambda^{3/2}+h^2\sum_{P\in\Omega_h^*}(\overline{M}_h-M_h)u\bar{v}_h+O(h^4) \tag{3.7}$$

or

$$\lambda-\lambda_h-\frac{h^2}{6}\lambda^{3/2}=-h^2\sum_{P\in\Omega_h^*}(\overline{M}_h-M_h)u\bar{v}_h+O(h^4)$$
$$=-h^2\sum_{P\in\Omega_h^*}(\overline{M}_h-M_h)u(\bar{v}_h-u)-h^2\sum_{P\in\Omega_h^*}(\overline{M}_h-M_h)uu+O(h^4)$$
$$=I_1+I_2+O(h^4). \tag{3.8}$$

Evidently I_1 and I_2 are of $O(h^2)$. In order to estimate I_1 and I_2 more precisely, we consider first how well that u is approximated by u_h or v_h.

The following system of linear equations

$$\begin{cases}(M_h-\lambda_h)(u-u_h)=O(h^2), & P\in\Omega_h', \\ (\overline{M}_h-\lambda_h)(u-u_h)=O(h^2), & P\in\Omega_h^*, \\ (u-u_h,\ v_h)=0, \end{cases} \tag{3.9}$$

because of the simplicity of λ_h, has a unique solution on the subspace $S=\{w\mid(w,\ v_h)_h=0\}$. Under this constraint, the spectrum of (3.9) is $\{\lambda_h^i-\lambda_h\mid\lambda_h^i\neq\lambda_h,\ i=1,\ 2,\ \cdots,\ n-1\}$. (3.8) shows that $\lambda_h^i-\lambda^i=O(h^2)$; therefore $|\lambda_h^i-\lambda_h|\geqslant|\lambda^i-\lambda|-O(h^2)>0$, and hence the linear system (3.9) is invertible and the inverse is uniformly bounded, that is

$$\|u-u_h\|_h=O(h^2). \tag{3.10}$$

Consider the projection operators $P=(\cdot,\ u)_h u$ and $P_h=(\cdot,\ v_h)_h u_h$. Evidently we have $PX_h=\{u\}$, $P_hX_h=\{u_h\}$, $\{(I-P)X_h\}^\perp=\{u\}$ and $\{(I-P_h)X_h\}^\perp=\{v_h\}$.

According to the 'gap' property of the Hilbert space (see [7], 86—87):

$$\sin(\widehat{u,\ u_h})=\sin(\widehat{u,\ v_h}) \tag{3.11}$$

(3.10) implies that $\sin(\widehat{u,u_h})$ and $\sin(\widehat{u,v_h})$ are both of $O(h^2)$. On the other hand, from $1=(v_h,\ u)_h=\|v_h\|_h\cos(\widehat{u,v_h})$ and $\cos(\widehat{u,v_h})=1-O(h^4)$, we deduce $\|v_h\|_h=1+O(h^4)$. It follows that

$$\|u-v_h\|_h=O(h^2). \tag{3.12}$$

We are going to prove that I_1 and I_2 are of $O(h^{9/2})$ and $O(h^5)$ respectively.

$$|I_1|=|h^2\sum_{P\in\Omega_h^*}(\overline{M}_h-M_h)u(\bar{v}_h-u)|$$
$$\leqslant\{h^2\sum_{P\in\Omega_h^*}((\overline{M}_h-M_h)u)^2\}^{1/2}\{h^2\sum_{P\in\Omega_h^*}(\bar{v}_h-u)^2\}^{1/2}$$
$$=O(h^{5/2})O(h^2)=O(h^{9/2}). \tag{3.13}$$

Here, we have used (2.1) to define the interpolating values on points which do not belong to $\Omega_h\cup\partial\Omega_h$.

Noting the boundary condition in (1.1), we have $u(P)=O(h^2)$ for $P\in\Omega_h^*$ and hence

$$|I_2|=|h^2\sum_{P\in\Omega_h^*}(\overline{M}_h-M_h)uu|=O(h^5). \tag{3.14}$$

Thus the proof is completed.

§ 4. Numerical Experiments

We consider the solution of (1.1), where Ω is the unit square, $0<x$, $y<1$. The first four significant figures of the minimum eigenvalue is known as 1296[8].

The biharmonic eigenvalue problem (1.1) is approximated by the linear system (2.3) which is solved on the IBM 3031 computer with double precision arithmetic. The subroutine LEQT2B which is used for solving linear systems is called from the computer library IMSL.

The numerical results are as follows.

	Minimum eigenvalue		
	Uncorrected	Corrected	Extrapolation
h	λ_h	$\hat{\lambda}_h=\lambda_h+\frac{h^2}{6}\lambda_h^{3/2}$	
1/10	1239.41	1312.13	
1/15	1271.14	1304.71	1296.52
1/20	1281.74	1300.87	1295.37
1/25	1286.55	1299.04	1295.10

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