A NOTE ON SIMPLE NON-ZERO SINGULAR VALUES*1)

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Abstract

The technique described in [4] is used to investigate the analyticity and to obtain second order perturbation expansions of simple non-zero singular values of a matrix analytically dependent on several parameters.

The object of this note is to use the technique described in [4] to investigate the analyticity and to obtain second order perturbation expansions of simple non-zero singular values of a matrix analytically dependent on several parameters. The results may be useful for investigating the performance and robustuess of multivariable feedback systems as well as design techniques (see [3] and the references contained therein).

Notation. The symbol $\mathbb{C}^{m\times n}$ denotes the set of complex $m\times n$ matrices and $\mathbb{R}^{m\times n}$ the set of real $m\times n$ matrices, $\mathbb{C}^n=\mathbb{C}^{n\times 1}$ and $\mathbb{R}=\mathbb{R}^1$. The superscript H is for conjugate transpose, and T for transpose. $\|x\|$ denotes the usual Euclidean vector norm of x and $\|A\|$ denotes the spectral norm of a matrix A.

§1. Singular Values of a Complex Matrix

Let $p=(p_1, \cdots, p_N)^T$ and $A(p)\in\mathbb{C}^{m\times n}$. We may assume without loss of generality that the parameters $p_1, \cdots, p_N$ are real and $m\geqslant n$ throughout this note.

Let $A=A(p^*)$ for some point $p^*\in\mathbb{R}^N$. Suppose that σ is a singular value of A. Then there exist two unit vectors, $v\in\mathbb{C}^n$ and $u\in\mathbb{C}^m$, such that

$$Av=\sigma u, \quad A^Hu=\sigma v.$$

Such v, u will be called unit right and unit left singular vectors of A corresponding to the singular value σ.

First, applying the Implicit Function Theorem we prove the following theorem.

Theorem 1.1. *Let* $p\in\mathbb{R}^n$ *and* $A(p)\in\mathbb{C}^{m\times n}$. *Suppose that* $\mathrm{Re}[A(p)]$ *and* $\mathrm{Im}[A(p)]$ *are real analytic matrix-valued functions of* p *in some neighbourhood* $\mathscr{B}(0)$ *of the origin. If* σ_1 *is a simple non-zero singular value of* $A(0)$, $v_1\in\mathbb{C}^n$ *and* $u_1\in\mathbb{C}^m$ *are associated unit right and unit left singular vectors, respectively, then*

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1) *there exists a simple singular value* $\sigma_1(p)$ *of* $A(p)$ *which is a real analytic function of* p *in some neighbourhood* $\mathscr{B}_0$ *of the origin, and* $\sigma_1(0)=\sigma_1$;

2) *the unit right singular vector* $v_1(p)$ *and the unit left singular vector* $u_1(p)$ *of* $A(p)$ *corresponding to* $\sigma_1(p)$ *may be so defined that* $\mathrm{Re}[v_1(p)]$, $\mathrm{Im}[v_1(p)]$, $\mathrm{Re}[u_1(p)]$ *and* $\mathrm{Im}[u_1(p)]$ *are real analytic functions of* p *in* $\mathscr{B}_0$, $v_1(0)=v_1$ *and* $u_1(0)=u_1$.

Proof. By hypotheses there exist two unitary matrices

$$U=(u_1, U_2)\in\mathbb{C}^{m\times m},\quad V=(v_1, V_2)\in\mathbb{C}^{n\times n} \tag{1.1}$$

such that

$$U^H A(0)V=\Sigma=\begin{pmatrix}\sigma_1 & 0\\ 0 & \Sigma_2\end{pmatrix},\qquad \Sigma_2=\begin{pmatrix}\sigma_2 & & \\ & \ddots & \\ & & \sigma_n\\ & 0 & \end{pmatrix}\in\mathbb{R}^{(m-1)\times(n-1)}, \tag{1.2}$$

where $\sigma_1,\cdots,\sigma_n\geqslant 0$ and $\sigma_j\neq\sigma_1>0$ for $j=2,\cdots,n$.

We set

$$\tilde{A}(p)=V^H A(p)^H A(p)V=\begin{pmatrix}\tilde{a}_{11}(p) & \tilde{a}_{21}(p)^H\\ \tilde{a}_{21}(p) & \tilde{A}_{22}(p)\end{pmatrix},\quad \tilde{a}_{11}(p)\in\mathbb{R} \tag{1.3}$$

and introduce a vector-valued function

$$f(z,p)=\tilde{a}_{21}(p)-\tilde{a}_{11}(p)z+\tilde{A}_{22}(p)z-z\tilde{a}_{21}(p)^H z, \tag{1.4}$$

where

$$f=(f_1\cdots,f_{n-1})^T,\quad z=(\zeta_1,\cdots,\zeta_{n-1})^T\in\mathbb{C}^{n-1},\quad p\in\mathbb{R}^N.$$

Let

$$f_j=\varphi_j+i\psi_j,\quad \zeta_j=\xi_j+i\eta_j,\quad i=\sqrt{-1},\ j=1,\cdots,n-1$$

and

$$x=(\xi_1,\cdots,\xi_{n-1})^T,\quad y=(\eta_1,\cdots,\eta_{n-1})\in\mathbb{R}^{n-1}.$$

Obviously, $\varphi_j(x,y,p)$ and $\psi_j(x,y,p)$ $(j=1,\cdots,n-1)$ are real analytic functions of real variables $x, y\in\mathbb{R}^{n-1}$ and $p\in\mathscr{B}(0)$, and the functions satisfy

$$\varphi_j(0,0,0)=0,\quad \psi_j(0,0,0)=0,\quad j=1,\cdots,n-1. \tag{1.5}$$

Since $f_1,\cdots,f_{n-1}$ are complex analytic functions of the complex variables $\zeta_1,\cdots,\zeta_{n-1}$ for any $p\in\mathscr{B}(0)$, we have ([1, p. 39, Theorem 8])

$$\det\frac{\partial(\varphi_1,\cdots,\varphi_{n-1},\psi_1,\cdots,\psi_{n-1})}{\partial(\xi_1,\cdots,\xi_{n-1},\eta_1,\cdots,\eta_{n-1})}=\left|\det\frac{\partial(f_1,\cdots,f_{n-1})}{\partial(\zeta_1,\cdots,\zeta_{n-1})}\right|^2.$$

Combining it with

$$\left(\frac{\partial(f_1,\cdots,f_{n-1})}{\partial(\zeta_1,\cdots,\zeta_{n-1})}\right)_{z=0,\ p=0}=\tilde{A}_{22}(0)-\tilde{a}_{11}(0)I=\Sigma_2^T\Sigma_2-\sigma_1^2 I$$

we get

$$\det\left(\frac{\partial(\varphi_1,\cdots,\varphi_{n-1},\psi_1,\cdots,\psi_{n-1})}{\partial(\xi_1,\cdots,\xi_{n-1},\eta_1,\cdots,\eta_{n-1})}\right)_{\substack{x=y=0\\ p=0}}=\prod_{l=2}^{n}(\sigma_l^2-\sigma_1^2)^2\neq 0.$$

Hence by the Implicit Function Theorem (see [2, p. 277] or [4, Theorem 1.2]) the system of equations

$$\varphi_j(x, y, p) = 0, \quad \psi_j(x, y, p) = 0, \quad j=1, \cdots, n-1, \text{ i.e., } f(z, p) = 0 \tag{1.6}$$

has a unique real analytic solution

$$x = x(p), \quad y = y(p), \quad \text{i.e., } z = z(p)$$

in some neighbourhood $\mathscr{B}_0 \subset \mathscr{B}(0)$ of the origin, and

$$x(0) = 0, \quad y(0) = 0, \quad \text{i.e., } z(0) = 0. \tag{1.7}$$

Observe that

$$\tilde{a}_{11}(p) + z(p)^H \tilde{a}_{21}(p) + \tilde{a}_{21}(p)^H z(p) + z(p)^H \tilde{A}_{22}(p) z(p) > 0$$

provided that $p \in \mathscr{B}_0$ and $\mathscr{B}_0$ is sufficiently small. Therefore we may define a positive valued function

$$\sigma_1(p) = [(\tilde{a}_{11}(p) + z(p)^H \tilde{a}_{21}(p) + \tilde{a}_{21}(p)^H z(p) + z(p)^H \tilde{A}_{22}(p) z(p))(1 + z(p)^H z(p))^{-1}]^{1/2}, \quad p \in \mathscr{B}_0. \tag{1.8}$$

Besides, we define two vector-valued functions

$$v_1(p) = V \begin{pmatrix} 1 \\ z(p) \end{pmatrix} (1 + z(p)^H z(p))^{-1/2}, \quad u_1(p) = A(p) v_1(p) / \sigma_1(p), \quad p \in \mathscr{B}_0. \tag{1.9}$$

From (1.3)—(1.9) and the above mentioned argument we see that the functions $\sigma_1(p)$, $v_1(p)$ and $u_1(p)$ are real analytic in $\mathscr{B}_0$ and satisfy

$$A(p) v_1(p) = \sigma_1(p) u_1(p), \quad A(p)^H u_1(p) = \sigma_1(p) v_1(p), \quad \|u_1(p)\| = \|v_1(p)\| = 1 \tag{1.10}$$

and

$$\sigma_1(0) = \sigma_1, \quad v_1(0) = v_1, \quad u_1(0) = u_1. \tag{1.11}$$

According to the well-known perturbation theorem for singular values the singular value $\sigma_1(p)$ of $A(p)$ is simple provided that the neighbourhood $\mathscr{B}_0$ is sufficiently small. ∎

Theorem 1.2. *Assume that the hypotheses of Theorem* 1.1 *are valid. Then there are the following formulas for the simple non-zero singular value* $\sigma_1(p)$ *and the associated singular vectors* $v_1(p)$ *and* $u_1(p)$ *defined by* (1.8) *and* (1.9):

$$\left(\frac{\partial \sigma_1(p)}{\partial p_j}\right)_{p=0} = \mathrm{Re}\left[u_1^H \left(\frac{\partial A(p)}{\partial p_j}\right)_{p=0} v_1\right], \tag{1.12}$$

$$\left(\frac{\partial v_1(p)}{\partial p_j}\right)_{p=0} = V_2 \left(\Phi_1 V_2^H \left(\frac{\partial A(p)}{\partial p_j}\right)_{p=0}^H, \ \Psi^T U_2^H \left(\frac{\partial A(p)}{\partial p_j}\right)_{p=0}\right) \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \tag{1.13}$$

$$\left(\frac{\partial u_1(p)}{\partial p_j}\right)_{p=0} = U_2 \left(\Psi V_2^H \left(\frac{\partial A(p)}{\partial p_j}\right)_{p=0}^H, \ \Phi_2 U_2^H \left(\frac{\partial A(p)}{\partial p_j}\right)_{p=0}\right) \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + \frac{i}{\sigma_1} \mathrm{Im}\left[u_1^H \left(\frac{\partial A(p)}{\partial p_j}\right)_{p=0} v_1\right] u_1, \quad i = \sqrt{-1}, \tag{1.14}$$

$$\left(\frac{\partial^2\sigma_1(p)}{\partial p_j\partial p_k}\right)_{p=0}=\operatorname{Re}\left[u_1^H\left(\frac{\partial^2 A(p)}{\partial p_j\partial p_k}\right)_{p=0}v_1\right]$$
$$+\operatorname{Re}\left[\begin{pmatrix}u_1\\ v_1\end{pmatrix}^H\begin{pmatrix}\left(\frac{\partial A(p)}{\partial p_k}\right)^H_{p=0} & 0\\ 0 & \left(\frac{\partial A(p)}{\partial p_k}\right)_{p=0}\end{pmatrix}^H\right.$$
$$\times\begin{pmatrix}V_2\Phi_1V_2^H & V_2\Psi^TU_2^H\\ U_2\Psi V_2^H & U_2\Phi_2U_2^H\end{pmatrix}\begin{pmatrix}\left(\frac{\partial A(p)}{\partial p_j}\right)^H & 0\\ 0 & \left(\frac{\partial A(p)}{\partial p_j}\right)_{p=0}\end{pmatrix}\begin{pmatrix}u_1\\ v_1\end{pmatrix}$$
$$+\frac{1}{\sigma_1}\operatorname{Im}\left[u_1^H\left(\frac{\partial A(p)}{\partial p_j}\right)_{p=0}v_1\right]\operatorname{Im}\left[u_1^H\left(\frac{\partial A(p)}{\partial p_k}\right)_{p=0}v_1\right],\quad(1.15)$$

where $j, k=1, \cdots, N$, u_1, v_1, U_2 and V_2 are defined by (1.1), *and*

$$\Phi_1=\sigma_1(\sigma_1^2I-\Sigma_2^T\Sigma_2)^{-1},\quad \Phi_2=\sigma_1(\sigma_1^2I-\Sigma_2\Sigma_2^T)^{-1},\quad \Psi=\Sigma_2(\sigma_1^2I-\Sigma_2^T\Sigma_2)^{-1},\quad(1.16)$$

in which Σ_2 is defined by (1.2).

Proof. 1) By Theorem 1.1 (see (1.10) and (1.11))

$$\sigma_1(p)=u_1(p)^HA(p)v_1(p)=v_1(p)^HA(p)^Hu_1(p),\quad(1.17)$$

and so we have

$$\frac{\partial\sigma_1(p)}{\partial p_j}=\sigma_1(p)\left(\frac{\partial u_1(p)}{\partial p_j}\right)^Hu_1(p)+u_1(p)^H\frac{\partial A(p)}{\partial p_j}v_1(p)$$
$$+\sigma_1(p)v_1(p)^H\frac{\partial v_1(p)}{\partial p_j}\quad(1.18)$$

and

$$\frac{\partial\sigma_1(p)}{\partial p_j}=\sigma_1(p)\left(\frac{\partial v_1(p)}{\partial p_j}\right)^Hv_1(p)+v_1(p)^H\left(\frac{\partial A(p)}{\partial p_j}\right)^Hu_1(p)$$
$$+\sigma_1(p)u_1(p)^H\frac{\partial u_1(p)}{\partial p_j}.\quad(1.19)$$

From (1.18), (1.19) and $\|u_1(p)\|=\|v_1(p)\|=1$ we obtain

$$\frac{\partial\sigma_1(p)}{\partial p}=\frac{1}{2}\left[u_1(p)^H\frac{\partial A(p)}{\partial p_j}v_1(p)+v_1(p)^H\left(\frac{\partial A(p)}{\partial p_j}\right)^Hu_1(p)\right].\quad(1.20)$$

Substituting $p=0$ into (1.20), we get (1.12) at once.

2) From

$$A(p)^HA(p)v_1(p)=\sigma_1(p)^2v_1(p)$$

it follows that

$$(\sigma_1^2I-A(0)^HA(0))\left(\frac{\partial v_1(p)}{\partial p_j}\right)_{p=0}$$
$$=\left[\left(\frac{\partial A(p)}{\partial p_j}\right)^H_{p=0}A(0)+A(0)^H\left(\frac{\partial A(p)}{\partial p_j}\right)_{p=0}-2\sigma_1\left(\frac{\partial\sigma_1(p)}{\partial p_j}\right)_{p=0}I\right]v_1.$$

Combining it with (1.2), (1.7) and (1.9) we get

$$\begin{pmatrix} 0 & 0 \\ 0 & \sigma_1^2 I - \Sigma_2^T \Sigma_2 \end{pmatrix} \begin{pmatrix} 0 \\ \left(\frac{\partial z(p)}{\partial p_j}\right)_{p=0} \end{pmatrix}$$

$$= \sigma_1 V^H \left(\frac{\partial A(p)}{\partial p_j}\right)^H_{p=0} u_1 + \begin{pmatrix} \sigma_1 & 0 \\ 0 & \Sigma_2^T \end{pmatrix} U^H \left(\frac{\partial A(p)}{\partial p_j}\right)_{p=0} v_1 - 2\sigma_1 \left(\frac{\partial \sigma_1(p)}{\partial p_j}\right)_{p=0} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\left(\frac{\partial z(p)}{\partial p_j}\right)_{p=0} = (\sigma_1^2 I - \Sigma_2^T \Sigma_2)^{-1} \left[\sigma_1 V_2^H \left(\frac{\partial A(p)}{\partial p_j}\right)^H_{p=0} u_1 + \Sigma_2^T U_2^H \left(\frac{\partial A(p)}{\partial p_j}\right)_{p=0} v_1\right]. \tag{1.21}$$

Substituting (1.21) into

$$\left(\frac{\partial v_1(p)}{\partial p_j}\right)_{p=0} = V_2 \left(\frac{\partial z(p)}{\partial p_j}\right)_{p=0}$$

and utilizing the symbols Φ_2 and Ψ defined by (1.16), we get the formula (1.13).

3) From $A(p)v_1(p) = \sigma_1(p)u_1(p)$ (see (1.10)) we obtain

$$\left(\frac{\partial u_1(p)}{\partial p_j}\right)_{p=0} = \frac{1}{\sigma_1}\left[\left(\frac{\partial A(p)}{\partial p_j}\right)_{p=0} v_1 + A(0)\left(\frac{\partial v_1(p)}{\partial p_j}\right)_{p=0} - \left(\frac{\partial \sigma_1(p)}{\partial p_j}\right)_{p=0} u_1\right].$$

Combining it with (1.12), (1.13) and utilizing the relations

$$\frac{1}{\sigma_1} A(0) V_2 \Phi_1 V_2^H = U_2 \Psi V_2^H, \quad \frac{1}{\sigma_1} A(0) V_2 \Psi^T U_2^H = U_2 \Phi_1 U_2^H - \frac{1}{\sigma_1} U_2 U_2^H$$

we get the formula (1.14).

4) From (1.20) it follows that

$$\left(\frac{\partial^2 \sigma_1(p)}{\partial p_j \partial p_k}\right)_{p=0} = \mathrm{Re}\left[u_1^H \left(\frac{\partial^2 A(p)}{\partial p_j \partial p_k}\right)_{p=0} v_1\right] + \mathrm{Re}\left[\left(\frac{\partial u_1(p)}{\partial p_k}\right)^H_{p=0} \left(\frac{\partial A(p)}{\partial p_j}\right)_{p=0} v_1 + \left(\frac{\partial v_1(p)}{\partial p_k}\right)^H_{p=0} \left(\frac{\partial A(p)}{\partial p_j}\right)^H_{p=0} u_1\right].$$

Combining it with (1.13) and (1.14) we obtain the formula (1.15). ▮

Remark 1.1. Let

$$S = \begin{pmatrix} \Phi_1 & \Psi^T \\ \Psi & \Phi_2 \end{pmatrix}, \tag{1.22}$$

in which Φ_1, Φ_2 and Ψ are defined by (1.16), and let

$$\hat{S} = \begin{pmatrix} V_2 & 0 \\ 0 & U_2 \end{pmatrix} S \begin{pmatrix} V_2 & 0 \\ 0 & U_2 \end{pmatrix}^H, \quad D_j = \begin{pmatrix} \left(\frac{\partial A(p)}{\partial p_j}\right)^H_{p=0} & 0 \\ 0 & \left(\frac{\partial A(p)}{\partial p_j}\right)_{p=0} \end{pmatrix}. \tag{1.23}$$

Then the formulas (1.13)—(1.15) may be rewritten as

$$\begin{pmatrix} \left(\frac{\partial v_1(p)}{\partial p_j}\right)_{p=0} \\ \left(\frac{\partial u_1(p)}{\partial p_j}\right)_{p=0} \end{pmatrix} = \hat{S} D_j \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + \frac{i}{\sigma_1} \begin{pmatrix} 0 \\ \mathrm{Im}\left[u_1^H \left(\frac{\partial A(p)}{\partial p_j}\right)_{p=0} v_1\right] u_1 \end{pmatrix} \tag{1.24}$$

and

$$\left(\frac{\partial^2\sigma_1(p)}{\partial p_j\partial p_k}\right)_{p=0}=\mathrm{Re}\left[u_1^H\left(\frac{\partial^2 A(p)}{\partial p_j\partial p_k}\right)_{p=0}v_1+\begin{pmatrix}u_1\\v_1\end{pmatrix}^H D_k^H\hat{S}D_j\begin{pmatrix}u_1\\v_1\end{pmatrix}\right]$$
$$+\frac{1}{\sigma_1}\mathrm{Im}\left[u_1^H\left(\frac{\partial A(p)}{\partial p_j}\right)_{p=0}v_1\right]\mathrm{Im}\left[u_1^H\left(\frac{\partial A(p)}{\partial p_k}\right)_{p=0}v_1\right]. \tag{1.25}$$

Example 1.1 [3, p.226–227]. Let

$$A(p)=\begin{pmatrix}1 & -1\\ \frac{1}{p_1+ip_2+2} & 1\end{pmatrix},\quad p=(p_1,p_2)^T\in\mathbb{R}^2,\quad i=\sqrt{-1}.$$

It is easy to see that the matrix $A(0)=\begin{pmatrix}1 & -1\\ \frac{1}{2} & 1\end{pmatrix}$ has a singular value decomposition $A(0)=U\Sigma V^H$ in which

$$U=\frac{1}{\sqrt{5}}\begin{pmatrix}2 & 1\\ -1 & 2\end{pmatrix},\quad V=\frac{1}{\sqrt{5}}\begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix},\quad \Sigma=\begin{pmatrix}\frac{3}{2} & 0\\ 0 & 1\end{pmatrix}.$$

Consequently, we have

$$\sigma_1=\frac{3}{2},\quad \sigma_2=1,\quad \Phi_1=\Phi_2=6/5,\quad \Psi=4/5,$$
$$u_1=\frac{1}{\sqrt{5}}(2,-1)^T,\quad u_2=\frac{1}{\sqrt{5}}(1,2)^T,\quad v_1=\frac{1}{\sqrt{5}}(1,-2)^T,\quad v_2=\frac{1}{\sqrt{5}}(2,1)^T,$$
$$\left(\frac{\partial A(p)}{\partial p_1}\right)_{p=0}=\begin{pmatrix}0 & 0\\ -\frac{1}{4} & 0\end{pmatrix},\quad \left(\frac{\partial A(p)}{\partial p_2}\right)_{p=0}=\begin{pmatrix}0 & 0\\ -\frac{i}{4} & 0\end{pmatrix},$$
$$\left(\frac{\partial^2 A(p)}{\partial p_1^2}\right)_{p=0}=\begin{pmatrix}0 & 0\\ \frac{1}{4} & 0\end{pmatrix},\quad \left(\frac{\partial^2 A(p)}{\partial p_1\partial p_2}\right)_{p=0}=\begin{pmatrix}0 & 0\\ \frac{i}{4} & 0\end{pmatrix},$$

and

$$\left(\frac{\partial^2 A(p)}{\partial p_2^2}\right)_{p=0}=\begin{pmatrix}0 & 0\\ -\frac{1}{4} & 0\end{pmatrix}.$$

Utilizing the formulas (1.12)—(1.15) after some straightforward calculations, we get

$$\left(\frac{\partial\sigma_1(p)}{\partial p_1}\right)_{p=0}=0.05,\quad \left(\frac{\partial\sigma_1(p)}{\partial p_2}\right)_{p=0}=0,$$
$$\left(\frac{\partial\sigma_2(p)}{\partial p_1}\right)_{p=0}=-0.2,\quad \left(\frac{\partial\sigma_2(p)}{\partial p_2}\right)_{p=0}=0,$$
$$\left(\frac{\partial^2\sigma_1(p)}{\partial p_1^2}\right)_{p=0}=-0.042,\quad \left(\frac{\partial^2\sigma_1(p)}{\partial p_1\partial p_2}\right)_{p=0}=0,\quad \left(\frac{\partial^2\sigma_1(p)}{\partial p_2^2}\right)_{p=0}=0.0916$$

and

$$\left(\frac{\partial^2\sigma_2(p)}{\partial p_1^2}\right)_{p=0}=0.208,\quad \left(\frac{\partial^2\sigma_2(p)}{\partial p_1\partial p_2}\right)_{p=0}=0,\quad \left(\frac{\partial^2\sigma_2(p)}{\partial p_2^2}\right)_{p=0}=-0.12.$$

Hence $\sigma_1(p)$ and $\sigma_2(p)$ have the expansions

$$\sigma_1(p)=1.5+0.05p_1-0.021p_1^2+0.0458\dot{3}p_2^2+O(\|p\|^3)$$

and

$$\sigma_2(p)=1.0-0.2p_1+0.104p_1^2-0.06p_2^2+O(\|p\|^3)$$

in a neighbourhood of the origin.

It is worth-while to point out that the above mentioned way for computing the second order Taylor expansions for simple singular values is feasible.

§ 2. Singular Values of a Real Matrix

From Theorems 1.1 and 1.2 we obtain the following result at once.

Theorem 2.1. *Let $p\in\mathbb{R}^N$ and $A(p)\in\mathbb{R}^{m\times n}$. Suppose that $A(p)$ is a real analytic matrix-valued function of p in some neighbourhood $\mathscr{B}(0)$ of the origin. If $A(0)$ has a singular value decomposition $A(0)=U\Sigma V^T$ in which U and V are real orthogonal matrices, and*

$$\begin{cases} U=(u_1, U_2)\in\mathbb{R}^{m\times m}, \quad V=(v_1, V_2)\in\mathbb{R}^{n\times n}, \\ \Sigma=\begin{pmatrix}\sigma_1 & 0\\ 0 & \Sigma_2\end{pmatrix} \quad \Sigma_2=\begin{pmatrix}\sigma_2 & & \\ & \ddots & \\ & & \sigma_n\\ & 0 & \end{pmatrix}\in\mathbb{R}^{(m-1)\times(n-1)} \end{cases} \tag{2.1}$$

with $\sigma_1, \cdots, \sigma_n\geqslant 0$ and $\sigma_j\neq\sigma_1>0$ *for $j=2, \cdots, n$, then*

1) *there exists a simple singular value $\sigma_1(p)$ of $A(p)$ which is a real analytic function of p in some neighbourhood $\mathscr{B}_0$ of the origin, and $\sigma_1(0)=\sigma_1$;*

2) *the unit right singular vector $v_1(p)$ and the unit left singular vector $u_1(p)$ of $A(p)$ corresponding to $\sigma_1(p)$ may be so defined that $v_1(p)$ and $u_1(p)$ are real analytic functions of p in $\mathscr{B}_0$, $v_1(0)=v_1$ and $u_1(0)=u_1$;*

3) *the following formulas are valid:*

$$\left(\frac{\partial\sigma_1(p)}{\partial p_j}\right)_{p=0}=u_1^T\left(\frac{\partial A(p)}{\partial p_j}\right)_{p=0}v_1, \tag{2.2}$$

$$\begin{pmatrix}\left(\frac{\partial v_1(p)}{\partial p_j}\right)_{p=0}\\ \left(\frac{\partial u_1(p)}{\partial p_j}\right)_{p=0}\end{pmatrix}=\hat{S}D_j\begin{pmatrix}u_1\\ v_1\end{pmatrix}, \tag{2.3}$$

$$\left(\frac{\partial^2\sigma_1(p)}{\partial p_j\partial p_k}\right)_{p=0}=u_1^T\left(\frac{\partial^2 A(p)}{\partial p_j\partial p_k}\right)_{p=0}v_1+\begin{pmatrix}u_1\\ v_1\end{pmatrix}^T D_k^T\hat{S}D_j\begin{pmatrix}u_1\\ v_1\end{pmatrix}, \tag{2.4}$$

where $j, k=1, \cdots, N$ and

$$\hat{S}=\begin{pmatrix}V_2 & 0\\ 0 & U_2\end{pmatrix}S\begin{pmatrix}V_2 & 0\\ 0 & U_2\end{pmatrix}^T, \quad D_j=\begin{pmatrix}\left(\frac{\partial A(p)}{\partial p_j}\right)_{p=0}^T & 0\\ 0 & \left(\frac{\partial A(p)}{\partial p_j}\right)_{p=0}\end{pmatrix}, \tag{2.5}$$

in which u_1, v_1, U_2, V_2 *and* S *are defined by* (2.1) *and* (1.22).

Corollary 2.1. Let $A_0, A_1, \cdots, A_N \in \mathbb{R}^{m\times n}$. Suppose that $A_0=U\Sigma V^T$ is a singular value decomposition of A_0, where U, V and Σ are represented by (2.1). If σ_1 is a simple non-zero singular value of A_0, then the matrix

$$A(p)=A_0+\sum_{j=1}^{N} p_j A_j, \quad p=(p_1, \cdots, p_N)^T \in \mathbb{R}^N$$

has a simple singular value $\sigma_1(p)$ which is a real analytic function in some neighbourhood of $p=0$, $\sigma_1(0)=\sigma_1$, and satisfies the following estimates:

$$\left|\left(\frac{\partial^2\sigma_1(p)}{\partial p_j \partial p_k}\right)_{p=0}\right| \leqslant 2\max\left\{\frac{1}{\min\limits_{2\leqslant l\leqslant n}|\sigma_1-\sigma_l|}, \frac{1}{\sigma_1}\right\}\|A_j\|\|A_k\|, \quad 1\leqslant j, k\leqslant N. \tag{2.6}$$

Proof. By the formula (2.4) we have

$$\left(\frac{\partial^2\sigma_1(p)}{\partial p_j \partial p_k}\right)_{p=0}=\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}^T \begin{pmatrix} A_k & 0 \\ 0 & A_k^T \end{pmatrix} \begin{pmatrix} V_2 & 0 \\ 0 & U_2 \end{pmatrix} S \begin{pmatrix} V_2^T & 0 \\ 0 & U_2^T \end{pmatrix} \begin{pmatrix} A_j^T & 0 \\ 0 & A_j \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \tag{2.7}$$

where S is defined by (1.22). Since

$$\left\|\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}\right\|=\sqrt{2}, \quad \left\|\begin{pmatrix} A_k & 0 \\ 0 & A_k^T \end{pmatrix}\right\|=\|A_k\|, \quad \left\|\begin{pmatrix} A_j^T & 0 \\ 0 & A_j \end{pmatrix}\right\|=\|A_j\|, \quad \left\|\begin{pmatrix} V_2 & 0 \\ 0 & U_2 \end{pmatrix}\right\|=1$$

and

$$\|S\|=\max\left\{\frac{1}{\min\limits_{2\leqslant l\leqslant n}|\sigma_1-\sigma_l|}, \frac{1}{\sigma_1}\right\},$$

from (2.7) we obtain (2.6) at once. ∎

Corollary 2.2. Let $A\in\mathbb{R}^{m\times n}$. Suppose that $A=U\Sigma V^T$ is a singular value decomposition of A, where U, V and Σ are represented by (2.1) in which $\sigma_1, \cdots, \sigma_n \geqslant 0$ and $\sigma_j \neq \sigma_1 > 0$ for $j=2, \cdots, n$. Let

$$\varepsilon=(\varepsilon_{11}, \cdots, \varepsilon_{1n}, \varepsilon_{21}, \cdots, \varepsilon_{2n}, \cdots, \varepsilon_{m1}, \cdots, \varepsilon_{mn})^T \in \mathbb{R}^{mn}$$

and

$$A(\varepsilon)=A+E, \quad E=(\varepsilon_{ij}). \tag{2.8}$$

Then there are a simple singular value $\sigma_1(\varepsilon)$ and corresponding unit singular vectors $v_1(\varepsilon)$ and $u_1(\varepsilon)$ of $A(\varepsilon)$ such that $\sigma_1(\varepsilon)$, $v_1(\varepsilon)$ and $u_1(\varepsilon)$ are real analytic functions of ε in some neighbourhood of the origin, and

$$\sigma_1(\varepsilon)=\sigma_1+u_1^T E v_1+\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}\begin{pmatrix} E & 0 \\ 0 & E^T \end{pmatrix}\hat{S}\begin{pmatrix} E^T & 0 \\ 0 & E \end{pmatrix}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}+O(\|E\|^3), \tag{2.9}$$

$$\begin{pmatrix} v_1(\varepsilon) \\ u_1(\varepsilon) \end{pmatrix}=\begin{pmatrix} v_1 \\ u_1 \end{pmatrix}+\hat{S}\begin{pmatrix} E^T & 0 \\ 0 & E \end{pmatrix}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}+O(\|E\|^2). \tag{2.10}$$

Where $\hat{S}$ is defined by (2.5).

Proof. Let $e_i^{(n)}$ denote the i-th column of the identity $I^{(n)}$, and let

$$A_{jk}=e_j^{(m)}e_k^{(n)}, \quad 1\leqslant j\leqslant m, 1\leqslant k\leqslant n.$$

Then

$$E=\sum_{j=1}^{m}\sum_{k=1}^{n}\varepsilon_{jk}A_{jk}. \tag{2.11}$$

By Theorem 2.1 there are a real analytic simple singular value $\sigma_1(s)$ and corresponding real analytic unit singular vectors $v_1(s)$ and $u_1(s)$ of $A(s)$ provided that $\|s\|$ is sufficiently small.

Utilizing the formulas (2.2)—(2.4) we obtain

$$\left(\frac{\partial\sigma_1(\varepsilon)}{\partial\varepsilon_{jk}}\right)_{\varepsilon=0}=u_1^T A_{jk}v_1, \tag{2.12}$$

$$\begin{pmatrix}\left(\frac{\partial v_1(\varepsilon)}{\partial\varepsilon_{jk}}\right)_{\varepsilon=0}\\ \left(\frac{\partial u_1(\varepsilon)}{\partial\varepsilon_{jk}}\right)_{\varepsilon=0}\end{pmatrix}=\hat{S}\begin{pmatrix}A_{jk}^T & 0\\ 0 & A_{jk}\end{pmatrix}\begin{pmatrix}u_1\\ v_1\end{pmatrix} \tag{2.13}$$

and

$$\left(\frac{\partial^2\sigma_1(\varepsilon)}{\partial\varepsilon_{jk}\partial\varepsilon_{st}}\right)_{\varepsilon=0}=\begin{pmatrix}u_1\\ v_1\end{pmatrix}^T\begin{pmatrix}A_{jk} & 0\\ 0 & A_{jk}^T\end{pmatrix}\hat{S}\begin{pmatrix}A_{st}^T & 0\\ 0 & A_{st}\end{pmatrix}\begin{pmatrix}u_1\\ v_1\end{pmatrix}.$$

Substituting (2.12)—(2.14) into the Taylor expansions of $\sigma_1(s)$, $v_1(s)$ and $u_1(s)$ and combining them with (2.11) we get (2.9) and (2.10). ▮

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