CONVERGENCE OF DIFFERENCE METHODS FOR INVERSE PROBLEMS OF A ONE-DIMENSIONAL HYPERBOLIC SYSTEM OF FIRST ORDER*1)

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Abstract

In this paper, the difference methods for solving the inverse problem of a one-dimensional hyperbolic system of first order are discussed. Some difference schemes are constructed and the convergence of these schemes is proved.

§ 1. Introduction and Summary

In [2], the inverse problem of a one-dimensional linear hyperbolic system of first order is discussed. This problem can be transformed into a semilinear initial-value problem by using a relation obtained from the propagation of singularity. The theorems of existence and stability are proved there. In this paper, we discuss the difference methods for solving this inverse problem as a semilinear initial-value problem.

Consider the following system

$$\begin{cases} \dfrac{\partial W}{\partial t}+c^{-1}(x)\dfrac{\partial P}{\partial x}=0, \\ \dfrac{\partial P}{\partial t}+c(x)\dfrac{\partial W}{\partial x}=0, \end{cases} \quad x>0,\ t>0 \tag{1.1}$$

with the initial conditions

$$W(x,0)=P(x,0)=0 \tag{1.2}$$

and the boundary conditions

$$\begin{cases} W(0,t)=\delta(t)+W_0(t), \\ P(0,t)=\delta(t)+P_0(t). \end{cases} \tag{1.3}$$

The inverse problem is to determine W, P and c satisfying (1.1) and (1.2) from the given data (1.3) and a given constant $c(0)$, here we assume $c(0)=1$.

Set $D=P+cW$ and $U=P-cW$. Then (1.1) becomes

$$\begin{cases} \dfrac{\partial D}{\partial t}+\dfrac{\partial D}{\partial x}=\beta(x)\cdot(D-U), \\ \dfrac{\partial U}{\partial t}-\dfrac{\partial U}{\partial x}=\beta(x)\cdot(D-U), \end{cases} \quad x>0,\ t>0, \tag{1.4}$$

* Received August 20, 1986.
1) The project supported by National Natural Science Foundation of China.

where

$$\beta(x)=\frac{c'(x)}{2c(x)} \tag{1.5}$$

and the corresponding initial and boundary conditions become

$$D(x,0)=U(x,0)=0 \tag{1.6}$$

and

$$\begin{cases} D(0,t)=2\delta(t)+D_0(t), \\ U(0,t)=U_0(t), \end{cases} \tag{1.7}$$

where $D_0(t)=P_0(t)+W_0(t)$, and $U_0(t)=P_0(t)-W_0(t)$. So we need only to solve (1.4) under the conditions (1.6) and (1.7). Obviously the solution of (1.4) with (1.6) satisfies

$$D(x,t)=U(x,t)=0, \quad \text{for } x>t>0. \tag{1.8}$$

By the theory of propagation of singularity (see [6], Ch. 6), we can get the important relation

$$U(x,x)=\beta(x)\exp\int_0^x\beta(s)ds, \quad x\geqslant 0 \tag{1.9}$$

and D can be decomposed as

$$D(x,t)=2\delta(t-x)\exp\int_0^x\beta(s)ds+\tilde{D}(x,t), \tag{1.10}$$

where $\tilde{D}(x,t)$ has a discontinuity of the second kind on $x=t$ (see Appendix).

Now we consider our problem only in the domain $S_{(0,T)}=\{(x,t)\,|\,t>x, 0<x<T\}$. Then the original inverse problem is transformed to the following initial value problem:

$$\begin{cases} \dfrac{\partial D}{\partial t}+\dfrac{\partial D}{\partial x}=\beta(x)\cdot(D-U), \\ \dfrac{\partial U}{\partial t}-\dfrac{\partial U}{\partial x}=\beta(x)\cdot(D-U) \end{cases} \tag{1.11}$$

with the initial conditions (in the x-direction)

$$\begin{cases} D(0,t)=D_0(t), \\ U(0,t)=U_0(t), \end{cases} \tag{1.12}$$

where $\beta(x)$ is determined by $U(x,x)=\beta(x)\exp\int_0^x\beta(s)ds$.

Set

$$d(x)=\exp\int_0^x\beta(s)ds. \tag{1.13}$$

Then by (1.5), we have

$$d(x)=\exp\int_0^x\frac{c'}{2c}ds=\sqrt{c(x)},$$

i.e.

$$d^2(x)=c(x).$$

On the other hand,

$$d'(x)-\beta(x)\exp\int_0^x\beta(s)ds-\beta(x)d(x)=U(x,x).$$

So

$$d(x)=1+\int_0^x U(s,s)ds \tag{1.14}$$

and

$$\beta(x)=d^{-1}(x)\cdot U(x,x). \tag{1.15}$$

Substituting (1.15) into (1.11) and integrating them along characteristics in the interval $[0, x]$, we get

$$\begin{cases} D(x,t)=D_0(t-x)+\int_0^x d^{-1}(s,s)U(s,s)(D-U)(s,t-x+s)ds, \\ U(x,t)=U_0(t+x)-\int_0^x d^{-1}(s,s)U(s,s)(D-U)(s,t+x-s)ds. \end{cases} \tag{1.16}$$

Set

$$D_1(x,t)=d^{-1}(x)D(x,t),\quad U_1(x,t)=d^{-1}(x)U(x,t).$$

From system (1.11) we have

$$\begin{cases} \frac{\partial D_1}{\partial t}+\frac{\partial D_1}{\partial x}=-\beta(x)d^{-1}(x)U(x,t)=-U_1(x,x)U_1(x,t), \\ \frac{\partial U_1}{\partial t}-\frac{\partial U_1}{\partial x}=\beta(x)d^{-1}(x)D(x,t)=U_1(x,x)D_1(x,t). \end{cases}$$

Consequently

$$\begin{cases} D_1(x,t)=D_0(t-x)-\int_0^x U_1(s,s)U_1(s,t-x+s)ds, \\ U_1(x,t)=U_0(t+x)-\int_0^x U_1(s,s)D_1(s,t+x-s)ds. \end{cases} \tag{1.17}$$

Definition. *x^* is called a singular point of problem* (1.11) *if $\int_0^x|\beta(s)|ds$ is finite for $x<x^*$ and tends to $+\infty$ as $x\to x^*$. The interval $[0, T)$ is called normal if there is no singular point in it. The normal interval $(0, T)$ is called the largest normal interval if T is a singular point.*

Remark. For (1.16), if $\left|\int_0^{x^*}\beta(s)\,ds\right|=+\infty$, then $\int_0^{x^*}\beta(s)ds=+\infty$ or $-\infty$. By (1.13) they correspond to $d(x^*)=+\infty$ or 0 respectively. If $\left|\int_0^{x^*}\beta(s)\,ds\right|$ is finite, then $d(x^*)$, similarly $c(x^*)$, is also finite. It implies that the variation of $c(x)$ is unbounded. For (1.17), as $\beta(x)=U_1(x,x)$, a singular point x^* is such that $\int_0^{x^*}|U_1(s,s)|ds=+\infty$.

The problem (1.17) has been studied in [2]. The following result is obtained.

For any $D_0(t), U_0(t)\in L^1(0,\infty)$, the problem (1.17) has a unique solution in $S_{(0,T)}$, where $(0, T)$ is the largest normal interval of this problem, and if $\|D_0\|_{L^1}$ and $\|U_0\|_{L^1}$ are sufficiently small, then $T=+\infty$.

§ 2. Difference Schemes and Lemmas

We now construct difference schemes for the problems (1.16) and (1.17) in the domain $S_T=\{(x, t)\,|\,0<x<T,\ x<t<2T-x\}$.

Take $\Delta x=\Delta t=h=T/J$, where J is an integer.

For the problem (1.17), we have

Scheme I.

$$\begin{cases} D_k^n=D_{k-1}^{n-1}-hU_{k-1}^{k-1}U_{k-1}^{n-1}, \\ U_k^n=U_{k-1}^{n+1}-hU_{k-1}^{k-1}D_{k-1}^{n+1}, \quad k=1, \cdots, J;\ n=k, \cdots, 2J-k, \end{cases} \tag{2.1}$$

where U_k^n and D_k^n are approximations of $U_1(k\Delta x, n\Delta t)$ and $D_1(k\Delta x, n\Delta t)$ respectively. The approximate value β_k of $\beta(k\Delta x)$ can be determined by

$$\beta_k^k=U_k^k. \tag{2.2}$$

For the problem (1.16) we have

Scheme II.

$$\begin{cases} D_k^n=D_{k-1}^{n-1}+hd_{k-1}^{-1}U_{k-1}^{k-1}(D_{k-1}^{n-1}-U_{k-1}^{n-1}), \\ U_k^n=U_{k-1}^{n+1}-hd_{k-1}^{-1}U_{k-1}^{k-1}(D_{k-1}^{n+1}-U_{k-1}^{n+1}), \\ d_k=d_{k-1}+hU_{k-1}^{k-1}, \quad k=1, \cdots, J;\ n=k, \cdots, 2J-k, \end{cases} \tag{2.3}$$

where U_k^n, D_k^n and d_k are approximations of $U(k\Delta x, n\Delta t)$, $D(k\Delta x, n\Delta t)$ and $d(k\Delta x)$ respectively.

Obviously, Schemes I and II are the first order approximations of (1.17) and (1.16). The numerical solution of these schemes can be obtained step by step in advancing in the x-direction. In the next section we will discuss the convergence of these schemes. Before that we introduce some lemmas.

Lemma 1. *Let a non-negative series $\{E_k\}$ satisfy the following inequality*

$$E_k\leqslant E_{k-1}+C_1h(C_2E_{k-1}^3+C_3E_{k-1}^2+C_4E_{k-1})+C_5h\cdot\eta \tag{2.4}$$

and

$$E_0\leqslant C_0\cdot\eta. \tag{2.5}$$

Then

$$E_k\leqslant C_0\cdot\lambda\cdot\eta, \quad \textit{for } k\cdot h\leqslant\tau_0, \tag{2.6}$$

where

$$\tau_0=\frac{(\lambda-1)C_0}{C_0C_1\lambda(C_2(C_0\lambda\eta)^2+C_3(C_0\lambda\eta)+C_4)+C_5}, \tag{2.7}$$

and $\lambda(>1)$ is a constant.

Proof. We shall prove this lemma by induction with respect to k.

Obviously (2.6) holds for $k=0$. Suppose that $(k+1)\cdot h\leqslant\tau_0$ and (2.6) holds for all $i\leqslant k$. Then by (2.4) we get

$$\begin{aligned} E_{k+1}&\leqslant E_k+C_1h(C_2E_k^3+C_3E_k^2+C_4E_k)+C_5h\eta \\ &\leqslant E_0+C_1h\sum_{i=0}^{k}(C_2E_i^3+C_3E_i^2+C_4E_i)+C_5(k+1)h\eta \\ &\leqslant C_0\eta+\tau_0\eta(C_1C_0\lambda(C_2(C_0\lambda\eta)^2+C_3C_0\lambda\eta+C_4)+C_5) \end{aligned}$$

$$\leqslant C_0\eta+(\lambda-1)C_0\eta=\lambda C_0\eta.$$

Lemma 2. *Let $T>0$ be an arbitrary positive number and $\{E_k\}$ satisfy (2.4) and (2.5) in Lemma 1. Assume $\eta<\lambda^{-\sigma}$. Then we have*

$$E_k\leqslant\lambda^\sigma C_0\eta,\quad for\ k\cdot h\leqslant T, \tag{2.8}$$

where $\sigma=\left[\frac{T}{\tau}\right]^{1)}+1$,

$$\tau=\frac{(\lambda-1)C_0}{C_0C_1\lambda(C_2C_0^2+C_3C_0+C_4)+C_5}. \tag{2.9}$$

Proof. Set

$$\tau^i=\frac{(\lambda-1)\lambda^{i-1}C_0}{C_1C_0\lambda^i(C_2(C_0\lambda^i\eta)^2+C_3C_0\lambda^i\eta+C_4)+C_5},\quad i=1,\cdots,\sigma.$$

By assumption, we have

$$\eta\cdot\lambda^i\leqslant\eta\cdot\lambda^\sigma\leqslant 1,$$

and

$$C_0C_1\lambda^i(C_2(C_0\lambda^i\eta)^2+C_3C_0\lambda^i\eta+C_4)+C_5$$
$$\leqslant\lambda^{i-1}(C_0C_1\lambda(C_2C_0^2+C_3C_0+C_4)+C_5).$$

Hence

$$\tau=\frac{(\lambda-1)C_0}{C_1C_0\lambda(C_2C_0^2+C_3C_0+C_4)+C_5}$$
$$\leqslant\frac{(\lambda-1)C_0\lambda^{i-1}}{C_1C_0\lambda^i(C_2(C_0\lambda^i\eta)^2+C_3C_0\lambda^i\eta+C_4)+C_5}=\tau^i.$$

Denote $\delta_0=0$, $\delta_i=i\cdot\tau$, $i=1, 2, \cdots, \sigma$. Now we shall prove that

$$E_k\leqslant\lambda^i C_0\eta,\quad \text{for } k\in[\delta_{i-1}/h,\ \delta_i/h],$$

by induction with respect to i.

For $i=1$, it is just Lemma 1. Suppose the assertion is true for $i-1$. Then there exists k_0 such that

$$k_0h\leqslant\delta_{i-1}\leqslant(k_0+1)h$$

and

$$E_{k_0}\leqslant\lambda^{i-1}C_0\eta.$$

Set $\tilde{E}_j=E_{k_0+j}$. Then $\{\tilde{E}_j\}$ satisfies (2.4) and (2.5) with $C_0\lambda^{i-1}$ as a new C_0. In this case, τ_0 is replaced by τ^i. Now from Lemma 1 it follows that

$$\tilde{E}_j\leqslant\lambda(\lambda^{i-1}C_0)\eta=\lambda^iC_0\eta,\quad \text{for } j\cdot h\leqslant\tau\leqslant\tau^i.$$

Returning to E_k, we have

$$E_k\leqslant C_0\lambda^i\eta,\quad \text{for } k\in[\delta_{i-1}/h,\ \delta_i/h].$$

Remark. This lemma is different from Lemma 1 by introducing the restriction $\eta<\lambda^{-\sigma}$ on η. Without this restriction we are only able to estimate E_k in the interval $[0, \tau_0]$. But in Lemma 2 we can make estimation on E_k in the interval $[0, T]$ for any $T>0$ so long as η is sufficiently small.

1) Here $[a]$ represents the integer part of the number a.

§ 3. Convergence

We now discuss the convergence and the stability of the difference schemes constructed above.

Denote briefly $D(k, n) = D(k\Delta x, n\Delta t)$, $U(k, n) = U(k\Delta x, n\Delta t)$ and $d(k) = d(k\Delta x)$,

$$\|D_k\| = \max_n |D_k^n|, \quad \|U_k\| = \max_n |U_k^n|,$$
$$\bar{D}_k^n = D_k^n - D(k, n), \quad \bar{U}_k^n = U_k^n - U(k, n),$$
$$\bar{d}_k = d_k - d(k).$$

First assume that $D(x, t)$, $U(x, t)$ is the solution of the problem (1.17) and D_k^n, U_k^n is the discrete solution of Scheme I.

Theorem 1. 1) *If $D(x, t)$ and $U(x, t)$ are continuous in $\bar{S}_T$, then $\forall \varepsilon \in (0, 2^{-\sigma})$, $\exists \delta > 0$, such that as $h < \delta$ and*

$$\|\bar{D}_0\| + \|\bar{U}_0\| \leqslant \varepsilon, \tag{3.1}$$

we have

$$\|\bar{D}_k\| + \|\bar{U}_k\| \leqslant 2^\sigma \varepsilon, \quad \text{for } k \cdot h < T, \tag{3.2}$$

where $\sigma = \left[\frac{T}{\tau}\right] + 1$, $\tau = (2 + 10M)^{-1}$, $M = \max(\|D\|_{C(\bar{S}_T)}, \|U\|_{C(\bar{S}_T)})$.

2) *If $D(x, t)$ and $U(x, t)$ are Hölder continuous of order α, then as $h^\alpha < 2^{-\sigma_1}$ and*

$$\|\bar{D}_0\| + \|\bar{U}_0\| \leqslant h^\alpha, \tag{3.3}$$

there holds the inequality

$$\|\bar{D}_k\| + \|\bar{U}_k\| \leqslant 2^{\sigma_1} h^\alpha, \quad \text{for } k \cdot h \leqslant T, \tag{3.4}$$

where $\sigma_1 = \left[\frac{T}{\tau_1}\right] + 1$, $\tau_1 = (2 + 6M + 4MM_1)^{-1}$, $M_1 = \max([D]_{\alpha, \bar{S}_T}, [U]_{\alpha, \bar{S}_T})$[1].

Proof.

$$\begin{aligned}
\bar{D}_k^n &= D_k^n - D(k, n) \\
&= D_{k-1}^{n-1} - hU_{k-1}^{k-1}U_{k-1}^{n-1} - D(k-1, n-1) + \int_{(k-1)h}^{kh} U(s, s)U(s, (n-k)h + s)\,ds \\
&= \bar{D}_{k-1}^{n-1} - h(\bar{U}_{k-1}^{k-1}\bar{U}_{k-1}^{n-1} + U(k-1, k-1)\bar{U}_{k-1}^{n-1} + \bar{U}_{k-1}^{k-1}U(k-1, n-1)) \\
&\quad - \int_0^h \{U(k-1, k-1) - U((k-1)h + s, (k-1)h + s)\}U(k-1, n-1)\,ds \\
&\quad - \int_0^h U((k-1)h + s, (k-1)h + s)\{U(k-1, n-1) \\
&\quad - U((k-1)h + s, (n-1)h + s)\}\,ds.
\end{aligned} \tag{3.5a}$$

Similarly,

$$\begin{aligned}
\bar{U}_k^n &= \bar{U}_{k-1}^{n+1} - h(\bar{U}_{k-1}^{k-1}\bar{D}_{k-1}^{n+1} + U(k-1, k-1)\bar{D}_{k-1}^{n+1} + \bar{U}_{k-1}^{k-1}D(k-1, n+1)) \\
&\quad - \int_0^h \{U(k-1, k-1) - U((k-1)h + s, (k-1)h + s)\}D(k-1, n+1)\,ds
\end{aligned}$$

1) $[f]_{\alpha, D}$ represents the Hölder coefficient of f in the domain D, i.e.

$$[f]_{\alpha, D} = \sup_{P, Q \in D, P \neq Q} \frac{|f(Q) - f(P)|}{|P - Q|^\alpha}.$$

$$-\int_0^h U((k-1)h+s,\ (k-1)h+s)\{D(k-1,\ n+1)-D((k-1)h+s,\ (n+1)h-s)\}ds. \tag{3.5b}$$

1) By assumption, D and U are uniformly continuous in S_T. So $\forall \varepsilon>0$, $\exists \delta>0$ such that as $|x_1-x_2|<\delta$ and $|t_1-t_2|<\delta$,

$$|D(x_1,\ t_1)-D(x_2,\ t_2)|<\varepsilon, \quad |U(x_1,\ t_1)-U(x_2,\ t_2)|<\varepsilon.$$

Take $h<\delta$. Then by (3.5)

$$|\overline{D}_k^n|\leqslant|\overline{D}_{k-1}^{n-1}|+h(|\overline{U}_{k-1}^{k-1}|\,|\overline{U}_{k-1}^{n-1}|+M(|\overline{U}_{k-1}^{n-1}|+|\overline{U}_{k-1}^{k-1}|))+2M\varepsilon h.$$

Hence

$$\|\overline{D}_k\|\leqslant\|\overline{D}_{k-1}\|+h(\|\overline{U}_{k-1}\|^2+2M\|\overline{U}_{k-1}\|)+2M\varepsilon h. \tag{3.6a}$$

Similarly we have

$$\|\overline{U}_k\|\leqslant\|\overline{U}_{k-1}\|+h(\|\overline{U}_{k-1}\|\cdot\|\overline{D}_{k-1}\|+M(\|\overline{D}_{k-1}\|+\|\overline{U}_{k-1}\|))+2M\varepsilon h. \tag{3.6b}$$

Denote $E_k=\|\overline{D}_k\|+\|\overline{U}_k\|$. Then by (3.6) we get

$$E_k\leqslant E_{k-1}+h(E_{k-1}^2+3ME_{k-1})+4Mh\varepsilon.$$

By assumption, $E_0\leqslant\varepsilon$, so E_k satisfies (2.4) and (2.5) with $C_0=C_1=C_3=1$, $C_2=0$, $C_4=3M$ and $C_5=4M$. Hence the corresponding $\tau=[2+10M]^{-1}(\lambda=2)$. If $\varepsilon<2^{-\sigma}$, then it follows from Lemma 2 that as $h<\delta(\varepsilon)$ we have

$$\|\overline{D}_k\|=\|\overline{U}_k\|<2^{\sigma}\varepsilon, \quad \text{for } k\cdot h\leqslant T,$$

where $\sigma=\left[\dfrac{T}{\tau}\right]+1$.

2) In this case, by (3.5) we have

$$|\overline{D}_k^n|\leqslant|\overline{D}_{k-1}^{n-1}|+h(|\overline{U}_{k-1}^{k-1}|\,|\overline{U}_{k-1}^{n-1}|+M(|\overline{U}_{k-1}^{k-1}|+|\overline{U}_{k-1}^{n-1}|))+2MM_1h^{\alpha+1}.$$

Therefore

$$\|\overline{D}_k\|\leqslant\|\overline{D}_{k-1}\|+h(\|\overline{U}_{k-1}\|^2+2M\|\overline{U}_{k-1}\|)+2MM_1h^{\alpha+1}.$$

Analogously,

$$\|\overline{U}_k\|<\|\overline{U}_{k-1}\|+h(\|\overline{D}_{k-1}\|\cdot\|\overline{U}_{k-1}\|+M(\|\overline{D}_{k-1}\|+\|\overline{U}_{k-1}\|))+2MM_1h^{\alpha+1}.$$

Then

$$E_k\leqslant E_{k-1}+h(E_{k-1}^2+3ME_{k-1})+4MM_1h^{\alpha+1}.$$

By assumption, $E_0=\|\overline{D}_0\|+\|\overline{U}_0\|\leqslant h^{\alpha}$. Thus the conditions in Lemma 2 are satisfied with $C_0=C_1=C_3=1$, $C_2=0$, $C_4=3M$ and $C_5=4MM_1$, $\eta=h^{\alpha}$, and $\tau_1=(2+6M+4MM_1)^{-1}$. Then as $h^{\alpha}<2^{-\sigma_1}$,

$$\|\overline{D}_k\|+\|\overline{U}_k\|\leqslant 2^{\sigma_1}h^{\alpha}, \quad \text{for } k\cdot h\leqslant T,$$

where $\sigma_1=\left[\dfrac{T}{\tau_1}\right]+1$.

Corollary. Under the assumption of Theorem 1, the solution D_k^n and U_k^n is uniformly bounded, i.e. $\exists\overline{M}>0$, such that

$$|D_k^n|\leqslant\overline{M}, \quad |U_k^n|\leqslant\overline{M},$$

for all k and h satisfying $k\cdot h\leqslant T$.

Theorem 2. *Let D_k^n, U_k^n and $\tilde{D}_k^n$, $\tilde{U}_k^n$ be solutions of equations* (2.1) *corresponding* *to different initial conditions* D_0, U_0 *and* $\tilde{D}_0$, $\tilde{U}_0$ *respectively. If one of these solutions has a uniform bound* $\overline{M}$, *then as* $h\leqslant 2^{-\sigma}$ *and*

$$\|D_0-\tilde{D}_0\|+\|U_0-\tilde{U}_0\|\leqslant h,$$

we have

$$\|D_k-\tilde{D}_k\|+\|U_k-\tilde{U}_k\|\leqslant 2^{\sigma}(\|D_0-\tilde{D}_0\|+\|U_0-\tilde{U}_0\|),\quad \textit{for } k\cdot h\leqslant T,$$

where $\sigma=\sigma(\overline{M}, T)$.

Proof. Assume D_k^n and U_k^n have a uniform bound $\overline{M}$. Since

$$D_k^n=D_{k-1}^{n-1}-hU_{k-1}^{k-1}U_{k-1}^{n-1},$$
$$\tilde{D}_k^n=\tilde{D}_{k-1}^{n-1}-h\tilde{U}_{k-1}^{k-1}\tilde{U}_{k-1}^{n-1},$$

then

$$\overline{D}_k^n=D_k^n-\tilde{D}_k^n=\overline{D}_{k-1}^{n-1}-h(\overline{U}_{k-1}^{k-1}\overline{U}_{k-1}^{n-1}+U_{k-1}^{k-1}\overline{U}_{k-1}^{n-1}+U_{k-1}^{n-1}\overline{U}_{k-1}^{k-1}).$$

Therefore

$$\|\overline{D}_k\|\leqslant\|\overline{D}_{k-1}\|+h(\|\overline{U}_{k-1}\|^2+2M\|\overline{U}_{k-1}\|).$$

Similarly,

$$\|\overline{U}_k\|\leqslant\|\overline{U}_{k-1}\|+h(\|\overline{U}_{k-1}\|\cdot\|\overline{D}_{k-1}\|+M(\|\overline{D}_{k-1}\|+\|\overline{U}_{k-1}\|)).$$

Set $E_k=\|\overline{D}_k\|+\|\overline{U}_k\|$. Then

$$E_k\leqslant E_{k-1}+h(E_{k-1}^2+3\overline{M}E_{k-1}).$$

Take $\sigma=\left[\frac{T}{\tau}\right]+1$ and $\tau=(2+6\overline{M})^{-1}$. Then the conditions of Lemma 2 are satisfied. The theorem follows immediately.

We now discuss the convergence of Scheme II. From now on, we always assume that $D(x, t)$, $U(x, t)$ and $d(x)$ is the solution of the problem (1.16) and D_k^n, U_k^n and d_k is the discrete solution of Scheme II. For given D_0 and U_0, the solutions D_k^n, U_k^n and d_k of the problem (2.3) depend on the step-size h as a parameter. We denote this one-parameter family of solutions by $\{D_k^n, U_k^n, d_k; h\}$.

A family of solutions $\{D_k^n, U_k^n, d_k; h\}$ is said to have the property PLB (positively lower bound) in the interval $[0, \tau]$ if $\exists\delta>0$, $c_0>0$ such that as $h\leqslant\delta$,

$$d_k\geqslant c_0>0,\quad \text{for } k\cdot h\leqslant\tau. \tag{3.7}$$

Sometimes in order to emphasize the parameters δ and c_0, we say that $\{D_k^n, U_k^n, d_k; h\}$ has the property PLB(δ, c_0) in the interval $[0, \tau]$. It is obvious that if $\{D_k^n, U_k^n, d_k; h\}$ has PLB(δ, c_0) in $[0, \tau]$ then so does it in $[0, \tau']$ for $\tau'<\tau$.

Lemma 3. *Let* D_k^n, U_k^n *and* d_k *be a solution of equation* (2.3). *If* $d_{j_0}>0$ *for some* j_0 *then*

$$\|D_{j_0+k}\|+\|U_{j_0+k}\|\leqslant 2(\|D_{j_0}\|+\|U_{j_0}\|),$$

$$d_{j_0+k}\geqslant\frac{1}{2}d_{j_0},\quad \textit{for } k\cdot h\leqslant\tau,$$

where

$$\tau=\frac{d_{j_0}}{16(\|D_{j_0}\|+\|U_{j_0}\|)}.$$

Proof. Set $E_k=\|D_k\|+\|U_k\|$. Suppose that $E_{j_0+i}\leqslant 2E_{j_0}$ and $d_{j_0+i}\geqslant\frac{1}{2}d_{j_0}$ for $i=1, 2, \cdots, k-1$, $i\cdot h\leqslant\tau$. Then by (2.3),

$$\|D_{j_0+k}\| \le \|D_{j_0+k-1}\| + hd_{j_0+k-1}^{-1}\|U_{j_0+k-1}\|(\|D_{j_0+k-1}\| + \|U_{j_0+k-1}\|)$$
$$\le \|D_{j_0}\| + h\sum_{i=0}^{k-1} d_{j_0+i}^{-1}\|U_{j_0+i}\| E_{j_0+i},$$
$$\|U_{j_0+k}\| \le \|U_{j_0}\| + h\sum_{i=0}^{k-1} d_{j_0+i}^{-1}\|U_{j_0+i}\| E_{j_0+i},$$
$$d_{j_0+k} \ge d_{j_0+k-1} - h\|U_{j_0+k-1}\| \ge d_{j_0} - h\sum_{i=0}^{k-1}\|U_{j_0+i}\|.$$

Hence

$$E_{j_0+k} \le E_{j_0} + 2h\sum_{i=0}^{k-1} d_{j_0+i}^{-1}E_{j_0+i}^2 \le E_{j_0} + 2h\cdot k\cdot 2d_{j_0}^{-1}(2E_{j_0})^2$$
$$\le E_{j_0} + 16\tau d_{j_0}^{-1}E_{j_0}^2 \le 2E_{j_0},$$
$$d_{j_0+k} \ge d_{j_0} - h\cdot k\cdot 2E_{j_0} \ge d_{j_0} - \frac{1}{8}d_{j_0} \ge \frac{1}{2}d_{j_0}.$$

Taking $j_0=0$ in Lemma 3, we may see that for any initial value there always exists τ_0 such that the family $\{D_k^n, U_k^n, d_k; h\}$ of the problem (2.3) has the property PLB in the interval $[0, \tau_0]$. On the other hand, for any τ_0, there always exists E^* such that as $\|D_0\| + \|U_0\| < E^*$ the family $\{D_k^n, U_k^n, d_k; h\}$ has the property PLB in the interval $[0, \tau_0]$. But in general, we do not know whether or not the family $\{D_k^n, U_k^n, d_k; h\}$ has the property PLB.

Theorem 3. *Let $[0, T]$ be the normal interval of the problem (1.16) and suppose $\{D_k^n, U_k^n, d_k; h\}$ has the property $PLB(\delta_0, c_0)$ in the interval $[0, T]$.*

1) *If D, U and d are continuous in $\bar{S}_T$, then $\forall s\in(0, 2^{-\sigma})$, $\exists\delta>0$ such that as $h\le\min(\delta, \delta_0)$ and $\|\bar{D}_0\| + \|\bar{U}_0\| \le s$,*

$$\|\bar{D}_k\| + \|\bar{U}_k\| + |\bar{d}_k| \le 2^\sigma s, \quad \text{for } k\cdot h\le T.$$

2) *If D, U and d are Hölder continuous of order α in $\bar{S}_T$, then as $h^\alpha\le\min(\delta_0, 2^{-\sigma_1})$ and $\|\bar{D}_0\| + \|\bar{U}_0\| \le h^\alpha$,*

$$\|\bar{D}_k\| + \|\bar{U}_k\| + |\bar{d}_k| \le 2^{\sigma_1}h^\alpha, \quad \text{for } k\cdot h\le T,$$

where $\sigma = \left[\frac{T}{\tau}\right]+1, \quad \sigma_1 = \left[\frac{T}{\tau_1}\right]+1,$

$$\tau = (4m^{-1}(c_0^{-1}(1+3M+4M^2)+2+3M)+3M(M+4m^{-1}))^{-1},$$
$$\tau_1 = (4m^{-1}(c_0^{-1}(1+3M+4M^2)+2+3M)+3MM_1(M+4m^{-1}))^{-1},$$
$$m = \inf_{x\in[0,T]} d(x), \quad M = \max(\|D\|_c, \|U\|_c, \|d\|_c),$$
$$M_1 = \max([D]_{\alpha,\bar{S}_T}, [U]_{\alpha,\bar{S}_T}, [d]_{\alpha,[0,T]}).$$

The proof is similar to that of Theorem 1 and is omitted here.

Theorem 4. *Let $[0, T^*)$ be the largest normal interval of the problem (1.16) and $T<T^*$. If the solution of the problem (1.16) is continuous in the domain $\bar{S}_T$ then the solution of Scheme II converges to it as $h\to 0$.*

Proof. By Theorem 3 we need only to prove that the family $\{D_k^n, U_k^n, d_k; h\}$ has the property PLB in the interval $[0, T]$.

By hypothesis,

$$M^* = \max(\|D\|_{C(\bar{S}_T)}, \|U\|_{C(\bar{S}_T)}, \|d\|_{C(0,T)}) < +\infty,$$

$$m^* = \inf_{x \in [0,T]} d(x) > 0.$$

Denote $I = \{T' \leqslant T \mid \{D_k^n, U_k^n, d_k; h\}$ has the property PLB in $[0, T']\}$. It is obvious that I is not empty and if $T' \in I$ then $[0, T'] \subset I$ with the same parameters δ' and c'.

Set $T^0 = \sup I$. Of course $T^0 > 0$. For any $T_1 < T^0$, Theorem 3 holds in the domain $\{(x, t) \mid 0 \leqslant x \leqslant T_1, x < t < 2T - x\}$. Notice that in Theorem 3 we may use the uniform bound M^* and m^*, i.e. τ_0, δ and σ depend only on M^*, m^*, $c_1(T_1)$ and $\delta_1(T_1)$, where $c_1(T_1)$ and $\delta_1(T_1)$ are parameters of PLB(δ_1, c_1) in the interval $[0, T_1]$.

We want to prove $T^0 = T$. If it is not true, i.e. $T^0 < T$, then take T_1 satisfying

$$T^0 - \frac{m^*}{128\, M^*} < T_1 < T^0.$$

By Theorem 3, $\forall \varepsilon \in (0, 2^{-\sigma})$, $\exists \delta \leqslant \delta_1(T_1)$, such that as $h \leqslant \delta$,

$$\|\bar{D}_k\| + \|\bar{U}_k\| + |\bar{d}_k| \leqslant 2^\sigma \varepsilon, \quad \text{for } k \cdot h \leqslant T_1,$$

where $\sigma = \sigma(M^*, m^*, c_1(T_1), T_1)$. Take ε_0 sufficiently small such that

$$2^\sigma \varepsilon_0 \leqslant \min\left(M^*, \frac{1}{2} m^*\right).$$

Then

$$\|D_k\| + \|U_k\| \leqslant 2\, M^*, \tag{3.8}$$

$$d_k \geqslant d(k) - 2^\sigma \varepsilon_0 \geqslant \frac{1}{2} m^*, \quad \text{for } k \cdot h \leqslant T_1. \tag{3.9}$$

For any fixed $h\left(\leqslant \min\left(\delta(\varepsilon_0, \delta_1), T_1 - T^0 + \frac{m^*}{128\, M^*}\right)\right)$, take k_0 satisfying $T^0 - \frac{m^*}{128 M^*} \leqslant k_0 \cdot h \leqslant T_1$. By (3.8) and (3.9),

$$\|D_{k_0}\| + \|U_{k_0}\| \leqslant 2M^*, \quad d_{k_0} \geqslant \frac{1}{2} m^*.$$

It follows from Lemma 3 that

$$\|D_{k_0+j}\| + \|U_{k_0+j}\| \leqslant 4M^*, \quad d_{k_0+j} \geqslant \frac{1}{4} m^*, \text{ for } j \cdot h \leqslant \tau,$$

where $\tau = \frac{m^*}{64\, M^*}$. It shows that for such h the difference equation (2.3) is solvable in $S_{k_0 h + \tau}$ and $d_k \geqslant \frac{1}{4} m^*$ for $k \cdot h \leqslant k_0 h + \tau$. In other words, $\{D_k^n, U_k^n, d_k; h\}$ has the property PLB in the interval $[0, k_0 h + \tau]$. Hence $k_0 h + \tau \in I$. But

$$k_0 \cdot h + \tau \geqslant T^0 - \frac{m^*}{128 M^*} + \frac{m^*}{64 M^*} = T^0 + \frac{m^*}{64 M^*} > T^0.$$

This contradicts $T^0 = \sup I$. It implies that $T^0 = T$.

Similarly, if D, U and d are Hölder continuous of order α, we can show that in $\bar{S}_T$ the solution of (2.3) converges to them in the rate of h^α as $h \to 0$. Of course, it is necessary to assume $T < T^*$.

Remark. Here we only analyse two difference schemes. Other similar difference schemes for solving the inverse problem numerically can be analysed in the same way. The numerical simulations of Schemes I and II are satisfactory. We will show them in another paper together with numerical results of other difference schemes.

Appendix. By the theory of propagation of singularity (see [6], Ch. 6), D, U can be expanded as

$$D(x,t)=\delta(t-x)g_1^{(1)}(x)+\eta(t-x)g_1^{(2)}(x)+g_1^{(3)}(x,t), \tag{A. 1a}$$

$$U(x,t)=\delta(t-x)g_2^{(1)}(x)+\eta(t-x)g_2^{(2)}+g_2^{(3)}(x,t), \tag{A. 1b}$$

where $g_j^{(i)}(x)$, $g_j^{(3)}(x,t)$ $(i,j=1,2)$ are suitably smooth functions and $g_j^{(3)}(x,t)$ $(j=1,2)$ are at least continuous in the domain $\{x>0,\ t>0\}$. Differentiating (A. 1a) with respect to t and x, we get

$$\frac{\partial D}{\partial t}=\delta'(t-x)g_1^{(1)}+\delta(t-x)g_1^{(2)}+\frac{\partial g_1^{(3)}}{\partial t},$$

$$\frac{\partial D}{\partial x}=-\delta'(t-x)g_1^{(1)}+\delta(t-x)\left(\frac{\partial g_1^{(1)}}{\partial x}-g_1^{(2)}\right)+\eta(t-x)\frac{\partial g_1^{(2)}}{\partial x}+\frac{\partial g_1^{(3)}}{\partial x}.$$

Substituting them into the first equation of (1.4), we get

$$\delta(t-x)\frac{\partial g_1^{(1)}}{\partial x}+\eta(t-x)\frac{\partial g_1^{(2)}}{\partial x}+\frac{\partial g_1^{(3)}}{\partial x}$$
$$=\beta(x)\delta(t-x)(g_1^{(1)}-g_2^{(1)})+\beta(x)\eta(t-x)(g_1^{(2)}-g_2^{(2)})+\beta(x)(g_1^{(3)}-g_2^{(3)}). \tag{A. 2a}$$

Similarly, we have

$$2\delta'(t-x)g_2^{(1)}+\delta(t-x)\left(2g_2^{(2)}-\frac{\partial g_2^{(1)}}{\partial x}\right)+\frac{\partial g_2^{(3)}}{\partial t}-\eta(t-x)\frac{\partial g_2^{(2)}}{\partial t}-\frac{\partial g_2^{(3)}}{\partial x}$$
$$=\beta(x)\delta(t-x)(g_1^{(1)}-g_2^{(1)})+\beta(x)\eta(t-x)(g_1^{(2)}-g_2^{(2)})+\beta(x)(g_1^{(3)}-g_2^{(3)}). \tag{A. 2b}$$

Hence,

$$g_2^{(1)}=0,$$
$$\frac{\partial g_1^{(1)}}{\partial x}=\beta(x)g_1^{(1)},$$
$$2g_2^{(2)}=\beta(x)g_1^{(1)}.$$

By the boundary condition (1.7), we know $g_1^{(1)}(0)=2$. Therefore from the above equations it follows that

$$g_1^{(1)}(x)=2\exp\int_0^x\beta(s)ds,$$

$$g_2^{(2)}(x)=\beta(x)\exp\int_0^x\beta(s)ds.$$

Then (A. 1) become

$$D(x,t)=\delta(t-x)\cdot 2\exp\int_0^x\beta(s)ds+\tilde{D}(x,t),$$

$$U(x,t)=\eta(t-x)\cdot\beta(x)\exp\int_0^x\beta(s)ds+\tilde{U}(x,t),$$

where $\tilde{D}(x,t)$ has a discontinuity of the second kind on $x=t$ and $\tilde{U}(x,t)$ is continuous in the domain $\{x>0,\ t>0\}$. By the initial condition (1.6),

$$D(x, t) = U(x, t) = 0, \quad \text{in } x > t > 0.$$

It implies $\bar{U}(x, x) = 0$, so

$$U(x, x) = \beta(x) \exp \int_0^x \beta(s) ds.$$

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